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# Holomorphic Vanishing Theorems on Finsler Holomorphic Vector Bundles and Complex Finsler Manifolds

Bin Shen

*Abstract.* In this paper, we investigate the holomorphic sections of holomorphic Finsler bundles over both compact and non-compact complete complex manifolds. We also inquire into the holomorphic vector fields on compact and non-compact complete complex Finsler manifolds. We get vanishing theorems in each case according to different certain curvature conditions. This work can be considered as generalizations of the classical results on Kähler manifolds and hermitian bundles.

## 1 History and Introduction

A *holomorphic section*, or specifically, a *holomorphic vector field*, is the dual of a holomorphic form, which is a special kind of harmonic form. The harmonic form, because of its role in Hodge theory, connects analysis, topology and geometry. The relation of a holomorphic form and a harmonic form is a classical topic that has achieved some tremendous results [11].

The vanishing theorem of holomorphic sections on complex vector bundles is meaningful and has attracted a lot of mathematicians' attention. In 1946, S. Bochner proved that a compact Kähler manifold with negative Ricci curvature admits no non-zero holomorphic vector fields [4]. This result may seem inconspicuous nowadays, however, it is the first time to use the so-called Bochner's technique on complex manifolds. In 1970, S. Kobayashi and H. Wu generalized the result to the case of holomorphic vector bundles over compact complex manifolds [8]. Precisely, they showed that a holomorphic vector bundle over a compact complex manifold with a hermitian fibre metric whose curvature satisfies that $(\sum_i K_{\bar{\gamma}\beta i\bar{i}})$ is a negative definite hermitian matrix at each point admits no non-zero holomorphic sections.

In 1976, S. Yau discussed the case of holomorphic $L^p$-sections of holomorphic vector bundles over complex Kähler manifolds, by studying the $L^p$-harmonic forms. Actually, he proved the non-existence of non-constant $L^p$ holomorphic functions for $p > 0$ on a complete Kähler manifold. Moreover, he also proved that there are no non-zero $L^2$ holomorphic $n$-forms on a complete Kähler manifold with positive total scalar curvature, which is supposed to be bounded from below by a constant [12].

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Although these results are more general in the sense of vector fields, they are focused on the Kähler manifold for the reason of analysis.

In addition to the analysis of $L^p$ behaviours of functions on Kähler manifolds by using the general maximum principle, there is another method to restrain the vector fields on non-compact complete complex manifold [3]. In 1981, S. Yorozu used the method to study the non-existence of non-zero holomorphic vector fields on complete Kähler manifolds [13]. He proved that every holomorphic vector field of type $(1,0)$ with finite global norm on a complete Kähler manifold with non-positive Ricci curvature is parallel. Moreover, there is no such vector field, provided the negativeness of the Ricci curvature. His result generalizes the results of Bochner [4] and Kobayashi and Wu [8] to non-compact cases on the Kähler manifolds.

The complex Finsler manifolds and complex Finsler vector fields are interesting, because some curvature properties of them are related to the ampleness of the tautological bundles [5]. Some important concepts have been introduced and their properties have been researched [1, 2, 7]. The curvature properties and different kinds of Chern forms on Finsler bundles are studied by K. Liu, H. Feng and X. Wang [6].

Recently, the author has obtained some results about vanishing theorems in Finsler geometry by generalizing the Bochner's technique [9, 10]. The same approach can be utilized to research the holomorphic vector fields on a complex Finsler manifold and holomorphic sections in a holomorphic Finsler vector bundle.

In this article, we will look into the holomorphic sections of vector bundles equipped with some Finsler metrics over a complex or Hermitian manifold and specially, the holomorphic vector fields on general Finsler manifolds. The main theorems we get are listed below. In the next statement and in all the rest of the paper, we always use the notation $(A_{i\bar{j}})$ to denote a hermitian matrix with entries $A_{i\bar{j}}$. On a Finsler holomorphic vector bundle $(E,G)$ over a complex manifold $M$, we have the following.

***Theorem 1.1*** *Let $E$ be a holomorphic vector bundle over a complex manifold $M$ with a Finsler fibre metric $G$ such that $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is a negative semi-definite hermitian matrix at each point of $M$. Then every holomorphic section of $E$ with finite global norm is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is negative definite, then $E$ admits no non-zero holomorphic sections with finite global norm.*

On a compact complex Finsler manifold, we get the vanishing theorem of holomorphic vector field of type $(1,0)$ as follows. More details are shown in Theorem 4.2 in Section 3.

***Theorem 1.2*** *Let $(M,G)$ be a compact complex Finsler manifold. If it has non-positive (first) G-average Ricci curvature ${}^{\mathcal{G}}\mathcal{R}ic$, then every holomorphic vector field of type $(1,0)$ on $M$ is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if the curvature ${}^{\mathcal{G}}\mathcal{R}ic$ is negative, then $M$ does not admit any non-zero holomorphic vector field of type $(1,0)$.*

On a non-compact complete complex Finsler manifold, the following vanishing theorem holds.

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***Theorem 1.3*** *Let $(M,G)$ be a complete complex Finsler manifold. If it has non-positive second or third G-average Ricci curvature ${}^{\mathfrak{g}}\mathrm{Ric}$ or ${}^{\mathfrak{G}}\mathrm{Ric}$, then every holomorphic vector field of type $(1,0)$ with finite global norm is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if the curvature ${}^{\mathfrak{g}}\mathrm{Ric}$ or ${}^{\mathfrak{G}}\mathrm{Ric}$ is negative, then there is no non-zero holomorphic vector field of type $(1,0)$ on $M$ with finite global norm.*

## 2 Concepts and Preliminaries

In this section, we will first introduce some basic concepts on holomorphic vector bundles with Finsler metrics over a complex manifold. Then we will briefly present the analog concepts on complex Finsler manifolds. Later, curvatures on bundles and manifolds are given.

### 2.1 Complex Finsler Vector Bundles and Complex Finsler Manifolds

Let $\pi: E \to M$ be a holomorphic vector bundle on a complex manifold $M$. A continuous function $G$ defined on the bundle $E$ is called a *Finsler metric* if it satisfies the following conditions:

(i) $G$ is smooth on the punched bundle $E^o = E \smallsetminus O$, where $O$ denotes the zero section of $E$,
(ii) $G(z,\lambda v) = |\lambda|^2 G(z,v)$ for any $\lambda \in \mathbb{C}$,
(iii) $G(z,v) \geq 0$ for all $(z,v) \in E$ with $z \in M$ and $v \in \pi^{-1}(z)$, where the equality holds if and only if $v = 0$.

A holomorphic vector bundle $E$ admitting a complex Finsler metric $G$ is called a holomorphic Finsler vector bundle. For the convenience of application, one often requires that $G$ is strongly pseudo-convex, that is,

(iv) the Levi form $\sqrt{-1}\partial\bar{\partial}G$ on $E^o$ is positive-definite along fibres $E_z = \pi^{-1}(z)$ for $z \in M$.

The quotient map $q: E^o \to P(E) = E^o/\mathbb{C}^*$ defines the holomorphic projective bundle $P(E)$. Let $z = (z^1,\dots,z^n)$ be the local coordinate system in $M$, and let $v = (v^1,\dots,v^r)$ be the fibre coordinate system on $E/M$ defined by a local holomorphic frame $s = \{s_1,\dots,s_r\}$ of $E$. Customary symbols used here are

$$\partial_\alpha := \frac{\partial}{\partial z^\alpha}, \quad \partial_{\bar\beta} := \frac{\partial}{\partial \bar z^\beta}, \quad \dot\partial_i := \frac{\partial}{\partial v^i}, \quad \dot\partial_{\bar j} := \frac{\partial}{\partial \bar v^j},$$

where $1 \leq \alpha,\beta \leq n$ and $1 \leq i,j \leq r$. By the strongly pseudo-convex condition (iv), the Hermitian matrices $\big(G_{i\bar j}(z,v)\big)$ is positive-definite and actually defines a Hermitian metric $h^G$ on the pull-back bundle $p: \pi^*E \to E^o$. The following identities proved by Kobayashi [7] tells us that $\big(G_{i\bar j}(z,v)\big)$ can be defined on $P(E)$, hence $h^G$ is a Hermitian metric on the pull-back bundle $p: \pi^*E \to P(E)$.

