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INTERSECTIONS OF PRIMARY IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

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Introduction. Let C be the ring of all real valued continuous functions on a completely regular topological space. This paper is an investigation of the ideals of C that are intersections of prime or of primary ideals.

C. W. Kohls has analyzed the prime ideals of C in [**3**; **4**] and the primary ideals of C in [**5**]. He showed that these ideals are absolutely convex. (An ideal I of C is called *absolutely convex* if $|f| \leqq |g|$ and $g \in I$ imply that $f \in I$.) It follows that any intersection of prime or of primary ideals is absolutely convex. We consider here the problem of finding a necessary and sufficient condition for an absolutely convex ideal I of C to be an intersection of prime ideals and the problem of finding a necessary and sufficient condition for I to be an intersection of primary ideals.

The solution to the first problem is given in Theorem 2.1: an absolutely convex ideal I of C is an intersection of prime ideals if and only if $I = I^2$.

The problem of characterizing the absolutely convex ideals that are intersections of primary ideals turns out to be considerably more difficult, and we are less successful. In Theorem 2.8 we show that for any absolutely convex ideal I of C the ideals $I \cdot I^{1/2}$ and $I : I^{1/2}$ are always intersections of primary ideals. Thus if I satisfies either $I = I \cdot I^{1/2}$ or $I = I : I^{1/2}$, then I is an intersection of primary ideals; it is not difficult, however, to find examples of ideals that are intersections of primary ideals but satisfy neither of these conditions (Example 2.11). A necessary condition for an absolutely convex ideal to be an intersection of primary ideals is given in Theorem 2.15: if I is an intersection of primary ideals, then $I^2 = I \cdot (I : I^{1/2})$. Finally, we show that although this condition is not, in general, a sufficient condition for an absolutely convex ideal to be an intersection of primary ideals (Example 2.21), it is sufficient for absolutely convex ideals that satisfy an additional hypothesis. In particular, we show in Theorem 2.17 that if I is an absolutely convex ideal such that the intersection of all the minimal primary ideals of I is irredundant, then I is an intersection of primary ideals if and only if $I^2 = I \cdot (I : I^{1/2})$.

1. Background and preliminary results. Our terminology and notation will, with only a few exceptions, be that of [**1**]. The symbol $\subset$ will denote set inclusion, while $<$ will denote proper inclusion. The term "ring" unmodified will mean a commutative ring with identity. The term "ideal" will always mean a proper ideal; i.e., an ideal can never be the whole ring. If I is an ideal of a ring, we will denote the radical of I by $I^{1/2}$.

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Let P be a prime ideal of C. We list now some facts about the prime and primary ideals of the rings C and C/P and prove some preliminary propositions. For the proofs of statements about the prime ideals of these rings see [**1**, Chapter 14]. For the proofs of statements about the primary ideals see [**5**].

If I is an absolutely convex ideal of C, the residue class ring C/I is a partially ordered ring, according to the definition: $I(f) \geqq 0$ if there exists $g \in C$ such that $g \geqq 0$ and $f \equiv g \pmod{I}$. The canonical homomorphism of C onto C/I, which is clearly order preserving, is a lattice homomorphism. If r is a real number, we denote by r both the constant function whose constant value is r and the residue class of this constant function modulo I. Thus we view the real field as a subfield of both C and C/I.

For a prime ideal P of C, the residue class ring C/P is totally ordered [**1**, Theorem 5.5]. Every positive element b of C/P has a unique positive nth root $b^{1/n}$: if f is any pre-image of b in C, with $f \geqq 0$, then $P(f^{1/n}) = b^{1/n}$. If a is a positive non-unit of C/P, then a is infinitely small, i.e., $a < 1/n$ for every positive integer n. In particular, then, $a < 1$, so if r and s are positive rational numbers, $r < s$ if and only if $a^s < a^r$.

A convex ideal in a totally ordered ring is a symmetric interval: with every two of its elements, it also contains all elements that lie between them. Hence the prime and primary ideals of C/P are symmetric intervals. If a is a positive non-unit of C/P, then in the ring C/P there is a smallest prime ideal P^a that contains a and a largest prime ideal P_a that does not contain a:

$$P^a = \{b \in C/P\colon |b| \leqq a^{1/n} \text{ for some } n \in N\},$$
$$P_a = \{b \in C/P\colon |b| < a^n \quad \text{for all } n \in N\}.$$

Also, there is a smallest primary ideal $P|^a$ that contains a and a largest primary ideal $P|_a$ that does not contain a:

$$P|^a = \{b \in C/P\colon |b| < a^{1-1/n} \text{ for all } n \in N\},$$
$$P|_a = \{b \in C/P\colon |b| \leqq a^{1+1/n} \text{ for some } n \in N\}.$$

Clearly $P_a < P|_a < P|^a < P^a$.

1.1. *Definition.* In any ring an *upper prime ideal* (respectively, *upper primary ideal*) is a prime (respectively, primary) ideal that has an immediate predecessor in the set of all prime (respectively, primary) ideals partially ordered by inclusion. A *lower prime ideal* (respectively, *lower primary ideal*) is a prime (respectively, primary) ideal that has an immediate successor in the set of all prime (respectively, primary) ideals partially ordered by inclusion.

The ideals of the form P^a are the upper prime ideals of the ring C/P, and the ideals P_a are the lower prime ideals. The ideals $P|^a$ are the upper primary ideals of C/P, and the ideals $P|_a$ are the lower primary ideals. There are, in general, many prime ideals of C/P that are neither upper nor lower prime ideals. The maximal ideals of C/P, for example, which are clearly not lower prime ideals, are not upper prime ideals either. Every non-prime primary ideal of C/P, however, is either an upper or a lower primary ideal. No prime ideal of C/P can be both an upper and a lower prime ideal. We show now that this is also true of the primary ideals of C/P.

