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Special morphisms for zero-set spaces

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The author obtains characterizations of the quotients, epimorphisms and extreme monomorphisms in the category of separated zero-set spaces and zero-set maps (defined by Hugh Gordon [*Pacific J. Math.* 36 (1971), 133-157]). The method employed, that of initiality constructions, is also used to elucidate the relationship between zero-set spaces and certain other topological structures by means of forgetful functors and their right inverses. Characterizations of pseudocompactness for zero-set spaces then follow.

1. Introduction

In [*9*] Gordon defines the separated zero-set spaces, discusses their relationship to other topological type structures and introduces the concepts of realcompactness and pseudocompactness for these spaces. Zero-set spaces have also been studied by Canfell [*6*], Speed [*16*], and in the form of the separable M-fine spaces by Tashjian [*17*] and in detail by Hager [*10*, *11*, *12*, *13*]. Hager also points out that the axioms for a zero-set structure were formulated by Alexandroff in [*1*, *2*, *3*].

In this paper we obtain characterizations of the quotients, epimorphisms and extreme monomorphisms in the category of separated zero-

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set spaces and zero-set maps. The method employed, that of initiality constructions, also enables us to elucidate the relationship between zero-set spaces and certain other topological structures by means of forgetful functors and their right inverses. We then deduce characterizations of pseudocompactness for zero-set spaces.

NOTATION. By "function" we shall mean "real-valued function". We denote by 0 the constant function which vanishes everywhere.

The zero-set of a function f is denoted by $Zf = f^{-1}(0)$.

The set of all morphisms between two objects X and Y in a category A is denoted by $A(X, Y)$.

2. Background

We briefly review some basic results of Gordon [9].

A *zero-set space* X is a pair $(|X|, zX)$ where $|X|$ is a set and zX , the *zero-set structure* of X , is a collection of subsets of $|X|$ satisfying properties (1), (2), and (3) below. The sets in zX are called the *zero-sets* of X and their complements with respect to $|X|$ are the *cozero-sets* of X .

(1) zX is closed under finite unions and countable intersections; $\emptyset$ and $|X|$ are in zX .

(2) Disjoint zero-sets are contained in disjoint cozero-sets.

(3) Each cozero-set is a countable union of zero-sets of X .

If for each pair of distinct points in X there is a zero-set containing just one of them, then we call X a *separated zero-set space*.

Given zero-set spaces X and Y , a map $f : |X| \to |Y|$ is a *zero-set map* if the preimage of each zero-set of Y is a zero-set of X . The composition of zero-set maps is a zero-set map. Thus we can form the category $\mathcal{Zero}$ (respectively $\mathcal{Zeros}$) of zero-set spaces (respectively the separated zero-set spaces) and zero-set maps.

Denote by $\mathbb{R}$ the real line with the zero-set structure consisting of the closed sets in the usual topology. The collections $S(X)$ of all real-valued zero-set functions and $S^*(X)$ of all bounded real-valued zero-set functions, from a zero-set space X to $\mathbb{R}$, are uniformly closed rings and

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lattices, under the usual pointwise operations, and contain the constants. If X is separated then $S(X)$ and $S^*(X)$ separate points.

By a Urysohn lemma type argument we have the following theorem of Gordon's which justifies terminology.

THEOREM 2.1. *If X is a (separated) zero-set space, then*

$$zX = \{Zf : f \in S(X)\} = \{Zf : f \in S^*(X)\} .$$

3. Initiality considerations

The zero-set spaces considered in this section are not necessarily separated.

Given any family of maps $\{f_\alpha\}$ from a set A to zero-set spaces Y_α, we can always find a unique zero-set space X, with $|X| = A$, such that

(i) each f_α lifts to a zero-set map $: X \to Y_\alpha$, and

(ii) for any zero-set space W and zero-set maps $h_\alpha : W \to Y_\alpha$, if there is a map $k : |W| \to |X|$ with $h_\alpha = f_\alpha k$, then k lifts to a zero-set map $: W \to X$.