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***Lemma 2.1*** *The following identities hold for any $(z,v)\in E^o$ and $\lambda\in\mathbb{C}$:*

$$G_i=\frac{\partial G}{\partial v^i},\quad G_i(z,\lambda v)=\bar\lambda G_i(z,v),$$
$$G_{i\bar j}=\frac{\partial^2 G}{\partial v^i\partial\bar v^j},\quad G_{i\bar j}(z,\lambda v)=G_{i\bar j}(z,v)=\bar G_{j\bar i}(z,v),$$
$$G(z,v)=G_i(z,v)v^i=G_{\bar j}(z,v)\bar v^j=G_{i\bar j}(z,v)v^i\bar v^j,$$
$$G_{ij}(z,v)v^i=G_{i\bar j k}(z,v)v^i=G_{i\bar j\bar k}(z,v)\bar v^j=0.$$

We also write

$$G_{i\alpha}=\frac{\partial^2 G}{\partial v^i\partial z^\alpha},\quad G_{\alpha\bar\beta}=\frac{\partial^2 G}{\partial z^\alpha\partial\bar z^\beta},\quad \textit{etc.}$$

Let $\nabla^{\pi^*E}$ be the Chern connection on the holomorphic vector bundle $p\colon(\pi^*E,h^G)\to E^o$. We still use the notation $s_k$ to denote the sections of $\pi^*E$, then the Chern connection $(1,0)$-forms $\theta_i^k$ with respect to the local holomorphic frame $s=\{s_1,\dots,s_r\}$ are given by

(1)
$$\nabla^{\pi^*E}s_i=\theta_i^k\otimes s_k,\quad \theta_i^k=(\partial G_{i\bar j}G^{\bar j k})=\Gamma_{i\alpha}^k dz^\alpha+\gamma_{il}^k dv^l,$$
$$\Gamma_{i\alpha}^k=\frac{\partial G_{i\bar j}}{\partial z^\alpha}G^{\bar j k},\quad \gamma_{il}^k=\frac{\partial G_{i\bar j}}{\partial v^l}G^{\bar j k},$$

where $(G^{\bar j k})$ denote the inverse of the matrix $(G_{i\bar j})$. Lemma 2.1 tells us that

$$\Gamma_{i\alpha}^k(z,\lambda v)=\Gamma_{i\alpha}^k(z,v),\quad \gamma_{il}^k(z,v)v^i=\gamma_{il}^k(z,v)v^l=0.$$

The Chern connection $\nabla^{\pi^*E}$ provides a smooth horizontal-vertical decomposition of the holomorphic tangent vector bundle $TE^o$ of $E^o$:

$$TE^o=\mathcal{H}\otimes\mathcal{V},$$

where $\mathcal{V}$ is called the *vertical subbundle* of $TE^o$ defined by

$$\mathcal{V}=\ker(p_*:TE^o\to TM),$$

and $\mathcal{H}$ is called the *horizontal subbundle* of $TE^o$ defined by

$$\mathcal{H}=\ker(\nabla_\bullet^{\pi^*E}P:TE^o\to\pi^*E).$$

Here $P$ is the tautological section of the bundle $p\colon\pi^*E\to E^o$ defined by $P(z,v)=v$. Canonically, the vertical subbundle $\mathcal{V}$ is holomorphically isomorphic to $\pi^*E$. On the other hand, the horizontal subbundle $\mathcal{H}$ is smoothly isomorphic to $\pi^*TM\to E$. In local coordinates,

$$\mathcal{H}=\mathrm{span}_{\mathbb{C}}\Big\{\frac{\delta}{\delta z^\alpha}=\frac{\partial}{\partial z^\alpha}-\Gamma_{j\alpha}^k v^j\frac{\partial}{\partial v^k},\ 1\le\alpha\le n\Big\},$$
$$\mathcal{V}=\mathrm{span}_{\mathbb{C}}\Big\{\frac{\partial}{\partial v^i},\ 1\le i\le r\Big\}.$$

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The expression (1) can be rewritten with respect to the horizontal and vertical bases as

$$\nabla^{\pi^* E} s_i = \theta_i^k \otimes s_k, \quad \theta_i^k = (\partial G_{i\bar{j}} G^{\bar{j}k}) = \Gamma_{i;\alpha}^k dz^\alpha + \gamma_{i;l}^k \delta v^l,$$
$$\Gamma_{i;\alpha}^k = G^{\bar{j}k} \frac{\delta G_{i\bar{j}}}{\delta z^\alpha} = G^{\bar{j}k}(G_{i\bar{j}\alpha} - G_{i\bar{j}l}\Gamma_\alpha^l), \quad \gamma_{i;l}^k = \gamma_{il}^k = \frac{\partial G_{i\bar{j}}}{\partial v^l} G^{\bar{j}k},$$

where $\Gamma_\alpha^l = G^{l\bar{m}} G_{\bar{m}\alpha} = G^{l\bar{m}} G_{i\bar{m}\alpha} v^i$.

According to the zero-homogeneous of the horizontal connection, the quotient map $q$ induces a smooth horizontal-vertical decompositions of $TP(E)$ and $T^*P(E)$, *i.e.*,

$$TP(E) = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{V}}, \quad T^* P(E) = \tilde{\mathcal{H}}^* \oplus \tilde{\mathcal{V}}^*,$$

where $\tilde{\mathcal{H}} = q_* \mathcal{H}$, and $\tilde{\mathcal{V}} = q_* \mathcal{V}$. The Finsler metric provides the *Fubini–Study form* on each fibre, defined by

$$\omega_{\mathcal{V}} = \frac{\sqrt{-1}}{2\pi} (\log G)_{i\bar{j}} \delta v^i \wedge \delta \bar{v}^j. \tag{2}$$

***Lemma 2.2*** *The vertical metric form $\omega_{\mathcal{V}}$ is compatible with the horizontal covariant derivative.*

**Proof** Denote the horizontal covariant derivative with respect to $z^\alpha$ by $\nabla_\alpha$. For any $1 \le k \le n$, we have

$$\begin{aligned}
&\nabla_\alpha\big((\log G)_{i\bar{j}}(z,v)\delta v^i \wedge \delta\bar{v}^j\big)\\
&= \Big(\frac{\delta}{\delta z^\alpha}(\log G)_{i\bar{j}}\Big)\delta v^i \wedge \delta\bar{v}^j + (\log G)_{i\bar{j}}(z,v)(\nabla_\alpha \delta v^i) \wedge \delta\bar{v}^j\\
&= \Big[\frac{\delta}{\delta z^\alpha}\Big(\frac{G_{i\bar{j}}}{G} - \frac{G_i G_{\bar{j}}}{G^2}\Big) - \Big(\frac{G_{l\bar{j}}}{G} - \frac{G_l G_{\bar{j}}}{G^2}\Big)\Gamma_{i;\alpha}^l\Big]\delta v^i \wedge \delta\bar{v}^j\\
&= \Big[\frac{\partial_\alpha G_{i\bar{j}}}{G} - \frac{G_{i\bar{j}}\partial_\alpha G}{G^2} - \frac{\partial_\alpha G_i G_{\bar{j}}}{G^2} - \frac{G_i \partial_\alpha G_{\bar{j}}}{G^2} + 2\frac{G_i G_{\bar{j}} \partial_\alpha G}{G^3}\\
&\qquad - \Gamma_\alpha^l\Big(\frac{G_{i\bar{l}}}{G} - \frac{G_{i\bar{j}} G_l}{G^2} - \frac{G_{il} G_{\bar{j}}}{G^2} - \frac{G_i G_{l\bar{j}}}{G^2} + 2\frac{G_i G_{\bar{j}} G_l}{G^3}\Big)\\
&\qquad - \Big(\frac{G_{l\bar{j}}}{G} - \frac{G_l G_{\bar{j}}}{G^2}\Big) G^{l\bar{m}}(\delta_\alpha G_{i\bar{m}})\Big]\delta v^i \wedge \delta\bar{v}^j\\
&= 0.
\end{aligned}$$

The covariant derivative $\nabla_{\bar{\beta}}\big((\log G)_{i\bar{j}}(z,v)\delta v^i \wedge \delta\bar{v}^j\big) = 0$ follows by taking the conjugation. ■

A *complex Finsler metric* on a complex manifold $M$ of complex $n$-dimension is a continuous function $F\colon T^{1,0}M \to [0,+\infty)$ satisfying

(i) $F(z,v) \in C^\infty(\widetilde{M})$, $\widetilde{M} = T^{1,0}M \backslash \{0\}$,
(ii) $F(z,\lambda v) = |\lambda| G(z,v)$ for $\lambda \in \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$,
(iii) $F(z,v) \ge 0$, where the equality holds if and only if $v = 0$.

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A complex Finsler metric is called *strongly pseudo-convex* if the Levi matrix $\big(G_{i\bar{j}}(z,v)\big)$ is positively definite, where

$$G = F^2, \quad G_{i\bar{j}} = [G]_{v^i\bar{v}^j} = \dot{\partial}_i\dot{\partial}_{\bar{j}}G.$$

Note that in this case the holomorphic bundle $\pi\colon T^{1,0}M \to M$, equipped with the function $G$, is a strongly pseudoconvex holomorphic Finsler vector bundle, as it has been defined at the beginning of this section. Since the fibre is the tangent space of $M$, customary symbols used here are

$$\partial_i := \frac{\partial}{\partial z^i}, \quad \partial_{\bar{j}} := \frac{\partial}{\partial \bar{z}^j}, \quad \dot{\partial}_i := \frac{\partial}{\partial v^i}, \quad \dot{\partial}_{\bar{j}} := \frac{\partial}{\partial \bar{v}^j}.$$

The *projective tangent bundle* $P\widetilde{M}$ over $M$ is defined by $P\widetilde{M} := \widetilde{M}/\mathbb{C}^*$, of which the fibre is an $n-1$ dimensional complex projective space $\mathbb{C}P^{n-1}$.