1.2. PROPOSITION. *No primary ideal of C/P can be both an upper and a lower primary ideal.*

Proof. We show that if $P|_b \subset P|^a$ for some positive non-units a and b of C/P, then $P|_b < P|^a$. If $P|_b \subset P|^a$, then for all $n, m \in N$, $b^{1+1/m} < a^{1-1/n}$. By [**1**, Lemma 14.15] there exists $c \in C/P$ such that $b^{1+1/m} \leqq c \leqq a^{1-1/n}$ for all $n, m \in N$. Clearly $c \in P|^a - P|_b$.

Now we turn our attention to the prime and primary ideals of the ring C. Every lower prime ideal of C has a unique immediate successor. Also, every upper prime ideal of C has a unique immediate predecessor; in fact, if Q is an upper prime ideal of C and P is an immediate predecessor, then every predecessor of Q is contained in P. From these facts we conclude that if P is any prime ideal contained in a prime ideal Q, then Q is a lower prime ideal of C if and only if Q/P is a lower prime ideal of C/P. And if P' is any prime ideal properly contained in a prime ideal Q', then Q' is an upper prime ideal of C if and only if Q'/P' is an upper prime ideal of C/P'.

We want to verify the analogous statements for the primary ideals of C. In addition to facts stated above, we will make use of the following information about the primary ideals of C: every primary ideal contains a prime ideal, and every primary ideal is absolutely convex.

1.3. PROPOSITION. (a) *Every lower primary ideal of C has a unique immediate successor.*

(b) *Every upper primary ideal of C has a unique immediate predecessor.*

Proof. (a) Let K be a lower primary ideal of C. Let P be any prime ideal contained in K. Since the convex ideals containing P form a chain (i.e., are totally ordered by inclusion), the primary ideals containing K form a chain, so (a) is clear.

(b) Let J be an upper primary ideal of C, and let I and I' be two primary ideals that are immediate predecessors of J. Let P be any prime ideal contained in I, and let P' be any prime ideal contained in I'. In C/P, $J/P = P|^a$ and $I/P = P|_a$ for some a. If π: $C \to C/P$, then we have $\pi^{-1}(P_a) \subset I \subset J \subset \pi^{-1}(P^a)$. Now $\pi^{-1}(P_a)$ is an immediate predecessor of $\pi^{-1}(P^a)$ in the set of all prime ideals of C. Therefore, since $P' \subset I' < J \subset \pi^{-1}(P^a)$, $P' \subset \pi^{-1}(P_a)$. Hence $P' \subset I$; since $P' \subset I'$ also and the primary ideals containing a given prime ideal form a chain, we must have $I = I'$.

1.4. COROLLARY. *Let J be an upper primary ideal of C, and let I be its immediate predecessor. If I' is any predecessor of J, then $I' \subset I$.*

Proof. Note that J is not a prime ideal since a prime ideal cannot be an upper or lower primary ideal in C. Let P' be any prime ideal contained in I'. J/P' is a non-prime primary ideal in C/P', so there are two possibilities: either (1) $J/P' = P'|^b$ for some b, or (2) $J/P' = P'|_c$ for some c. Let π: $C \to C/P'$. In case (1) it is clear that $I' \subset \pi^{-1}(P'|_b)$, and $\pi^{-1}(P'|_b)$ is an immediate predecessor of J. By Proposition 1.3, $\pi^{-1}(P'|_b) = I$, and so $I' \subset I$. We conclude the proof by showing that case (2) is impossible. Assume, on the contrary, that $J/P' = P'|_c$. Then in C, J is an immediate predecessor of $\pi^{-1}(P'|^c)$. Let P be any prime ideal contained in I. In C/P, I/P is the immediate predecessor of J/P, and J/P is the immediate predecessor of $\pi^{-1}(P'|^c)/P$. This contradicts Proposition 1.2.

1.5. Corollary. *Let J be an ideal in C, and let P be any prime ideal contained in J. Then:*

(a) *J is an upper primary ideal in C if and only if J/P is an upper primary ideal in C/P.*

(b) *J is a lower primary ideal in C if and only if J/P is a lower primary ideal in C/P.*

By the remarks following Proposition 1.2 the upper prime ideals of C are of the form $\pi^{-1}(P^a)$, and the lower prime ideals of C are of the form $\pi^{-1}(P_a)$, where P is a prime ideal of C, π: $C \to C/P$, and a is a positive non-unit of C/P. By Corollary 1.5 the upper primary ideals of C are of the form $\pi^{-1}(P|^a)$, and the lower primary ideals of C are of the form $\pi^{-1}(P|_a)$. If $\pi(|f|) = a$, we will denote $\pi^{-1}(P^a)$ by P^f, $\pi^{-1}(P|^a)$ by $P|^f$, etc.

We will also want to know about intersections of chains of prime and primary ideals of C. In any ring the intersection of a chain of prime ideals is prime, so in particular this holds for the ring C. It is not true that in any ring the intersection of a chain of primary ideals is primary: for example, in the polynomial ring $F[X, Y]$ in two indeterminates over a field F the ideals (X^k, XY, Y^2) for $k \in N$, $k \geqq 2$, are all primary, and they clearly form a chain; but

$$I = \bigcap_{k=2}^{\infty} (X^k, XY, Y^2)$$

is not primary since $XY \in I$, $Y \notin I$, and $X \notin I^{1/2}$. Fortunately, however, in the ring C the intersection of a chain of primary ideals is in fact primary. For by [**2**, 2.10 and 4.1] the intersection of a chain of primary ideals of C contains a prime ideal, and it easily follows that the intersection is primary.

Finally, we will make much use of a theorem of L. Gillman and C. W. Kohls on pseudoprime ideals.

1.6. *Definition*. An ideal of C is called a *pseudoprime ideal* if it contains a prime ideal.

We remark that in [**2**] the pseudoprime ideals of C are defined as those ideals I satisfying the condition: $fg = 0$ implies $f \in I$ or $g \in I$. This definition is then shown to be equivalent to Definition 1.6.