We say X *is initial for the* f_α *to the* Y_α. In fact zX is the collection of all countable intersections of finite unions of preimages of zero-sets in the Y_α under the f_α. That zX is a zero-set structure on A is a consequence of [9, 2.5].

If X and Y are zero-set spaces with $|X| = |Y|$, and if $zX \subset zY$, then we say X *is coarser than* Y, Y *is finer than* X. We have:

The initial zero-set structure defined above is the coarsest making each f_α a zero-set map.

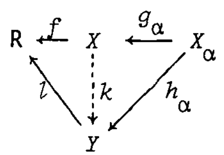
REMARKS. (1) The categorical dual concept to initiality is that of *coinitiality*. It follows by [4, 5] that *Zero* admits the formation of coinitial structures. Hence in particular *Zero* is complete, cocomplete, and has products, coproducts, and quotients.

(2) A *zero-set subspace* A of a zero-set space X is a subset $|A|$

of $|X|$ having the initial zero-set structure for the inclusion map $: |A| \to X$. Thus zA consists of sets which are the intersection of A with zero-sets of X [9].

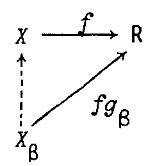
(3) A zero-set space X is initial for $S(X)$ to R .

Let $\{g_\alpha\}$ be a family of maps on zero-set spaces X_α to a set A . The existence of a finest zero-set space X , with $|X| = A$, which lifts each g_α to a zero-set map is ensured by (1) above. That is, X *is coinitial for the* g_α *from the* X_α . The following theorem gives a useful characterization in terms of initiality.



THEOREM 3.1. *Given zero-set spaces* X_α *and maps* $g_\alpha : |X_\alpha| \to |X|$, X *is coinitial for the* g_α *from the* X_α *if and only if* X *is initial to* R *for those functions* $f : |X| \to |\mathsf{R}|$ *which satisfy* $fg_\alpha \in S(X_\alpha)$ *for each* α .

Proof. Sufficiency. Fix β . Consider the following diagram where f ranges through the class of functions defined in our hypothesis.



Then, by initiality of X for the f , each g_β lifts to a zero-set map $: X_\beta \to X$. This deals with the dual of condition (i) of the definition. To check the dual of condition (ii), suppose $h_\alpha \in Zero(X_\alpha, Y)$ and let $k : |X| \to |Y|$ be such that $kg_\alpha = h_\alpha$ for each α . Y is initial for all l in $S(Y)$ to R . For each l and α , $(lk)g_\alpha = lh_\alpha \in S(X_\alpha)$, so lk is one of the f . Hence, by initiality of Y for the l , k is lifted to a zero-set map $: X \to Y$. Thus X is

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coinitial for the g_α from the X_α .

Necessity. We show that $\{f : fg_\alpha \in S(X_\alpha)$ for each $\alpha\} = S(X)$. The inclusion "$\supset$" is trivial and "$\subset$" follows by coinitiality of X for the g_α . Now it is known that X is initial for $S(X)$ to $\mathbb{R}$. The result follows.

REMARK. The zero-sets of the coinitial zero-set structure on X characterized in the above theorem are the countable intersections of finite unions of sets A with the following property: there exists an $f : |X| \to |\mathbb{R}|$ such that $A = Zf$ and $fg_\alpha \in S(X_\alpha)$ for each α .

In the terminology of Brümmer [5] an object X of a concrete category $\mathcal{A}$ is *separated* if, given that X is initial for a map f , then f is an injection and thus an embedding in $\mathcal{A}$.

PROPOSITION 3.2. *The objects of* $\mathcal{Zeros}$ *are precisely the separated objects of* $\mathcal{Zero}$.

We omit the straightforward proof.