Canonically, the vertical subbundle $\mathcal{V}$ is holomorphically isomorphic to $\pi^*TM$. On the other hand, the horizontal subbundle $\mathcal{H}$ is smoothly isomorphic to the pull-back bundle $\pi^*TM \to TM$. More details can be found in [1]. In local coordinates,

$$\mathcal{H} = \operatorname{span}_{\mathbb{C}}\Big\{\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j\frac{\partial}{\partial v^j}, 1 \le i, j \le n\Big\},$$
$$\mathcal{V} = \operatorname{span}_{\mathbb{C}}\Big\{\frac{\partial}{\partial v^i}, 1 \le i \le n\Big\},$$

where $N_j^i = G^{i\bar{l}}\dot{\partial}_{\bar{l}}\partial_j G$. Shortly, we denote $\delta_i := \frac{\delta}{\delta z^i}$, $\delta_{\bar{j}} := \frac{\delta}{\delta \bar{z}^j}$, $\dot{\partial}_i := \frac{\partial}{\partial v^i}$, $\dot{\partial}_{\bar{j}} := \frac{\partial}{\partial \bar{v}^j}$, and their dualities by $dz^i$, $d\bar{z}^i$, $\delta v^j$, $\delta\bar{v}^j$ with $\delta v^i = dv^i + N_j^i dz^j$. The horizontal covariant derivative with respect to $z^k$ is denoted by $\nabla_k$. Moreover, the non-linear connection coefficients are $N_j^i = \Gamma_{j;k}^i v^k$.

As $\pi\colon T^{1,0}M \to M$ is a holomorphic Finsler vector bundle, it is naturally equipped with the Fubini–Study form $\omega_{\mathcal{V}}$, defined in (2). Besides, as a hermitian manifold, we can define the *Kähler form* by the Finsler metric on $M$, *i.e.*,

$$\omega_{\mathcal{H}} = \sqrt{-1}G_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

where $G_{i\bar{j}}$ is the Levi matrix. The *invariant volume form* of $P\widetilde{M}$ is

$$d\mu_{P\widetilde{M}} = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \wedge \frac{\omega_{\mathcal{H}}^n}{n!}.$$

The volume form $\frac{\omega_{\mathcal{H}}^n}{n!} = \det(G_{i\bar{j}})dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$ does not only depend on the point coordinates $z$, but also on the fibre coordinates $v$. However, we can define a mean Hermitian metric by $g = g_{i\bar{j}}dz^i \otimes d\bar{z}^j$, with

$$g_{i\bar{j}} = \int_{P\widetilde{M}/M} G_{i\bar{j}}\frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}. \tag{3}$$

We denote $dM = (\det g_{i\bar{j}})dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$ as the volume form with respect to the average Hermitian metric $g$, which only depends on the point coordinates $z$.

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## 2.2 Curvatures on Complex Finsler Bundles and on Complex Finsler Manifolds

Using the Chern–Finsler connection, we can define the Chern curvature form $R^{\pi^* E}$. The Chern curvature $(1,1)$-forms are given by

$$R^{\pi^* E} s_i = \Omega_i^k \otimes s_k, \quad \Omega_i^k = d\theta_i^k - \theta_i^j \wedge \theta_j^k = \bar{\partial}\theta_i^k.$$

In local coordinates, the curvature can be written as

$$\Omega_j^i = R^i_{j\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + R^i_{jk\bar{\beta}} \delta v^k \wedge d\bar{z}^\beta + R^i_{j\bar{k}\alpha} dz^\alpha \wedge \delta\bar{v}^k + R^i_{jk\bar{l}} \delta v^k \wedge \delta\bar{v}^l,$$

where

$$\begin{aligned} R^i_{j\alpha\bar{\beta}} &= -\delta_{\bar{\beta}} \Gamma^i_{j;\alpha} - \gamma^i_{jk} \delta_{\bar{\beta}} \Gamma^k_\alpha, \\ R^i_{jk\bar{\beta}} &= -\delta_{\bar{\beta}} \gamma^i_{jk} = R^i_{kj\bar{\beta}}, \\ R^i_{j\bar{k}\alpha} &= -\dot{\partial}_{\bar{k}} \Gamma^i_{j;\alpha} - \gamma^i_{jh} \Gamma^h_{\bar{k};\alpha}, \\ R^i_{jk\bar{l}} &= -\dot{\partial}_{\bar{l}} \gamma^i_{jk} = R^i_{kj\bar{l}}. \end{aligned}$$

Moreover, we set

$$R^i_{\alpha\bar{\beta}} := R^i_{j\alpha\bar{\beta}} v^j = -\delta_{\bar{\beta}}(\Gamma^i_{j;\alpha}) v^j = -\delta_{\bar{\beta}}(\Gamma^i_{j;\alpha} v^j) = -\delta_{\bar{\beta}} \Gamma^i_\alpha.$$

There is a well-defined real $(1,1)$-form $\Psi$ on $E^o$, called the *Kobayashi curvature* of the holomorphic Finsler vector bundle $(E, G)$, which is defined by [7]

$$\Psi = \sqrt{-1} \frac{h^G(R^{\pi^* E} P, P)}{h^G(P, P)} = \sqrt{-1} K_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta,$$

where $P$ is the tautological section of the bundle $p\colon \pi^* E \to E^o$ defined by $P(z, v) = v$. Direct calculation shows that

$$K_{\alpha\bar{\beta}} = K_{i\bar{j}\alpha\bar{\beta}} \frac{v^i \bar{v}^j}{G} dz^\alpha \wedge d\bar{z}^\beta,$$

if we set $K_{i\bar{j}\alpha\bar{\beta}} = -G_{i\bar{j}\alpha\bar{\beta}} + G^{k\bar{l}} G_{i\bar{l}\alpha} G_{k\bar{j}\bar{\beta}}$. Moreover, it is also equal to

$$K_{\alpha\bar{\beta}} = R_{i\bar{j}\alpha\bar{\beta}} \frac{v^i \bar{v}^j}{G} dz^\alpha \wedge d\bar{z}^\beta.$$

$\Psi$ is actually a well-defined horizontal $(1,1)$-form on $P(E)$ by the homogeneity.

$\partial\bar{\partial} \log G$ gives the curvature of $(h^G)^{-1}$ on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$, hence it can provide the first Chern form associated to it. That is,

$$\Xi := c_1\big(\mathcal{O}_{P(E)}(-1), (h^G)^{-1}\big) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log G.$$

The following lemma due to Kobayashi [7] and Aikou [2] shows the decomposition of the first Chern form of the tautological line bundle of $P(E)$.

***Lemma 2.3*** *Let $\pi\colon E \to M$ be a holomorphic vector bundle with a strongly pseudo-convex Finsler metric $G$. Then*

$$\Xi := c_1\big(\mathcal{O}_{P(E)}(-1), (h^G)^{-1}\big) = -\frac{1}{2\pi} \Psi + \omega_{\mathcal{V}},$$

*where $\omega_{\mathcal{V}}$ is the Fubini–Study $(1,1)$-form on $E^o$ or $P(E)$.*

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Suppose the underlying manifold $M$ admits a Hermitian metric with the Kähler form $\omega = g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$, then the *volume form* on the projective vector bundle $P(E)$ can be defined by

$$d\mu_{P(E)} = \frac{\omega_{\mathcal{V}}^{r-1}}{(r-1)!} \wedge \frac{\omega^n}{n!},$$

where $\omega$ can be lifted canonically onto the horizontal projective bundle.

Now we define the *average curvature* here. Here and later, we use $P(E)/M$ to denote the fibre of $P(E)$.

***Definition 2.4*** The tensor ${}^{\mathcal{G}}\mathcal{R} = \frac{\sqrt{-1}}{G} {}^{\mathcal{G}}\mathcal{R}_{i\bar j\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta \otimes \delta v^i \otimes \delta \bar v^j$ is a well-defined curvature on the underlying manifold $M$, which is called the *G-average curvature*, where the components are

$$ {}^{\mathcal{G}}\mathcal{R}_{i\bar j\alpha\bar\beta} = \int_{P(E)/M} (R_{i\bar j\alpha\bar\beta} - G_{i\bar j l} R^l{}_{\alpha\bar\beta}) \omega_{\mathcal{V}}^{r-1}. \tag{4}$$

Due to Lemma 2.2, we can define another kind of average curvature.