1.7. PROPOSITION [**2**, Theorem 4.7]. *An ideal of C is absolutely convex if and only if it is an intersection of absolutely convex pseudoprime ideals.*

2. The main results. Using Proposition 1.7 we can immediately obtain our characterization of the absolutely convex ideals of C that are intersections of prime ideals.

2.1. THEOREM. *Let I be an absolutely convex ideal of C. I is an intersection of prime ideals if and only if $I = I^2$ (i.e., I is idempotent).*

Proof. Necessity: If $f \in I$, then $f^{1/3} \in I$, so

$$f = f^{1/3} \cdot (f^{1/3})^2 \in I^2.$$

Sufficiency: Let $f \in C - I$. We are to find a prime ideal that contains I but not f. By Proposition 1.7 there exists an absolutely convex pseudoprime ideal J that contains I but not f. Let P be any prime ideal contained in J. If M is the unique maximal ideal that contains P and $f \notin M$, we are done; so assume that $f \in M$. Since $f \notin P_f$, the proof will be complete if we show that $I \subset P_f$. Now since $f \in P|^f - J$ and the convex ideals containing P form a chain, we must have $J \subset P|^f$. Therefore $I \subset P|^f$. Let $g \in I$, and let $n \in N$. Since

$$g \in I^{2(n+1)} = I,$$

we have

$$g = \sum_{i=1}^{k} g_{1i} \ldots g_{2(n+1)i}$$

for some $\{g_{ji}\} \subset I$ and some $k \in N$. Therefore

$$\begin{aligned} |P(g)| &\leqq \sum_{i=1}^{k} |P(g_{1i})| \ldots |P(g_{2(n+1)i})| \\ &\leqq k \cdot \max_{1 \leqq i \leqq k} \{|P(g_{1i}) \ldots P(g_{2(n+1)i})|\} \\ &\leqq k\, |P(f)|^{n+1} \text{ since } |P(g_{ji})| < |P(f)|^{1/2} \text{ for all } j, i \\ &< |P(f)|^{n}. \end{aligned}$$

Since g and n are arbitrary, $I \subset P_f$.

2.2. COROLLARY. *Every pseudoprime idempotent ideal of C is prime.*

Proof. Let I be a pseudoprime idempotent ideal of C. It is sufficient to show that I is absolutely convex. For then by Theorem 2.1 I is an intersection of prime ideals, and a pseudoprime ideal that is an intersection of prime ideals is prime. Suppose that $|g| \leqq |f|$ for some $g \in C$ and $f \in I$. Then by [**1**, 1D.3], $g^{5/3} \in I$, so $g^{5/3} \in I^2$. Therefore for some $k \in N$ there exists

$$\{i_1, \ldots, i_k, j_1, \ldots, j_k\} \subset I$$

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such that

$$g^{5/3} = \sum_{n=1}^{k} i_n j_n.$$

Let P be a prime ideal contained in I. Then if

$$P(|i_m j_m|) = \max_{1 \leqq n \leqq k} \{P(|i_n j_n|)\},$$

and

$$P(|i_m|) = \max\{P(|i_m|), P(|j_m|)\},$$

we have

$$\begin{aligned} P(|g|)^{5/3} &\leqq \sum_{n=1}^{k} P(|i_n j_n|) \\ &\leqq k\, P(|i_m j_m|) \\ &\leqq k\, P(|i_m|)^2 \\ &= P(|i|)^2, \end{aligned}$$

where $i = \sqrt{k} i_m \in I$. Hence $P(|g|) \leqq P(|i|)^{6/5}$. It follows that there exists $h \in C$ such that $g \equiv h \pmod P$, and $|h| \leqq |i|^{6/5}$. By [**1**, 1D.3] $h \in I$, so since $g - h \in P \subset I$, $g \in I$.

We now turn our attention to the study of the absolutely convex ideals of C that are intersections of primary ideals. Our first theorem gives a sufficient condition for an absolutely convex ideal to be an intersection of primary ideals.

2.3. THEOREM. *Let I be an absolutely convex ideal of C. If $I = I \cdot I^{1/2}$, then I is an intersection of primary ideals.*

Proof. We show first that for any ideal J of C, any prime ideal P, and any $f \in M - P$, where M is the maximal ideal that contains P,

(*) $$J \subset P|^f \text{ implies } J \cdot J^{1/2} \subset P|_f.$$

It is sufficient to show that if $g \in J$ and $h \in J^{1/2}$, then $gh \in P|_f$. Since $h \in J^{1/2}$, $h^m \in J$ for some m. If we set $P(g) = b$, $P(h) = c$, and $P(|f|) = a$, we have $|b| < a^{1-1/n}$ and $|c|^m < a^{1-1/n}$ for all $n \in N$. Hence $|bc| \leqq a^{(1+1/m)(1-1/n)}$ for all $n \in N$. Since $(1 + 1/m)(1 - 1/n) > 1$ when $n > m + 1$, $|bc| \leqq a^{1+1/k}$ for some positive integer k, and we have $gh \in P|_f$.

Now let I be the ideal of the theorem, and let $f \in C - I$. We want to find a primary ideal that contains I but not f. Let Q be an absolutely convex pseudo-prime ideal that contains I but not f. Let P be a prime ideal contained in Q. If M is the maximal ideal that contains P and $f \notin M$, we are done; so assume that $f \in M$. Since $f \notin P|_f$, the proof will be complete if we show that $I \subset P|_f$. Now the convex ideals containing P form a chain, so since $f \in P|^f - Q$, we must have $Q \subset P|^f$. Therefore $I \subset P|^f$, so by (*) $I = I \cdot I^{1/2} \subset P|_f$.

We will show in Example 2.11 that the condition $I = I \cdot I^{1/2}$ of Theorem 2.3 is not a necessary condition for an absolutely convex ideal I to be an intersection of primary ideals. The next theorem gives information about the ideal $I \cdot I^{1/2}$ for any absolutely convex ideal I of C.