4. Special morphisms in $\mathcal{Zeros}$

To characterise the epimorphisms and the extreme monomorphisms in $\mathcal{Zero}$ and $\mathcal{Zeros}$ we use the familiar notion of the *double of a space* Y *along a subset* U . That is we glue two copies of Y along U . In the case of the separated zero-set spaces it is necessary to impose some condition on U to ensure that the double is separated. For the non-separated case no such condition is necessary.

Let $i_1, i_2 : Y \to Y \amalg Y$ be the two inclusions into the coproduct of two copies of a zero-set space Y . Define an equivalence relation on $Y \amalg Y$ by: $a \equiv b$ if and only if either $a = b$, $i_1^{-1}(a) = i_2^{-1}(b) \in U$ or $i_1^{-1}(b) = i_2^{-1}(a) \in U$. Let $q : Y \amalg Y \to Q$ be the natural map defined onto the space of equivalence classes with the quotient (that is, coinitial) zero-set structure. Let $r_n = qi_n$, $n = 1, 2$.

It is easy to show that the double Q of Y along U is the following pushout in $\mathcal{Zero}$:

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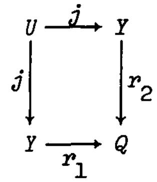
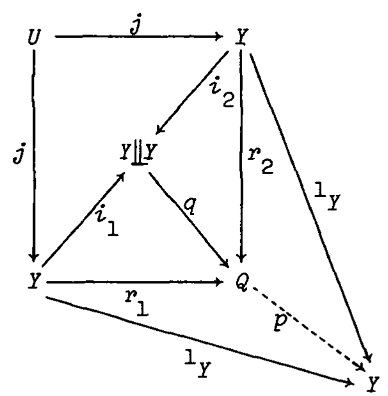
(A)



where j is the inclusion map.

LEMMA 4.1. *If Y is a separated zero-set space and U is a zero-set of Y then the diagram* (A) *is a pushout in* $\mathcal{Zeros}$.

Proof. We need to show that Q is separated. Consider the following diagram:



By the pushout property in $\mathcal{Zero}$ there exists a unique p in $\mathcal{Zero}(Q,\ Y)$ with $pr_1 = pr_2 = 1_Y$.

If $x \neq y$ in Q and $p(x) \neq p(y)$, then there is an A in zY containing only one of $p(x)$ and $p(y)$, since Y is separated. Then $p^{-1}(A) \in zQ$ and contains just one of x and y .

If $x \neq y$ in Q and $p(x) = p(y)$, we may assume that $x \in r_1(Y-U)$ and $y \in r_2(Y-U)$. By hypothesis $U \in zY$, hence there exists an f in $S(Y)$ with $U = Zf$. Define $g : |Q| \to |R|$ by:

$$gr_1 = f \text{ on } Y - U ,$$

$$g = 0 \text{ otherwise.}$$

Then $Q - r_1(Y-U) = Zg$, and it only remains to show that $g \in S(Q)$.

It is sufficient to show that $g^{-1}F \in zQ$ for each F closed in the real line with its usual topology. Every such F is the zero-set of some continuous and hence zero-set map $k : \mathbb{R} \to \mathbb{R}$. Thus $g^{-1}F = Z(kg)$ and, by definition of the quotient structure on Q, we need only show that $(kg)q \in S(Y \amalg Y)$. By coinitiality of $Y \amalg Y$ for i_1 and i_2 this is equivalent to showing that $(kg)r_n \in S(Y)$ for $n = 1$ and $n = 2$. Now $(kg)r_1 = kf \in S(Y)$ and $(kg)r_2$ is a constant function. This completes the proof.

The monomorphisms in *Zero* and *Zeros* are the injections. It follows from the pushout property that the epimorphisms in *Zero* are the surjections; the case for *Zeros* requires further consideration.

The co-zero-sets of a zero-set space X form a base for a topology on X which is completely regular and, if X is separated, is Hausdorff [9]. This topology is delivered by a forgetful functor F. By a *dense* subset of a zero-set space X we mean dense in the completely regular space FX.