***Definition 2.5*** Let $(M, g)$ be a Hermitian manifold, and $(E, G)$ a Finsler vector bundle over $M$. The *G-average bundle curvature of E* is defined by ${}^{\mathcal{B}}\mathcal{K} = \frac{\sqrt{-1}}{G} {}^{\mathcal{B}}\mathcal{K}_{i\bar j} dv^i \otimes d\bar v^j$, with

$$ {}^{\mathcal{B}}\mathcal{K}_{i\bar j}(z) = \int_{P(E)/M} g^{\bar\beta\alpha} (R_{i\bar j\alpha\bar\beta} - G_{i\bar j h} R^h{}_{\alpha\bar\beta}) \frac{\omega_{\mathcal{V}}^{r-1}}{(r-1)!}.$$

On Finsler manifolds, the Chern curvature $(1,1)$-forms are given by

$$\Omega e_i = \Omega_i^k \otimes e_k,$$

$$\Omega_j^i = R^i_{j;k\bar l} dz^k \wedge d\bar z^l + R^i_{jk;\bar l} \delta v^k \wedge d\bar z^l + R^i_{j\bar l;k} dz^k \wedge \delta \bar v^l + R^i_{jk\bar l} \delta v^k \wedge \delta \bar v^l.$$

Some Bianchi identities on complex Finsler manifolds can be derived from $\nabla \mathcal{G} = \nabla(G_{i\bar j} \delta v^i \otimes \delta \bar v^j) = 0$, which is equal to $dG_{i\bar j} - G_{k\bar j}\theta_i^k - G_{i\bar k}\overline{\theta_j^k} = 0$. Taking another derivative gives that

$$\begin{aligned} 0 &= dG_{k\bar j} \wedge \theta_i^k + G_{k\bar j} d\theta_i^k + dG_{i\bar k} \wedge \overline{\theta_j^k} + G_{i\bar k} d\overline{\theta_j^k} \\ &= G_{k\bar j}(d\theta_i^k + \theta_l^k \wedge \theta_i^l) + G_{i\bar k}(d\overline{\theta_j^k} + \overline{\theta_l^k} \wedge \overline{\theta_j^l}) \\ &= G_{k\bar j}\Theta_i^k + \overline{G_{\bar i k}\Theta_j^k} \\ &= (R_{i\bar j;k\bar l} - R_{\bar j i;\bar l k}) dz^k \wedge d\bar z^l + (R_{i\bar j k;\bar l} - R_{\bar j i k;\bar l}) \delta v^k \wedge d\bar z^l \\ &\qquad + (R_{i\bar j\bar l;k} - R_{\bar j i\bar l;k}) dz^k \wedge \delta\bar v^l + (R_{i\bar j k\bar l} - R_{\bar j i\bar l k}) \delta v^k \wedge \delta \bar v^l. \end{aligned}$$

It asserts that

$$\begin{aligned} &R_{i\bar j;k\bar l} - R_{\bar j i;\bar l k} = 0, \quad R_{i\bar j k;\bar l} - R_{\bar j i k;\bar l} = 0, \\ &R_{i\bar j\bar l;k} - R_{\bar j i\bar l;k} = 0, \quad R_{i\bar j k\bar l} - R_{\bar j i\bar l k} = 0. \end{aligned}$$

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As in [10], some special combinations of Finsler curvatures and their integrals are needed in the vanishing theorems.

The *G-average h-h curvature of the Finsler manifold* $(M,F)$, denoted by ${}^{\mathcal{G}}\mathcal{R} = \frac{\sqrt{-1}}{G}{}^{\mathcal{G}}\mathcal{R}_{i\bar{j};k\bar{l}}dz^k \wedge d\bar{z}^l \otimes \delta v^i \otimes \delta\bar{v}^j$, is the curvature ${}^{\mathcal{G}}\mathcal{R}$, defined in (4), of the holomorphic Finsler bundle $T^{1,0}M$.

***Definition 2.6*** We call ${}^{\mathcal{G}}\,\mathrm{Ric} = \sqrt{-1}{}^{\mathcal{G}}\mathcal{R}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ the (first) $G$-average Ricci curvature, where

$$ {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}} = \int_{P\widetilde{M}/M} G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h_{\ ;k\bar{l}})\omega_{\mathcal{V}}^{n-1}. $$

Using (3), we can define two more kinds of average curvatures.

***Definition 2.7*** The *second G-average Ricci curvature* is defined by ${}^{\mathfrak{g}}\,\mathrm{Ric} = \sqrt{-1}{}^{\mathfrak{g}}\mathfrak{R}_{i\bar{j}}dz^i \wedge d\bar{z}^j$, with

$$ {}^{\mathfrak{g}}\mathfrak{R}_{i\bar{j}}(z) = \int_{P\widetilde{M}/M} G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h_{\ ;k\bar{l}})\frac{\det G_{i\bar{j}}}{\det g_{i\bar{j}}}\frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}. $$

By the same method, we also can define the following curvature.

***Definition 2.8*** The *third G-average Ricci curvature* is defined by ${}^{\mathfrak{G}}\,\mathrm{Ric} = \sqrt{-1}{}^{\mathfrak{G}}\mathfrak{R}_{i\bar{j}}dz^i \wedge d\bar{z}^j$, with

$$ {}^{\mathfrak{G}}\mathfrak{R}_{i\bar{j}}(z) = \int_{P\widetilde{M}/M} G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h_{\ ;k\bar{l}})\frac{\det G_{i\bar{j}}}{\sigma(z)}\frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}, $$

where $\sigma(z) = \frac{1}{(2\pi)^{n-1}}\int_{P\widetilde{M}/M}\det G_{i\bar{j}}\frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}$.

***Remark 2.9*** All the first, second and third $G$-average Ricci curvatures are reduced to the Ricci curvature when the Finsler manifold is a Kähler manifold.

## 3 Vanishing Theorem of Holomorphic Sections of Certain Holomorphic Finsler Bundles

In this section, we will discuss the vanishing theorems of holomorphic sections of holomorphic bundles equipped with some Finsler metrics. The situations over compact manifolds and non-compact complete manifolds are different.

### 3.1 On Finsler Vector Bundles over Compact Complex Manifolds

In this subsection, we aim to prove that a certain holomorphic vector bundle $E$ over a complex or Hermitian manifold $M$ admits no non-zero holomorphic sections. This vanishing theorem generalizes the results of Kobayashi and Wu [8].

Suppose $E$ is a holomorphic vector bundle over a compact complex manifold $M$. Let $(z^1,\dots,z^n)$ be a local coordinate system of $M$, and $\{s_1,\dots,s_r\}$ be holomorphic

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local sections of $E$. A holomorphic section of $E$ is locally expressed by $\xi = \xi^i s_i$. We first derive the Ricci identity for $\xi$. It shows that

$$\begin{aligned}\nabla_\alpha \xi &= \frac{\delta \xi^i}{\delta z^\alpha} s_i + \xi^i \nabla_\alpha s_i = \Big(\frac{\delta \xi^i}{\delta z^\alpha} + \xi^k \theta^i_k\Big(\frac{\delta}{\delta z^\alpha}\Big)\Big) s_i \\ &= \Big(\frac{\delta \xi^i}{\delta z^\alpha} + \xi^k \frac{\delta G_{k\bar{j}}}{\delta z^\alpha} G^{\bar{j}i}\Big) s_i = \Big(\frac{\delta \xi^i}{\delta z^\alpha} + \xi^k \Gamma^i_{k;\alpha}\Big) s_i\end{aligned}$$

and

$$\nabla_{\bar{\beta}} \xi = \frac{\delta \xi^i}{\delta \bar{z}^\beta} s_i + \xi^i \nabla_{\bar{\beta}} s_i = \frac{\delta \xi^i}{\delta \bar{z}^\beta} s_i = 0.$$

Therefore,

$$\nabla_{\bar{\beta}} \nabla_\alpha \xi = \nabla_{\bar{\beta}} \Big[\Big(\frac{\delta \xi^i}{\delta z^\alpha} + \xi^k \Gamma^i_{k;\alpha}\Big) s_i\Big] = (\xi^j \delta_{\bar{\beta}} \Gamma^i_{j;\alpha}) s_i,$$

and

$$\nabla_{\bar{\beta}} \nabla_\alpha \xi^i - \nabla_\alpha \nabla_{\bar{\beta}} \xi^i = \xi^j \delta_{\bar{\beta}} \Gamma^i_{j;\alpha} = \xi^j (\gamma^i_{jk} R^k{}_{\alpha\bar{\beta}} - R^i_{j\alpha\bar{\beta}}).$$

We now get the Ricci identity for any holomorphic section $\xi$.

***Lemma 3.1*** *For any holomorphic section $\xi = \xi^i s_i$ of a Finsler vector bundle $E$ on a complex manifold $M$, the Ricci identity reduces to*

$$\nabla_{\bar{\beta}} \nabla_\alpha \xi^i = -\xi^k G^{i\bar{j}} (R_{k\bar{j}\alpha\bar{\beta}} - G_{k\bar{j}l} R^l{}_{\alpha\bar{\beta}}).$$

The Ricci identity can be used to prove the following vanishing theorem.