2.4. THEOREM. *Let I be an absolutely convex ideal of C. Then:*

(a) $I \cdot I^{1/2} = \{f \in C\colon |f|^{1-1/n} \in I$ *for some* $n \in N\}$, *so* $I \cdot I^{1/2}$ *is absolutely convex.*

(b) $I \cdot I^{1/2} \cdot (I \cdot I^{1/2})^{1/2} = I \cdot I^{1/2}$.

(c) *If P is a prime ideal of C, then* $I \cdot I^{1/2} \subset P|_f$ *if and only if* $I \subset P|^f$.

2.5. LEMMA. *Let J be an absolutely convex pseudoprime ideal of C. If* $0 = J(|g|)^m < J(|f|)^m$ *for some* $m \in N$, *then* $J(|g|) \leqq J(|f|)$.

Proof. Assume, on the contrary, that $J(|g|) \nleqq J(|f|)$. Then since C/J is totally ordered, $J(|g|) > J(|f|)$. Hence $0 = J(|g|)^m \geqq J(|f|)^m \geqq 0$, and so $J(|f|)^m = 0$, which is a contradiction.

Proof of Theorem 2.4: (a) Set $J = \{f \in C\colon |f|^{1-1/n} \in I$ for some $n \in N\}$. Note that since J is clearly closed under multiplication by elements of C, [**1**, 1E.1] implies that $f \in J$ if and only if $|f| \wedge 1 \in J$. We show first that $J \subset I$. Let $f \in J$; then $|f| \wedge 1 \in J$, so $(|f| \wedge 1)^{1-1/n} \in I$ for some $n \in N$. Therefore since $|f| \wedge 1 \leqq (|f| \wedge 1)^{1-1/n}$, $|f| \wedge 1 \in I$. By [**1**, 1E.1] this implies that $|f| \in I$. Hence $f \in I$, and so $J \subset I$. It follows from this that $J \subset I \cdot I^{1/2}$. For, let $f \in J$; then $f^{1-1/m} \in I$ for some odd $m \in N$. Also, since $f \in I$, $f^{1/m} \in I^{1/2}$, and we have $f = f^{1-1/m} \cdot f^{1/m} \in I \cdot I^{1/2}$.

Before verifying the reverse inclusion we show that J is closed under addition. Let $f, g \in J$, and suppose first that $|f| \leqq 1$ and $|g| \leqq 1$. Then $|f|^{1-1/n} \in I$ and $|g|^{1-1/m} \in I$ for some $n, m \in N$, so if $k = \max\{n, m\}$, $|f|^{1-1/k} + |g|^{1-1/k} \in I$. Since

$$(|f| + |g|)^{1-1/k} \leqq |f|^{1-1/k} + |g|^{1-1/k},$$

$(|f| + |g|)^{1-1/k} \in I$. Therefore since

$$|f + g|^{1-1/k} \leqq (|f| + |g|)^{1-1/k},$$

$|f + g|^{1-1/k} \in I$, and we have $f + g \in J$. In the general case, if $f, g \in J$, then $|f| \wedge 1, |g| \wedge 1 \in J$, so by the above $|f| \wedge 1 + |g| \wedge 1 \in J$. Since

$$|f + g| \wedge 1 \leqq (|f| \wedge 1) + (|g| \wedge 1),$$

$|f + g| \wedge 1 \in J$, and so $f + g \in J$.

Now since J is closed under addition, to verify the reverse inclusion it is sufficient to show that if $h \in I$ and $g \in I^{1/2}$, then $gh \in J$ (equivalently, $|gh| \wedge 1 \in J$). Let u be a unit of C such that $u|gh| = |gh| \wedge 1$. Suppose, on the contrary, that $f = u|gh| \notin J$. Since $u|g| \in I^{1/2}$, $(u|g|)^m \in I$ for some $m \in N$. Pick $n \in N$ such that $1 < 1 + 1/m - (1/n)(1 + 1/m)$. Since $f \notin J$, $f^{1-1/n} \notin I$, so there exists an absolutely convex pseudoprime ideal K that contains I but not $f^{1-1/n}$. Then $0 = K(u|g|)^m < K(f)^{1-1/n}$, so $K(u|g|) \leqq K(f)^{(1-1/n)1/m}$ by Lemma 2.5; and $0 = K(|h|) < K(f)^{1-1/n}$ since $|h| \in I$, so

$$K(f) = K(u|gh|) \leqq K(f)^{(1-1/n)(1+1/m)}.$$

Therefore since $0 \leqq f \leqq 1$,

$$1 \geqq (1 - 1/n)(1 + 1/m)) = 1 + 1/m - (1/n)(1 + 1/m),$$

which is a contradiction.

(b) Clearly $J \cdot J^{1/2} \subset J$. For the reverse inclusion let $f \in J$ with $|f| \leqq 1$. Then $|f|^{1-1/n} \in I$ for some even integer n. We claim that $f^{1-1/(n+1)} \in J$. Let $k \in N$ be such that

$$1 - 1/n \leqq (1 - 1/(n + 1))(1 - 1/k).$$

Then

$$(|f|^{1-1/(n+1)})^{1-1/k} \leqq |f|^{1-1/n} \in I.$$

Therefore

$$(|f|^{1-1/(n+1)})^{1-1/k} \in I,$$

and so

$$f^{1-1/(n+1)} \in J.$$

Now we have

$$f = f^{1-1/(n+1)} \cdot f^{1/(n+1)} \in J \cdot J^{1/2}.$$

(c) We showed in the proof of Theorem 2.3 that if $I \subset P|^f$, then $I \cdot I^{1/2} \subset P|_f$, for any ideal I. For the converse, suppose that $J \subset P|_f$, and let $g \in I$ with $|g| \leqq 1$. Note that for all $m \in N$ there exists $n \in N$ such that

$$(|g|^{1+1/m})^{1-1/n} \leqq |g|.$$

Therefore

$$|g|^{1+1/m} \in J$$

for all $m \in N$, so

$$|g|^{1+1/m} \in P|_f$$

for all $m \in N$, and it follows that $g \in P|^f$.