PROPOSITION 4.2. *A morphism in Zeros is an epimorphism if and only if it is dense.*

Proof. Sufficiency follows by a standard argument.

Necessity. If $f : X \to Y$ in *Zeros* is not dense then the closure $\mathrm{cl}\, f(X) = \bigcap\{A : A \in zY,\ A \supset f(X)\}$ is not the whole of Y. Hence there exists an A in zY with $f(X) \subset A$ and $A \neq Y$. By the above lemma there is a separated zero-set space Q with maps r_1 and r_2 in *Zeros*(Y, Q) such that $A = \{y \in Y : r_1(y) = r_2(y)\}$. Then $r_1 f = r_2 f$ but $r_1 \neq r_2$.

It follows that *Zero* and *Zeros* are well- and cowell-powered.

The following theorem has well-known analogues for other topological-type structures.

THEOREM 4.3. *If* $f \in$ *Zero*(X, Y) (*respectively* $f \in$ *Zeros*(X, Y)) *then the following are equivalent:*

(1) f *is an embedding (respectively a closed embedding);*

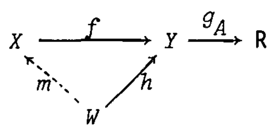
(2) f *is an equalizer (respectively a multiple equalizer);*

(3) f *is an extreme monomorphism.*

Proof. We prove (1) $\Rightarrow$ (2) for the separated case. The remaining implications have proofs analogous to those for the topological case.

If f is a closed embedding then $f(X) = \cap\mathcal{A}$ where $\mathcal{A} = \{A : A \in zY \text{ and } f(X) \subset A\}$. Then $A = Zg_A$ for each A in $\mathcal{A}$ and some g_A in $S(Y)$. We show that f equalizes the g_A.

Now certainly $g_Af = g_Bf = 0$ for each A and B in $\mathcal{A}$. Let h in *Zeros*(W, Y) be given.



Suppose $g_Ah = g_Bh$ for all A and B in $\mathcal{A}$. Then $Z(g_Ah) = Z(g_Bh)$ for each A and B. Thus, for a fixed A in $\mathcal{A}$,

$h^{-1}A = h^{-1}Zg_A = \cap\left\{h^{-1}Zg_B : B \in \mathcal{A}\right\} = h^{-1}\left[\cap\{Zg_B : B \in \mathcal{A}\}\right] = h^{-1}[f(X)]$.

In particular $Y \in \mathcal{A}$, thus $W = h^{-1}Y = h^{-1}[f(X)]$.

We can thus define $m : |W| \to |X|$ by $m(w) = x$ where $f(x) = h(w)$. Then m is well-defined and unique since f is an injection, and m lifts to a zero-set map since X is initial for f.

It should be remarked here that we have characterized the multiple equalizers only. We have been unable to characterize the equalizers (of pairs) in *Zeros*.

Since *Zero* is complete, well- and cowell-powered, a full subcategory is epireflective in *Zero* if and only if it is productive and hereditary [14, 10.2.1]. It is easily seen that *Zeros* is epireflective in *Zero* and is hence complete. Thus an epireflective subcategory of *Zeros* is one which is productive and closed hereditary.

5. Admissible structures and pseudocompactness

In [9] Gordon defines and characterizes pseudocompactness for zero-set spaces and shows, notably, that the product of pseudocompact zero-set spaces is again pseudocompact.

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The relationship between zero-set spaces and uniform spaces, as for topological and uniform spaces, leads to further characterizations of pseudocompactness. We briefly summarise the relationship of *Zero* to other topological type categories by means of functors.

We have seen that there is a forgetful functor F from *Zero* to *Crg* , the category of completely regular spaces and continuous maps. Now F has a unique right inverse $R : Crg \to Zero$ where the zero-sets of RX are the zero-sets of the continuous functions on X . Furthermore R is left adjoint to F and *Crg* is thus embedded as a full bicoreflective subcategory of *Zero* .