***Lemma 3.2*** *Let $E$ be a holomorphic vector bundle over a compact complex manifold $M$ with a Finsler fibre metric $F$ such that $\big(\sum_\alpha (R_{k\bar{j}\alpha\bar{\alpha}} - G_{k\bar{j}l} R^l{}_{\alpha\bar{\alpha}})\big)$ is a negative semi-definite hermitian matrix at each point of $P(E)$. Then every holomorphic section of $E$ is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if $\big(\sum_\alpha (R_{k\bar{j}\alpha\bar{\alpha}} - G_{k\bar{j}l} R^l{}_{\alpha\bar{\alpha}})\big)$ is negative definite, then $E$ admits no non-zero holomorphic sections.*

**Proof** Denote a holomorphic section $\xi$ locally by $\xi = \xi^i s_i$. Let $f = |\xi|^2 = G_{i\bar{j}} \xi^i \bar{\xi}^j$. We have

$$\nabla_\alpha f = G_{i\bar{j}} (\nabla_\alpha \xi^i) \bar{\xi}^j.$$

Hence by the Ricci identity,

$$\begin{aligned}\nabla_{\bar{\beta}} \nabla_\alpha f &= G_{i\bar{j}} (\nabla_{\bar{\beta}} \nabla_\alpha \xi^i) \bar{\xi}^j + G_{i\bar{j}} (\nabla_\alpha \xi^i)(\nabla_{\bar{\beta}} \bar{\xi}^j) \\ &= -(R_{i\bar{j}\alpha\bar{\beta}} - G_{i\bar{j}l} R^l{}_{\alpha\bar{\beta}}) \xi^i \bar{\xi}^j + G_{i\bar{j}} (\nabla_\alpha \xi^i)(\nabla_{\bar{\beta}} \bar{\xi}^j).\end{aligned} \tag{5}$$

Taking the trace by $\delta_{\alpha\beta}$ in (5), the left-hand side is an elliptic operator on $f$, hence must be negative semi-definite at the maximum point of $f$. To see this, we expand the

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left-hand side of (5) as

$$\begin{aligned}\nabla_{\bar\beta}\nabla_\alpha f &= \frac{\partial^2 f}{\partial\bar z^\beta\partial z^\alpha} - \frac{\partial}{\partial\bar z^\beta}(G^{i\bar j}G_{\bar j\alpha})\frac{\partial f}{\partial v^i} - G^{\bar p q}G_{q\bar\beta}\frac{\partial^2 f}{\partial\bar v^p\partial z^\alpha} - G^{i\bar j}G_{\bar j\alpha}\frac{\partial^2 f}{\partial\bar z^\beta\partial v^i}\\ &\quad + G^{\bar p q}G_{q\bar\beta}\frac{\partial}{\partial\bar v^p}(G^{i\bar j}G_{\bar j\alpha})\frac{\partial f}{\partial v^i} + G^{\bar p q}G_{q\bar\beta}G^{i\bar j}G_{\bar j\alpha}\frac{\partial^2 f}{\partial\bar v^p\partial v^i}\\ &= \mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f) + \mathrm{FOP}(\nabla_{\bar\beta}\nabla_\alpha f),\end{aligned}$$

where

$$\begin{aligned}\mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f) &= \frac{\partial^2 f}{\partial\bar z^\beta\partial z^\alpha} - G^{\bar p q}G_{q\bar\beta}\frac{\partial^2 f}{\partial\bar v^p\partial z^\alpha}\\ &\qquad - G^{i\bar j}G_{\bar j\alpha}\frac{\partial^2 f}{\partial\bar z^\beta\partial v^i} + G^{\bar p q}G_{q\bar\beta}G^{i\bar j}G_{\bar j\alpha}\frac{\partial^2 f}{\partial\bar v^p\partial v^i},\\ \mathrm{FOP}(\nabla_{\bar\beta}\nabla_\alpha f) &= -\frac{\partial}{\partial\bar z^\beta}(G^{i\bar j}G_{\bar j\alpha})\frac{\partial f}{\partial v^i} + G^{\bar p q}G_{q\bar\beta}\frac{\partial}{\partial\bar v^p}(G^{i\bar j}G_{\bar j\alpha})\frac{\partial f}{\partial v^i}.\end{aligned}$$

The second order part can be expressed by

$$\mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f) = \begin{pmatrix}\delta_{\alpha\gamma}\\ -G_{\alpha\bar j}G^{\bar j i}\end{pmatrix}\begin{pmatrix}\frac{\partial^2 f}{\partial z^\gamma\partial\bar z^\xi} & \frac{\partial^2 f}{\partial z^\gamma\partial\bar v^p}\\ \frac{\partial^2 f}{\partial v^i\partial\bar z^\xi} & \frac{\partial^2 f}{\partial v^i\partial\bar v^p}\end{pmatrix}\begin{pmatrix}\delta_{\xi\beta}\\ -G^{\bar p q}G_{q\bar\beta}\end{pmatrix},$$

with $1 \le \alpha,\beta,\gamma,\xi \le n$, $1 \le i,j,p,q \le r$. Let $A = (A_1, A_2)$ be a complex matrix, where $A_1$ is an $n\times n$ identity matrix and $A_2$ is an $n\times r$ matrix whose entries are $a_{\alpha i} = -\sum_j G_{\alpha\bar j}G^{\bar j i}$, and let the matrix of second derivatives of $f$ be $H(f)$. Then it deduces that

$$\mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f) = AH(f)A^*.$$

It is easy to see that $H(f)$ is the locally complex Hessian of $f$ with respect to $z, v$. Therefore, $\mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f)$ is a Hermitian matrix and $\mathrm{SOP}(\nabla_{\bar\beta}\nabla_\alpha f)$ is an elliptic operator of $f$. It indicates that $\nabla_{\bar\alpha}\nabla_\alpha f = \delta_{\alpha\beta}\nabla_{\bar\beta}\nabla_\alpha f$ is an elliptic operator of $f$.

On the other hand, the last term is always non-negative. It follows that $\xi$ must vanish identically when $\left(\sum_\alpha(R_{k\bar j\alpha\bar\alpha} - G_{k\bar j l}R^l{}_{\alpha\bar\alpha})\right)$ is a negative definite hermitian matrix at each point of $E^o$, hence of $P(E)$ by the homogeneity. Furthermore, $\xi$ must be parallel with respect to the horizontal Chern–Finsler connection when it is negative semi-definite. ■

Indeed we have proved that

$$\nabla_{\bar\alpha}\nabla_\alpha f = -(R_{i\bar j\alpha\bar\beta} - G_{i\bar j l}R^l{}_{\alpha\bar\beta})\xi^i\bar\xi^j + G_{i\bar j}(\nabla_\alpha\xi^i)(\nabla_{\bar\alpha}\bar\xi^j). \tag{6}$$

The curvature condition is related both on the underlying manifold and the fibre, which cannot be considered as the direct generalization of Kobayashi and Wu's result[8]. Before improving the above vanishing lemma, we need the following lemma.

***Lemma 3.3*** *The volume of each fibre is invariant under the horizontal parallel translation.*

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**Proof** We only need to prove that the volume form $\omega_{\mathcal{V}}^{r-1}$ is invariant under the horizontal parallel translation, *i.e.*, $\nabla_\alpha \omega_{\mathcal{V}} = 0$. It is because in local coordinates,

$$\begin{aligned}
\nabla_\alpha\Big(\frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j}\Big) &= \nabla_\alpha\Big(\frac{G_{i\bar{j}}}{G} - \frac{G_i G_{\bar{j}}}{G^2}\Big)\\
&= \frac{\delta}{\delta z^\alpha}\Big(\frac{G_{i\bar{j}}}{G} - \frac{G_i G_{\bar{j}}}{G^2}\Big) - \Big(\frac{G_{k\bar{j}}}{G} - \frac{G_i G_{\bar{j}}}{G^2}\Big)\Gamma^k_{i;\alpha}\\
&= \frac{G_{i\bar{j}\alpha}}{G} - \frac{G_{i\bar{j}}G_\alpha}{G^2} - \frac{G_{i\alpha}G_{\bar{j}} + G_i G_{\bar{j}\alpha}}{G^2} + 2\frac{G_i G_{\bar{j}} G_\alpha}{G^3}\\
&\quad - \Gamma^k_\alpha\Big(\frac{G_{i\bar{j}k}}{G} - \frac{G_{i\bar{j}}G_k}{G^2} - \frac{G_{ik}G_{\bar{j}} + G_i G_{\bar{j}k}}{G^2} + 2\frac{G_i G_{\bar{j}} G_k}{G^3}\Big)\\
&\quad - \Big(\frac{G_{k\bar{j}}}{G} - \frac{G_k G_{\bar{j}}}{G^2}\Big)(G^{k\bar{l}}G_{i\bar{l}\alpha} - G^{h\bar{l}}G_{\bar{l}\alpha}G_{i\bar{m}h}G^{k\bar{m}})\\
&= -\frac{G_{i\bar{j}}G_\alpha}{G^2} - \frac{G_i G_{\bar{j}\alpha}}{G^2} + 2\frac{G_i G_{\bar{j}} G_\alpha}{G^3}\\
&\quad - G^{k\bar{l}}G_{\bar{l}\alpha}\Big(-\frac{G_{i\bar{j}}G_k}{G^2} - \frac{G_i G_{\bar{j}k}}{G^2} + 2\frac{G_i G_{\bar{j}} G_k}{G^3}\Big)\\
&= 0. \qquad \blacksquare
\end{aligned}$$

Using Lemmas 3.2 and 3.3 and Definition 2.4, we can prove the following theorem.