Combining Theorems 2.3 and 2.4 we have:

2.6. COROLLARY. *If I is an absolutely convex ideal of C, then $I \cdot I^{1/2}$ is an intersection of primary ideals. In fact, $I \cdot I^{1/2}$ is the intersection of all the maximal ideals containing I and all the lower primary ideals whose corresponding upper primary ideals contain I.*

Proof: By Theorems 2.3 and 2.4 (b), $I \cdot I^{1/2}$ is an intersection of primary ideals. Let K be the intersection of all the maximal ideals containing I and all the lower primary ideals whose corresponding upper primary ideals contain I. By Theorem 2.4 (c), $I \cdot I^{1/2} \subset K$. For the reverse inclusion suppose that $f \notin I \cdot I^{1/2}$. Since $I \cdot I^{1/2}$ is an intersection of primary ideals, there exists a primary ideal J such that $I \cdot I^{1/2} \subset J$ and $f \notin J$. Let P be a prime ideal contained in J, and let M be the maximal ideal that contains P. If $f \notin M$, then since clearly $I \subset M, f \notin K$. So assume that $f \in M$. Since $f \notin J$, $J \subset P|_f$, and so $I \cdot I^{1/2} \subset P|_f$. By Theorem 2.4 (c), $I \subset P|^f$, so again we have $f \notin K$.

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2.7. *Definitions.* (a) If I and J are ideals of a ring R, the *quotient* $I : J$ is defined as follows: $I : J = \{r \in R: rJ \subset I\}$. (It is clear that $I : J = R$ if and only if $J \subset I$; and if $J \not\subset I$, then $I : J$ is an ideal of R which contains I.)

(b) If I is an ideal of a ring R and K is a primary ideal of R which contains I, then K is called a *minimal primary ideal of* I if there does not exist a primary ideal J of R such that $I \subset J < K$.

If I is an absolutely convex ideal of C, we denote by I^* the intersection of all the primary ideals containing I and by $I^\#$ the set $I^\# = \{f \in C: |f|^{1+1/n} \in I$ for all $n \in N\}$. The next theorem shows that the ideals $I : I^{1/2}$ and $I^\#$ are always intersections of primary ideals and provides descriptions of the ideals $I \cdot I^{1/2}$, I, I^*, and $I^{1/2}$ which will be frequently used throughout the remainder of this paper.

2.8. Theorem. *Let I be an absolutely convex ideal of C. Let $\{Q_\alpha: \alpha \in \mathrm{A}\}$ be the set of minimal primary ideals of I that are prime, $\{P_\beta|^{f_\beta}: \beta \in \mathrm{B}\}$ be the set of minimal primary ideals of I that are upper primary ideals, and $\{P_\gamma|_{f_\gamma}: \gamma \in \Gamma\}$ be the set of minimal primary ideals of I that are lower primary ideals. Then:*

(a) $I \cdot I^{1/2} = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta|_{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$.

(*b*) $I = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta + I) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$.

(c) $I^* = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma})$.

(d) $I^\# = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|^{f_\gamma})$.

(e) $I^{1/2} = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta{}^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma{}^{f_\gamma})$.

(f) $I : I^{1/2} = (\bigcap_{\beta \in \mathrm{B}} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|^{f_\gamma})$.

Proof. We will need the following fact several times:

(*) Every primary ideal containing I contains a minimal primary ideal of I.

Since the intersection of a chain of primary ideals of C is primary, this follows from a Zorn's Lemma argument.

(a) This is easily verified by using Corollary 2.6 and (*).

(b) $\subset$ is clear. Let $f \in C - I$. If f is not in every minimal primary ideal of I, then f is not in the given intersection; so assume that f is in every minimal primary ideal of I. Let K be an absolutely convex pseudoprime ideal that contains I but not f, and let P be a prime ideal contained in K. $P|^f$ is a minimal primary ideal of I. (Otherwise, by (*), $P|^f$ properly contains a minimal primary ideal of I; but no primary ideal properly contained in $P|^f$ contains f.) Therefore $I + P$ appears in the given intersection. And since $I + P \subset K, f \notin I + P$.

(c) This follows from (*).

(d) First we note that for a lower primary ideal $P|_g$ of C, $(P|_g)^\# = P|^g$: for, $f \in P|^g$ if and only if either $f \in P$ or $P|^f \subset P|^g$, if and only if either $f \in P$ or

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$P|_f \subset P|_g$, if and only if $|f|^{1+1/n} \in P|_g$ for all $n \in N$. It follows from this that if h is in the given intersection, then

$$|h|^{1+1/n} \in I \cdot I^{1/2} \subset I$$

for all $n \in N$, so $h \in I^\#$. For the reverse inclusion suppose that $f \in I^\#$. Then f is in every upper primary ideal that contains I: for, if $|f|^{1+1/n} \in P|^g$ for all $n \in N$, then either $f \in P$ or $P|_f \subset P|^g$; hence either $f \in P$ or $P|^f \subset P|^g$, and so $f \in P|^g$. Since clearly

$$f \in \bigcap_{\alpha \in A} Q_\alpha,$$

f is in the given intersection.

(e) $\subset$ is clear. For the reverse inclusion let P be a prime ideal containing I. If P is a minimal primary ideal of I, then P appears in the given intersection. If P is not a minimal primary ideal of I, then P contains an ideal that appears in the given intersection.