The above holds, *mutatis mutandis*, for the forgetful functor F' from the category of proximity spaces and proximity maps to *Zero* , which assigns to a proximity space the initial zero-set structure for the proximity functions (to the real line with its standard proximity structure) to $\mathbb{R}$. Here the unique right inverse P of F' , also left adjoint to F' , has the defining property: if X is a zero-set space, then sets in PX are distal if they are contained in disjoint zero-sets of X .

The functor $F : Zero \to Crg$ preserves initiality (in the obvious sense) and thus products. Also F clearly preserves extreme monomorphisms. It is then straightforward to prove the following special case of a theorem of Brümmer [5, 1.9.2].

PROPOSITION 5.1. *If* A *is an epireflective subcategory of* *Crg* , *then the class of all zero-set spaces* X , *with* FX *an object of* A , *forms an epireflective subcategory of* *Zero* .

EXAMPLES. **(1)** Gordon [9] defines a zero-set space X to be *compact* if FX is compact. A compact zero-set space Y is a *compactification* of a zero-set space X if X is a dense zero-set subspace of Y . Thus the separated compact zero-set spaces form an epireflective subcategory of *Zeros* .

(2) A z-ultrafilter on a separated zero-set space X is *real* if it has the countable intersection property; X is *realcompact* if every real z-ultrafilter is fixed [9]. The realcompact spaces form an epireflective subcategory of *Zeros* . In [8] I raised the question of whether the induced topology of a realcompact zero-set space is realcompact. This was

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answered in the affirmative by Hager, in a personal letter, using the Shirota theorem. A direct and simple proof is given by Salbany in [15]. Zero-set spaces with realcompact induced topology are not necessarily realcompact. The following counter-example was communicated by Hager.

Let X be an uncountable set of nonmeasurable cardinality equipped with the separated zero-set structure consisting of the countable and co-countable sets. The co-countable sets form a real z-ultrafilter on X which is not fixed. But FX is a discrete topological space and thus realcompact.

Let UX (respectively U^*X) be the initial uniform structure on a zero-set space X for $S(X)$ (respectively $S^*(X)$) to the real line with its usual uniformity. UX and U^*X are functorial. There is a forgetful functor F'' from the category of uniform spaces and uniformly continuous maps to *Zero* which assigns to a uniform space Y the initial zero-set structure for the real-valued uniformly continuous functions on Y, to $\mathbb{R}$. Both U and U^* are right inverses of F''. U^* has the further properties: for a zero-set space X,

(1) U^*X is the finest precompact uniform structure admitted by X;

(2) U^*X is the coarsest uniform structure on X that is delivered by a functor right inverse to F'';

(3) U^*X is the unique admissible precompact uniform structure on X in which any two disjoint zero-sets of X can be separated by a uniformly continuous function; (*cf.* [7 ; 15 I 3, 15 J 6]).

A separated zero-set space X is *pseudocompact* if every z-ultrafilter on X is real [9]. We have:

THEOREM 5.2. *The following are equivalent for a separated zero-set space* X :

(1) X *is pseudocompact;*

(2) $S(X) = S^*(X)$;

(3) $UX = U^*X$;

(4) *every admissible uniform structure on* X *is precompact;*

(5) *MX is the same for each right inverse M of F" ;*

(6) *X admits a unique compactification;*

(7) *X admits a unique precompact uniform structure;*

(8) *X admits a unique uniform structure.*

The analogues of conditions (1)-(5) are equivalent for a completely regular Hausdorff space X, as are the analogues of conditions (6)-(8), [7; 15 Q, R]. In general however the analogue of the implication "(1) $\Rightarrow$ (6)" is not true for these spaces. Gordon showed that for zero-set spaces conditions (1), (2), and (6) are equivalent [9]. The proofs of the remaining implications correspond to those for the topological case and may utilise the properties of U^* listed above.

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