***Theorem 3.4*** *Let $E$ be a holomorphic vector bundle over a compact complex manifold $M$ with a Finsler fibre metric $G$ such that $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is a negative semi-definite hermitian matrix at each point of $M$. Then every holomorphic section of $E$ is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is negative definite, then $E$ admits no non-zero holomorphic sections.*

**Proof** Taking the integral of (6) on each fibre of $P(E)$ yields that

$$\begin{aligned}
(7)\quad \nabla_{\bar{\beta}}\nabla_\alpha\Big(\int_{P(E)/M} f\omega_{\mathcal{V}}^{r-1}\Big)
&= \int_{P(E)/M} \nabla_{\bar{\beta}}\nabla_\alpha f\omega_{\mathcal{V}}^{r-1}\\
&= -\int_{P(E)/M}[R_{i\bar{j}\alpha\bar{\beta}} - G_{i\bar{j}l}R^l{}_{\alpha\bar{\beta}}]\xi^i\bar{\xi}^j\omega_{\mathcal{V}}^{r-1} + \int_{P(E)/M} G_{i\bar{j}}(\nabla_\alpha\xi^i)(\nabla_{\bar{\beta}}\bar{\xi}^j)\omega_{\mathcal{V}}^{r-1}\\
&= -\int_{P(E)/M}[R_{i\bar{j}\alpha\bar{\beta}} - G_{i\bar{j}l}R^l{}_{\alpha\bar{\beta}}]\omega_{\mathcal{V}}^{r-1}\xi^i\bar{\xi}^j + \int_{P(E)/M} G_{i\bar{j}}(\nabla_\alpha\xi^i)(\nabla_{\bar{\beta}}\bar{\xi}^j)\omega_{\mathcal{V}}^{r-1}\\
&= -{}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\beta}}\xi^i\bar{\xi}^j + \int_{P(E)/M} G_{i\bar{j}}(\nabla_\alpha\xi^i)(\nabla_{\bar{\beta}}\bar{\xi}^j)\omega_{\mathcal{V}}^{r-1}.
\end{aligned}$$

Taking the trace by $\delta_{\alpha\beta}$ as in Lemma 4.1 yields that

$$\nabla_{\bar{\alpha}}\nabla_\alpha\Big(\int_{P(E)/M} f\omega_{\mathcal{V}}^{r-1}\Big) \geq -{}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}}\xi^i\bar{\xi}^j,$$

whose left-hand side is an elliptic operator on $\int_{P(E)/M} f\omega_{\mathcal{V}}^{r-1}$. On the other hand, the right-hand side must be positive when $\xi$ does not vanish, provided that $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$

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is a negative definite hermitian matrix at each point of $M$. Then $\int_{P(E)/M} f\omega_{\mathcal{V}}^{r-1}$ must be constant on $M$, and ${}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}}\xi^i\bar{\xi}^j = 0$. Since $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is negative definite, $\xi$ must vanish.

Moreover, $\int_{P(E)/M} f\omega_{\mathcal{V}}^{r-1}$ is a constant on $M$, if $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is negative semi-definite. Thus, $\int_{P(E)/M} G_{i\bar{j}}(\nabla_\alpha \xi^i)(\nabla_{\bar{\beta}}\bar{\xi}^j)\omega_{\mathcal{V}}^{r-1} = 0$ hence $\nabla\xi = 0$ in this case. ∎

## 3.2 On Finsler Vector Bundles over Non-compact Complete Complex Manifolds

In this subsection, we aim to obtain the results in the last section for non-compact complete complex manifolds. This work is based on the method used in [13]. However, it is more general. First, we deal with the Hermitian manifolds case.

Firstly, we need the global scalar product of bundle sections. For any two holomorphic sections $\xi, \eta$ of a Finsler bundle, we define the *global scalar product* $\langle\langle\ ,\ \rangle\rangle$ by

$$\langle\langle \xi, \eta\rangle\rangle = \int_{P(E)} \langle \xi, \eta\rangle\, d\mu_{P\widetilde{M}} = \int_{P(E)} \sum_{i,j} G^{i\bar{j}}\xi_i\overline{\eta_j}\, d\mu_{P(E)}.$$

Let $L_2(\Gamma E)$ denote the completion of the space of all holomorphic sections with compact support, with respect to the scalar product $\langle\langle\ ,\ \rangle\rangle$. A complex holomorphic section $\xi$ of $M$ is said to have a *finite global norm* if the following integral is finite,

$$\|\xi\|^2 = \langle\langle \xi, \xi\rangle\rangle = \int_{P(E)} |\xi|_G^2\, d\mu_{P(E)} = \int_{P(E)} \sum_{i,j} G^{i\bar{j}}\xi_i\overline{\xi_j}\, d\mu_{P(E)} < \infty.$$

Secondly, a cut-off function and the distance function are needed in taking the integral on a non-compact manifold. Let $O$ be a fixed point of $M$. For each point $p \in M$, we denote $\rho(p)$ by the geodesic distance from $O$ to $p$. Since any complex Finsler metric is absolutely homogeneous, hence reversible, there is no difference between forward and backward distances (or completeness). Let $B(t) = \{p \in M \mid \rho(p) < t\}$ for $t > 0$. We choose a $C^\infty$-function $\mu$ on $\mathbb{R}$ satisfying

(i) $0 \le \mu(s) \le 1$ on $\mathbb{R}$,
(ii) $\mu(s) = 1$ for $s \le 1$,
(iii) $\mu(s) = \mu'(s) = 0$ for $s \ge 2$.

We set $w_t(p) = \mu(\rho(p)/t)$ for $t$ being some positive real numbers. Then we have the following lemma.

***Lemma 3.5*** *There exists a positive number A, depending only on $\mu$, such that*

$$|\nabla w_t|_\omega \le \frac{A}{t},$$

*where $|\cdot|_\omega$ denotes the norm with respect to the Kähler from $\omega$.*

**Proof** From the definition,

$$|\nabla w_t|_\omega = \left|\mu'(s)\frac{\nabla\rho}{t}\right|_\omega \le \frac{|\mu'(s)|}{t}|\nabla\rho|_\omega \le \frac{A}{t},$$

where $A$ is the upper bound of $\mu'(s)$ on $[1, 2]$. ∎

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Now we can prove the following theorem.

***Theorem 3.6*** *Let $(M,\omega)$ be a complete Hermitian manifold with the Kähler form $\omega = \sqrt{-1}g_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$, and let E be a holomorphic vector bundle over M with a strongly pseudo-convex Finsler metric G. If it has non-positive G-average bundle curvature ${}^{\mathcal{B}}\mathcal{K}$, then every holomorphic section of E with finite global norm is parallel with respect to the Chern–Finsler connection. Moreover, if the curvature ${}^{\mathcal{B}}\mathcal{K}$ is negative, then there is no non-zero holomorphic section of E with finite global norm.*