(f) First we record the following fact, which is routine to verify: if P is a prime ideal of C and $f \in M - P$, where M is the maximal ideal containing P, then

$$(**) \qquad P|_f \colon P^f = P|^f \colon P^f = P|^f.$$

Now set

$$J = (\bigcap_{\beta \in B} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|^{f_\gamma}).$$

By (**), (a), and (e)

$$J \cdot I^{1/2} \subset I \cdot I^{1/2} \subset I,$$

and so

$$J \subset I : I^{1/2}.$$

For the reverse inclusion suppose that $h \in C - J$. Then $h \notin P_\beta|^{f_\beta}$ for some $\beta \in B$ (or $h \notin P_\gamma|^{f_\gamma}$ for some $\gamma \in \Gamma$ in which case the proof is the same). By (**) there exists $g \in P_\beta{}^{f_\beta}$ such that $g \geqq 0$ and $hg \notin P_\beta|^{f_\beta}$. Choose $f \in I - P_{\beta f_\beta}$ such that $f \geqq 0$. Since $f \in P_\beta{}^{f_\beta} - P_{\beta f_\beta}$, $P_\beta{}^{f_\beta} = P_\beta{}^f$. Therefore $g \in P_\beta{}^f$, and so $P_\beta(g) \leqq P_\beta(f)^{1/k}$ for some $k \in N$. Set $t = g \wedge f^{1/k}$. Since $0 \leqq t \leqq f^{1/k}$, $0 \leqq t^k \leqq f$; hence $t^k \in I$, so $t \in I^{1/2}$. Also, since $P_\beta(t) = P_\beta(g) \wedge P_\beta(f)^{1/k} = P_\beta(g)$ and $hg \notin P_\beta|^{f_\beta}$, we have $ht \notin P_\beta|^{f_\beta}$. Therefore $ht \notin I$, and it follows that $h \notin I : I^{1/2}$.

Applying Theorem 2.8 to the class of absolutely convex pseudoprime ideals we have:

2.9. COROLLARY. *If I is an absolutely convex pseudoprime ideal of C that is not prime, then $I^\# = I : I^{1/2}$ is an upper primary ideal, and $I \cdot I^{1/2}$ is the corresponding lower primary ideal.*

Proof: Since I is pseudoprime, I has a unique minimal primary ideal J; and since I is not prime, it is easy to see that J is not prime and consequently is either an upper or a lower primary ideal. In either case the conclusion of the Corollary follows immediately from Theorem 2.8.

We can now give a characterization of the absolutely convex pseudoprime ideals that are primary (equivalently, are intersections of primary ideals).

2.10. COROLLARY. *Let I be an absolutely convex pseudoprime ideal of C. I is primary if and only if either $I = I : I^{1/2}$ or $I = I \cdot I^{1/2}$.*

Proof: Suppose that I is primary. If I is prime, then $I = I^{1/2}$ and $I = I^2$, so $I = I \cdot I^{1/2}$. If I is not prime, then

$$I \cdot I^{1/2} < I < I : I^{1/2}$$

is impossible by Corollary 2.9. Conversely, suppose that either $I = I : I^{1/2}$ or $I = I \cdot I^{1/2}$. If I is prime, then I is primary. If I is not prime, then I is primary by Corollary 2.9.

We have seen that for any absolutely convex ideal I of C the ideals $I \cdot I^{1/2}$, $I^\#$, and $I : I^{1/2}$ are always intersections of primary ideals. Thus each of the conditions $I = I \cdot I^{1/2}$, $I = I^\#$, and $I = I : I^{1/2}$ is a sufficient condition for an absolutely convex ideal I to be an intersection of primary ideals. An obvious question arises: if I is an intersection of primary ideals, does I necessarily satisfy one of these conditions? If I is pseudoprime, then by Corollary 2.10, the answer is "yes". We show now that the answer in general, however, is "no".

2.11. *Example.* Let $N^* = N \cup \{\omega\}$ be the one point compactification of the discrete space N of positive integers. We construct an ideal I of $C(N^*)$ such that I is an intersection of primary ideals, but

$$I \cdot I^{1/2} < I < I^\# \subset I : I^{1/2}.$$

In our construction we will make use of *free ultrafilters* on N: an *ultrafilter* is a maximal filter, and an ultrafilter $\mathscr{U}$ is called *free* if for every $n \in N$ there exists $U \in \mathscr{U}$ such that $n \notin U$. Every infinite subset of N belongs to 2^c free ultrafilters on N [**1**, Section 9.3].

Let $\mathscr{U}_P$ and $\mathscr{U}_Q$ be free ultrafilters on N such that $\{n \in N: n \text{ odd}\} \in \mathscr{U}_P$ and $\{n \in N: n \text{ even}\} \in \mathscr{U}_Q$. Set

$$P = \{f \in C(N^*): Z(f) - \{\omega\} \in \mathscr{U}_P\},$$

and

$$Q = \{f \in C(N^*): Z(f) - \{\omega\} \in \mathscr{U}_Q\}.$$

By [**1**, 14G.3] P and Q are non-maximal prime ideals contained in M_ω. Let j be the function $j(n) = 1/n$ for $n \in N$, $j(\omega) = 0$, and set $I = P|_j \cap Q|^j$. By Theorem 2.8, $(P|_j)^\# = P|^j$, $(Q|^j)^\# = Q|^j$, $(P|_j) \cdot (P|_j)^{1/2} = P|_j$, and $(Q|^j) \cdot (Q|^j)^{1/2} = Q|_j$. Using these facts, the definition of $I^\#$, and Theorem 2.4(a), a straightforward computation shows that $I^\# = P|^j \cap Q|^j$ and $I \cdot I^{1/2} = P|_j \cap Q|_j$. Now I is an intersection of primary ideals, but we have $j \in I^\# - I$. And if f is defined by $f(n) = 1/n$ for n even, $f(n) = 0$ for n odd, and $f(\omega) = 0$, then $f \in P \cap Q|^j \subset I$; but $f \notin Q|_j$, so $f \notin I \cdot I^{1/2}$.

The next theorem, which gives a description of the square of an absolutely convex ideal, will be useful in obtaining a necessary condition for an absolutely convex ideal to be an intersection of primary ideals.