**Proof** For any holomorphic section $\xi$ of $E$, let $f_t = w_t^2 f$, where $f = G_{i\bar j}\xi^i\bar\xi^j$. Considering the Schwartz and Young's inequalities, we get

$$\begin{aligned}
g^{\bar\beta\alpha}w_t\nabla_{\bar\beta}w_t\nabla_\alpha f &\geq -w_t|\nabla w_t|_\omega|\nabla f|_\omega\\
&\geq -\frac{Aw_t}{t}|\nabla\xi\cdot\xi|_\omega\\
&= -\frac{Aw_t}{t}\sqrt{g^{\bar\beta\alpha}(\nabla_{\bar\beta}\bar\xi^i G_{i\bar j}\xi^i)(\nabla_\alpha\xi^p G_{p\bar q}\bar\xi^q)}\\
&\geq -\frac{1}{4}w_t^2(g^{\bar\beta\alpha}\nabla_{\bar\beta}\bar\xi^i G_{i\bar j}\nabla_\alpha\xi^j) - \frac{4A^2}{t^2}(G_{p\bar q}\xi^p\bar\xi^q)\\
&= -\frac{1}{4}w_t^2|\nabla\xi|^2 - \frac{4A^2}{t^2}|\xi|_G^2,
\end{aligned}$$

where $|\nabla\xi|^2 = g^{\bar\beta\alpha}\nabla_{\bar\beta}\bar\xi^i G_{i\bar j}\nabla_\alpha\xi^j$ and $|\xi|_G^2 = G_{p\bar q}\xi^p\bar\xi^q$. So it implies that

$$\begin{aligned}
&g^{\bar\beta\alpha}\nabla_{\bar\beta}\nabla_\alpha f_t - 2g^{\bar\beta\alpha}\nabla_{\bar\beta}[(w_t\nabla_\alpha w_t)f]\\
&\quad= g^{\bar\beta\alpha}\nabla_{\bar\beta}(w_t^2\nabla_\alpha f)\\
&\quad\geq \frac{1}{2}w_t^2|\nabla\xi|^2 - \frac{8A^2}{t}|\xi|_G^2 - w_t^2 g^{\bar\beta\alpha}(R_{i\bar j\alpha\bar\beta} - g_{i\bar j h}R^h{}_{\alpha\bar\beta})\xi^i\bar\xi^j.
\end{aligned}$$

Taking the integral on both sides gives that

$$\begin{aligned}
0 &= \int_{B(2t)}\int_{P(E)/M}\{g^{\bar\beta\alpha}\nabla_{\bar\beta}w_t\nabla_\alpha f - 2g^{\bar\beta\alpha}\nabla_{\bar\beta}[(w_t\nabla_\alpha w_t)f]\}\,d\mu_{P(E)}\\
&\geq \frac{1}{2}\int_{B(2t)}w_t^2\frac{\omega^n}{n!}\int_{P(E)/M}|\nabla\xi|^2\frac{\omega_{\mathcal{V}}^{r-1}}{(r-1)!} - \frac{8A^2}{t}\int_{B(2t)}\int_{P(E)/M}|\xi|_G^2\,d\mu_{P(E)}\\
&\quad-\int_{B(2t)}w_t^2\frac{\omega^n}{n!}\int_{P(E)/M}g^{\bar\beta\alpha}(R_{i\bar j\alpha\bar\beta} - G_{i\bar j h}R^h{}_{\alpha\bar\beta})\xi^i\bar\xi^j\frac{\omega_{\mathcal{V}}^{r-1}}{(r-1)!}\\
&= \frac{1}{2}\int_{B(2t)}w_t^2\frac{\omega^n}{n!}\int_{P(E)/M}|\nabla\xi|^2\frac{\omega_{\mathcal{V}}^{r-1}}{(r-1)!}\\
&\quad-\frac{8A^2}{t}\int_{B(2t)}\int_{P(E)/M}|\xi|_G^2\,d\mu_{P(E)} - \int_{B(2t)}w_t^2\,{}^{\mathcal{B}}\mathcal{K}(\xi,\xi)\frac{\omega^n}{n!}.
\end{aligned}$$

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Since $\xi$ has finite global norm and the average curvature is non-positive by the assumption, $t$ tending to infinity yields that

$$0 \geq \limsup_{t\to\infty} \int_{B(2t)} {}^{\mathcal{B}}\mathcal{K}(w_t\xi, w_t\xi)\frac{\omega^n}{n!} \geq \frac{1}{2}\int_{P(E)} |\nabla\xi|^2\, d\mu_{P(E)} \geq 0.$$

Therefore, $\xi$ vanishes when ${}^{\mathcal{B}}\mathcal{K}$ is negative, and is parallel when ${}^{\mathcal{B}}\mathcal{K}$ is non-positive. ∎

By replacing $g^{\bar{\beta}\alpha}$ by $\delta^{\bar{\beta}\alpha}$ in the proof of Theorem 3.6, we can prove the following.

***Theorem 3.7*** *Let $M$ be a complete complex manifold, and let $E$ be a holomorphic vector bundle over $M$ with a strongly pseudo-convex Finsler metric $G$. If $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is a negative semi-definite hermitian matrix at each point of $M$, then every holomorphic section of $E$ with finite global norm is parallel with respect to the Chern–Finsler connection. Moreover, if the matrix $(\sum_\alpha {}^{\mathcal{G}}\mathcal{R}_{i\bar{j}\alpha\bar{\alpha}})$ is negative definite, then there is no non-zero holomorphic section of $E$ with finite global norm.*

Theorem 1.1 follows from Theorems 3.4 and 3.7 obviously.

## 4 Vanishing Theorems of Holomorphic Vector Fields on Complex Finsler Manifolds

Vanishing theorems of some holomorphic sections of certain bundles can be traced back to Bochner [4], Kobayashi and Wu [8], and Yau [12]. In this section, we will discuss the vanishing theorems of holomorphic vector fields on both complex and non-compact complete Finsler manifolds.

### 4.1 On Compact Complex Finsler Manifolds

In this subsection, we want to show the results on Finsler manifolds, namely, to prove that any holomorphic vector field over a complex Finsler manifold with negative definite Ricci curvature is a zero vector field.

A complex vector field $\xi$ of type $(1,0)$ on $M$ is called a holomorphic vector field if its components $\xi^i$ are all holomorphic functions, for all $1 \leq i \leq n$. It is denoted by $\xi = \xi^i(z)\partial_i$. Since we are working with the Finsler metric, which is defined on the pullback bundle, a holomorphic vector field can be horizontal lifting to a section in $\mathcal{H}$. Still denoted by $\xi$, it is locally expressed by $\xi = \xi^i(z)\delta_i$. The following lemma is just a corollary of Lemma 3.2 for the holomorphic bundle $\pi\colon T^{1,0}M \to M$.

***Lemma 4.1*** *Let $(M,G)$ be a compact complex Finsler manifold, on which the curvature $\big(\sum_m (R_{k\bar{j};m\bar{m}} - G_{k\bar{j}l}R^l{}_{;m\bar{m}})\big)$ is a negative semi-definite hermitian matrix at each point of $P\widetilde{M}$. Then every holomorphic vector field of type $(1,0)$ is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if $\big(\sum_m (R_{k\bar{j};m\bar{m}} - G_{k\bar{j}l}R^l{}_{;m\bar{m}})\big)$ is negative definite, then $M$ does not admit any non-zero holomorphic vector field of type $(1,0)$.*

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Indeed we obtain that

$$\nabla_{\bar{l}}\nabla_k f = -(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h_{\;;k\bar{l}})\xi^i\bar{\xi}^j + G_{i\bar{j}}(\nabla_k\xi^i)(\nabla_{\bar{l}}\bar{\xi}^j). \tag{8}$$

The curvature condition is related both to the underlying manifold and the fibre, which cannot be considered as the direct generalization of Kobayashi and Wu's result [8] in the case of vector fields.

By Lemma 4.1 and Definition 2.6, we can prove the following theorem, which is a generalization of the main result in [8] on the tangent bundle case. Here $(A_{i\bar{j}})$ denotes a Hermitian matrix.

***Theorem 4.2*** *Let $(M, G)$ be a compact complex Finsler manifold, on which the curvature $(\sum_m {}^{\mathcal{G}}\mathcal{R}_{i\bar{j};m\bar{m}})$ or $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j}})$ is a negative semi-definite hermitian matrix at each point of $M$. Then every holomorphic vector field of type $(1,0)$ on $M$ is parallel with respect to the horizontal Chern–Finsler connection. Moreover, if $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j};m\bar{m}})$ or $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j}})$ is negative definite, then $M$ does not admit any non-zero holomorphic vector field of type $(1,0)$.*

**Proof** By the same method in the proof of Lemma 4.1, the claim concerning $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j};m\bar{m}})$ is simply Theorem 3.4 for the holomorphic Finsler bundle $\pi\colon E = T^{1,0}M \to M$.

To prove the result in the condition of $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j}})$, we only need to find an inequality like (7). Taking the trace of (8) by $G^{i\bar{j}}$, which are the components of the inverse of Levi matrix $G_{i\bar{j}}$, yields that

$$G^{\bar{l}k}\nabla_{\bar{l}}\nabla_k f = -G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h_{\;;k\bar{l}})\xi^i\bar{\xi}^j + G^{\bar{l}k}G_{i\bar{j}}(\nabla_k\xi^i)(\nabla_{\bar{l}}\bar{\xi}^j). \tag{9}$$

It follows by taking the integral of (9) on each fibre of $T\widetilde{M}$ that

$$\int_{P\widetilde{M}/M} G^{\bar{l}k}\nabla_{\bar{l}}\nabla_k f\,\omega_{\mathcal{V}}^{n-1} = -{}^{\mathcal{G}}\mathcal{R}_{i\bar{j}}\xi^i\bar{\xi}^j + \int_{P\widetilde{M}/M} G^{\bar{l}k}G_{i\bar{j}}(\nabla_k\xi^i)(\nabla_{\bar{l}}\bar{\xi}^j)\omega_{\mathcal{V}}^{n-1},$$

whose left-hand side is equal to zero. So we get

$${}^{\mathcal{G}}\mathcal{R}_{i\bar{j}}\xi^i\bar{\xi}^j = \int_{P\widetilde{M}/M} G^{\bar{l}k}G_{i\bar{j}}(\nabla_k\xi^i)(\nabla_{\bar{l}}\bar{\xi}^j)\omega_{\mathcal{V}}^{n-1} \geq 0.$$

It follows from the assuption that $\xi$ must vanish when $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j}})$ is negative definite, and that $\xi$ must be parallel when $({}^{\mathcal{G}}\mathcal{R}_{i\bar{j}})$ is negative semi-definite. ∎

Theorem 1.2 follows from Theorem 4.2 immediately.