2.12. THEOREM. *Let I be an absolutely convex ideal of C. Then*

$$I^2 = \{f \in C: |f|^{1/2} \in I\}.$$

Proof. Let $f \in I^2$. Then

$$f = \sum_{i=1}^{n} g_i h_i$$

for some g_i, $h_i \in I$ and some $n \in N$. Note that since g_i, $h_i \in I$, $|g_i| \vee |h_i| \in I$ for all i. We have

$$\begin{aligned} |f|^{1/2} &\leqq \Big(\sum_{i=1}^{n} |g_i|\,|h_i|\Big)^{1/2} \\ &\leqq \sum_{i=1}^{n} |g_i|^{1/2}|h_i|^{1/2} \\ &\leqq \sum_{i=1}^{n} (|g_i| \vee |h_i|)^{1/2} \cdot (|g_i| \vee |h_i|)^{1/2} \\ &= \sum_{i=1}^{n} |g_i| \vee |h_i| \in I. \end{aligned}$$

Therefore, $|f|^{1/2} \in I$. Hence $I^2 \subset \{f \in C: |f|^{1/2} \in I\}$.

Conversely, let $|f|^{1/2} \in I$. Then $|f| = |f|^{1/2}|f|^{1/2} \in I^2$, so it is sufficient to show that I^2 is convex. Suppose that $0 \leqq g \leqq h$, with $h \in I^2$. By the first part of the proof $h^{1/2} = |h|^{1/2} \in I$. Therefore since $0 \leqq g^{1/2} \leqq h^{1/2}$, $g^{1/2} \in I$, and so $g = g^{1/2}g^{1/2} \in I^2$.

2.13. COROLLARY. *If I is an absolutely convex ideal of C, then I^2 is an absolutely convex ideal of C.*

2.14. COROLLARY. *If*

$$I = \bigcap_{\alpha \in A} I_\alpha,$$

where the I_α are absolutely convex ideals of C, then

$$I^2 = \bigcap_{\alpha \in A} I_\alpha^{\,2}.$$

Now we give a necessary condition for an absolutely convex ideal to be an intersection of primary ideals. This theorem and Theorem 2.17 are examples of how Theorem 2.8 can be used to obtain information about the ideal theory of C.

2.15. THEOREM. *If an absolutely convex ideal I of C is an intersection of primary ideals, then $I^2 = I \cdot (I : I^{1/2})$.*

Proof. First we show that if P is a prime ideal of C and a is a positive non-unit of C/P, then

$$(*) \qquad P|^a \cdot P|_a = (P|_a)^2.$$

Since $P|_a \subset P|^a$, $(P|_a)^2 \subset P|^a \cdot P|_a$. For the reverse inclusion it is sufficient to show that if $b \in P|^a$ and $c \in P|_a$, then $bc \in (P|_a)^2$. Now $|b| \leqq a^{1-1/n}$ for all $n \in N$, and $|c| \leqq a^{1+1/m}$ for some $m \in N$, so $|bc| \leqq a^{1-1/(m+1)}a^{1+1/m} = a^{2+1/m(m+1)}$. Hence $|bc|^{1/2} \leqq a^{1+1/2m(m+1)}$, so $|bc|^{1/2} \in P|_a$, and we have $bc \in (P|_a)^2$.

To prove the theorem it is clearly sufficient to show that

$$I \cdot (I : I^{1/2}) \subset I^2.$$

Using the notation of Theorem 2.8 we have

$$I = I^* = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|_{f_\gamma}),$$

and

$$I : I^{1/2} = (\bigcap_{\beta \in \mathrm{B}} P_\beta|^{f_\beta}) \cap (\bigcap_{\gamma \in \Gamma} P_\gamma|^{f_\gamma}).$$

By Corollary 2.14

$$I^2 = (\bigcap_{\alpha \in \mathrm{A}} Q_\alpha) \cap (\bigcap_{\beta \in \mathrm{B}} (P_\beta|^{f_\beta})^2) \cap (\bigcap_{\gamma \in \Gamma} (P_\gamma|_{f_\gamma})^2).$$

The inclusion $I \cdot (I : I^{1/2}) \subset I^2$ follows from these expressions and (*) above.

We note that since $I^* \subset I^\# \subset I : I^{1/2}$ for any absolutely convex ideal I by Theorem 2.8, it follows from Theorem 2.15 that if I is an intersection of primary ideals, then

$$I^2 = I \cdot (I : I^{1/2}) = I \cdot I^\# = I \cdot I^*.$$

2.16. *Definition.* Let $\{J_\alpha : \alpha \in \mathrm{A}\}$ be a collection of ideals of a ring. We shall say that the intersection of the collection $\{J_\alpha : \alpha \in A\}$ is *irredundant* if for all $\beta \in \mathrm{A}$ we have

$$\bigcap_{\alpha \in \mathrm{A},\, \alpha \neq \beta} J_\alpha \not\subset J_\beta.$$

If I is an absolutely convex ideal of C such that the intersection of all the minimal primary ideals of I is irredundant, then the condition $I^2 = I \cdot (I : I^{1/2})$ of Theorem 2.15 turns out to be a sufficient condition for I to be an intersection of primary ideals. In fact, we have:

2.17. THEOREM. *Let I be an absolutely convex ideal of C such that the intersection of all the minimal primary ideals of I is irredundant. Then the following are equivalent:*

(a) $I = I^*$.
(b) $I^2 = I \cdot (I : I^{1/2})$.
(c) $I^2 = I \cdot I^\#$.
(d) $I^2 = I \cdot I^*$.

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2.18. Lemma. *If I is an absolutely convex ideal of C such that $P|_f < I < P|^f$, then $I^2 < I \cdot P|^f$.*

Proof. We may assume that $f \notin I$ and $0 \leqq f \leqq 1$: for, otherwise choose any $t \in P|^f - I$ such that $0 \leqq t \leqq 1$; then $P|_f = P|_t$ and $P|^f = P|^t$, so we may replace f by t. Now take $h \in I - P|_f$ such that $0 \leqq h \leqq 1$. Since $f \notin P|_f$, $h \wedge f \notin P|_f$, so by replacing h by $h \wedge f$, we may assume that $0 \leqq h \leqq f \leqq 1$. And since $f^{3/2} \in P|_f, f^{3/2} \in I$; so by replacing h by $h \vee f^{3/2}$, we may assume that $0 \leqq f^{3/2} \leqq h \leqq f \leqq 1$. We define a function g as follows:

$$g(x) = f^2(x)/h(x) \qquad \text{if } x \notin Z(h);$$
$$g(x) = 0 \qquad \text{if } x \in Z(h).$$

For $x \notin Z(h), 0 \leqq g(x) \leqq f^{1/2}(x)$, so since $Z(g) = Z(h) = Z(f)$, g is continuous. Now $f^2 = hg$, so since $f \notin I$, $hg \notin I^2$ by Theorem 2.12. But for all $n \in N$, $P(f)^2 = P(g)P(h) > P(g)P(f)^{1+1/n}$, so $P(f)^{1-1/n} > P(g)$ for all $n \in N$. Therefore $g \in P|^f$, and we have $hg \in I \cdot P|^f - I^2$.