***Remark 4.3*** The curvature tensor $\sum_m(R_{k\bar{j};m\bar{m}} - G_{k\bar{j}l}R^l_{\;;m\bar{m}})\delta v^k \wedge \delta\bar{v}^l$ appearing in Lemma 4.1 and ${}^{\mathcal{G}}\mathcal{R}_{i\bar{j};m\bar{m}}\delta v^i \wedge \delta\bar{v}^j$ appearing in Theorem 4.2 reduce to the same one, $R_{k\bar{j};m\bar{m}}dv^k \wedge d\bar{v}^j$, when $G$ is a Hermitian metric.

Moreover, if the hermitian metric is Kähler, then the $G$-average curvature ${}^G\mathrm{Ric}$ is the Ricci curvature of $M$. By the analogous method of [9], one can use the Ricci (Kobayashi) curvature to get the same result as in Theorem 4.2 in this case.

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## 4.2 On Non-compact Complete Complex Finsler Manifolds

Let $\Lambda^{p,q}(M)$ be the space of all $(p,q)$-forms on $M$, and $\Lambda_o^{p,q}(M)$ be the subspace of $\Lambda^{p,q}(M)$ composed of forms with compact support. On a complex Finsler manifold, a $(p,q)$-form $\eta \in \Lambda^{p,q}(M)$ may be expressed locally as

$$\eta = \frac{1}{p!q!}\sum \eta_{i_1,\dots i_p\bar{j}_1\cdots\bar{j}_q}(z)dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}.$$

For any $\xi,\eta \in \Lambda_o^{p,q}(M)$, we define the global scalar product $\langle\langle\ ,\ \rangle\rangle$ by

$$\begin{aligned}\langle\langle \xi,\eta\rangle\rangle &= \int_{P\widetilde{M}}\langle \xi,\eta\rangle\, d\mu_{P\widetilde{M}}\\ &= \int_{P\widetilde{M}}\frac{1}{p!q!}\sum G^{\bar{k}_1 i_1}\cdots G^{\bar{k}_p i_p}G^{\bar{j}_1 l_1}\cdots G^{\bar{j}_q l_q}\\ &\qquad\qquad \times \xi_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\overline{\eta_{k_1\cdots k_p\bar{l}_1\cdots\bar{l}_q}}\, d\mu_{P\widetilde{M}}.\end{aligned}$$

Let $L_2^{p,q}(M)$ denote the completion of $\Lambda_o^{p,q}(M)$ with respect to the scalar product $\langle\langle\ ,\ \rangle\rangle$. A complex vector field $\xi$ of type $(1,0)$ on $M$ is said to have a finite global norm if the corresponding holomorphic $(0,1)$-form $\xi$ satisfies $\xi \in L_2^{0,1}(M)\cap\Lambda^{0,1}(M)$. That is, the following integral is finite,

$$\|\xi\|^2 = \langle\langle \xi,\xi\rangle\rangle = \int_{P\widetilde{M}}\langle \xi,\xi\rangle\, d\mu_{P\widetilde{M}} < \infty.$$

The following lemma is the tangent bundle version of Lemma 3.5.

***Lemma 4.4*** *There exists a positive number A, depending only on μ, such that*

$$|\nabla w_t| \le \frac{A}{t}.$$

Now we can prove Theorem 1.3 according to Definitions 2.7 and 2.8.

**Proof** For any holomorphic vector field $\xi$ of type $(1,0)$, let $f_t = w_t^2 f$, where $f = G_{i\bar{j}}\xi^i\bar{\xi}^j$. Using the Schwartz and Young inequalities, we get

$$\begin{aligned}G^{\bar{l}k}w_t\nabla_{\bar{l}}w_t\nabla_k f &\ge -w_t|\nabla w_t|\,|\nabla f|\\ &\ge -\frac{Aw_t}{t}|\nabla\xi\cdot\xi|\\ &= -\frac{Aw_t}{t}\sqrt{G^{\bar{l}k}(\nabla_{\bar{l}}\bar{\xi}^i G_{i\bar{j}}\bar{\xi}^j)(\nabla_k\xi^p G_{p\bar{q}}\bar{\xi}^q)}\\ &\ge -\frac{1}{4}w_t^2|\nabla\xi|^2 - \frac{4A^2}{t^2}|\xi|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&G^{\bar{l}k}\nabla_{\bar{l}}\nabla_k f_t - 2G^{\bar{l}k}\nabla_{\bar{l}}[(w_t\nabla_k w_t)f]\\ &= G^{\bar{l}k}\nabla_{\bar{l}}(w_t^2\nabla_k f)\\ &\ge \frac{1}{2}w_t^2|\nabla\xi|^2 - \frac{8A^2}{t}|\xi|^2 - w_t^2 G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}h}R^h{}_{;k\bar{l}})\xi^i\bar{\xi}^j.\end{aligned}$$

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Taking the integral on both side gives that

$$
\begin{aligned}
0 &= \int_{P\widetilde{B}(2t)} \{G^{\bar{l}k}\nabla_{\bar{l}} w_t \nabla_k f - 2G^{\bar{l}k}\nabla_{\bar{l}}[(w_t\nabla_k w_t)f]\}\, d\mu_{P\widetilde{M}} \\
&\geq \frac{1}{2}\int_{B(2t)} w_t^2 \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} |\nabla\xi|^2 \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} - \frac{8A^2}{t}\int_{P\widetilde{B}(2t)} |\xi|^2\, d\mu_{P\widetilde{M}} \\
&\quad - \int_{B(2t)} w_t^2 \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} G^{\bar{l}k}(R_{i\bar{j};k\bar{l}} - G_{i\bar{j}}R^h{}_{;k\bar{l}})\xi^i\bar{\xi}^j \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \\
&= \frac{1}{2}\int_{B(2t)} w_t^2 \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} |\nabla\xi|^2 \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \\
&\quad - \frac{8A^2}{t}\int_{P\widetilde{B}(2t)} |\xi|^2\, d\mu_{P\widetilde{M}} - \int_{B(2t)} w_t^2\, {}^{\mathfrak{g}}\mathrm{Ric}(\xi,\xi)\, dM \\
&= \frac{1}{2}\int_{B(2t)} w_t^2 \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} |\nabla\xi|^2 \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \\
&\quad - \frac{8A^2}{t}\int_{P\widetilde{B}(2t)} |\xi|^2\, d\mu_{P\widetilde{M}} - \int_{B(2t)} w_t^2\, {}^{\mathfrak{G}}\mathrm{Ric}(\xi,\xi)\, d\tilde{M},
\end{aligned}
$$

where $dM = (\det g_{i\bar{j}})dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n$ and $d\tilde{M} = \sigma(z)dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n$. Since $\xi$ has finite global norm and the average curvature is non-positive by the assumption, $t$ tending to infinity yields that

$$
0 \geq \limsup_{t\to\infty}\int_{B(2t)} {}^{\mathfrak{g}}\mathrm{Ric}(w_t\xi, w_t\xi)\, dM \geq \frac{1}{2}\int_M \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} |\nabla\xi|^2 \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \geq 0,
$$

and

$$
0 \geq \limsup_{t\to\infty}\int_{B(2t)} {}^{\mathfrak{G}}\mathrm{Ric}(w_t\xi, w_t\xi)\, d\tilde{M} \geq \frac{1}{2}\int_M \frac{\omega_{\mathcal{H}}^n}{n!}\int_{P\widetilde{M}/M} |\nabla\xi|^2 \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \geq 0,
$$

Therefore, $\xi$ vanishes when ${}^{\mathfrak{g}}\mathrm{Ric}$ or ${}^{\mathfrak{G}}\mathrm{Ric}$ is negative, and is parallel when ${}^{\mathfrak{g}}\mathrm{Ric}$ or ${}^{\mathfrak{G}}\mathrm{Ric}$ is non-positive. ∎

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*School of Mathematics, Southeast University, 211189, Nanjing, P. R. China*
*e-mail:* shenbin@seu.edu.cn

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