Proof of Theorem 2.17. By Theorems 2.15 and 2.8 the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) hold for any absolutely convex ideal of C. To complete the cycle, suppose that $I \neq I^*$; we will show that $I^2 \neq I \cdot I^*$. Let $f \in I^* - I$. Let J be an absolutely convex pseudoprime ideal that contains I but not f, and let P be a prime ideal contained in J. If M is the maximal ideal that contains P, then $f \in M$ since f is in every primary ideal containing I. Since $f \in P|^f - J$ and the convex ideals containing P form a chain, $J < P|^f$, and so $I + P < P|^f$. Also, $I \not\subset P|_f$ (since f is in every primary ideal containing I), so $I + P \not\subset P|_f$. Since the sum of two absolutely convex ideals of C is absolutely convex [**2**, Proposition 2.6], $I + P$ is absolutely convex, so we must have $P|_f < I + P$. By Lemma 2.18 there exists $g \in P|^f$, $i \in I$, and $p \in P$ such that

$$g(i + p) \notin (I + P)^2 = I^2 + P.$$

Now $P|^f$ is a minimal primary ideal of I since $I \not\subset P|_f$. If $P|^f$ is the only minimal primary ideal of I, then $g \in I^* = P|^f$, so that $gi \in I^* \cdot I - I^2$, and we are done. If $P|^f$ is not the only minimal primary ideal of I, let K be the intersection of all the other minimal primary ideals of I. By hypothesis there exists $h \in K - P|^f$. Since $h \notin P|^f$, $|P(g)| < |P(h)|$. Now $(K + P)/P$ is absolutely convex, so since $P(h) \in (K + P)/P$, we have $P(g) \in (K + P)/P$. Hence $g = k + p'$ for some $k \in K$ and $p' \in P$. Now $k + p' \in P|^f$ and $p' \in P \subset P|^f$, so $k \in P|^f$, and therefore $k \in K \cap P|^f = I^*$. Finally, since

$$(k + p')(i + p) = g(i + p) \notin I^2 + P,$$

$ki \notin I^2$, and we have

$$ki \in I^* \cdot I - I^2.$$

2.19. Corollary. *Let I be an absolutely convex ideal of C such that I^* is a finite intersection of primary ideals. Then the following are equivalent:*

(a) $I = I^*$.
(b) $I^2 = I \cdot (I : I^{1/2})$.
(c) $I^2 = I \cdot I^\#$.
(d) $I^2 = I \cdot I^*$.

Proof. Clearly we can write I^* as a finite irredundant intersection of minimal primary ideals of I, $I^* = J_1 \cap \ldots \cap J_n$. By Theorem 2.17 it is sufficient to show that every minimal primary ideal of I appears in this expression. Let J be a minimal primary ideal of I, and let $P = J^{1/2}$. Since $I^* \subset P$ and P is prime, we must have $J_i \subset P$ for some i. If $J_i = P$, then $J \subset J_i$, so $J = J_i$, and we are done. If $J = P$, then $J_i \subset J$, so $J_i = J$, and we are done. We assume therefore that $J_i < P$ and $J < P$. Let Q be a prime ideal contained in J. Then $P = Q^f$ and $Q_f \subset J$ for some f. Now let Q' be a prime ideal contained in J_i. Since $J_i < Q^f$, $Q' < Q^f$, and so $Q' \subset Q_f$. Therefore Q' is contained in J also. Since the primary ideals containing a given prime ideal form a chain, either $J \subset J_i$ or $J_i \subset J$, and in either case $J = J_i$.

We have characterized the absolutely convex pseudoprime ideals of C that are primary in Corollary 2.10. In view of Corollary 2.19 we can now say considerably more about such ideals. We have:

2.20. COROLLARY. *Let I be an absolutely convex pseudoprime ideal of C. Then the following are equivalent:*
(a) *I is primary.*
(b) $I^2 = I \cdot (I : I^{1/2})$.
(c) $I^2 = I \cdot I^\#$.
(d) $I^2 = I \cdot I^*$.

We have seen that the conditions $I^2 = I \cdot (I : I^{1/2})$, $I^2 = I \cdot I^\#$, and $I^2 = I \cdot I^*$ are necessary conditions for an absolutely convex ideal I to be an intersection of primary ideals. And if the intersection of all the minimal primary ideals of I is irredundant, each of these conditions is a sufficient condition as well. It may seem reasonable to conjecture that at least one of these conditions is a sufficient condition for an arbitrary absolutely convex ideal to be an intersection of primary ideals. Such a conjecture would be false, however; in the following example we show that even the strongest of these conditions, $I^2 = I \cdot (I : I^{1/2})$, does not in general imply that I is an intersection of primary ideals.

2.21. *Example*. Let $N^* = N \cup \{\omega\}$ be the one point compactification of the space N of positive integers. We will construct an absolutely convex ideal I of $C(N^*)$ such that $I^2 = I \cdot (I : I^{1/2})$, but $I \neq I^*$. In this construction we will use the points of $\beta N - N$, where βN is the Stone-Čech compactification of N, to index the set of all free ultrafilters on N: if $\alpha \in \beta N - N$, the unique free ultrafilter on N that converges to α in the topology of βN will be denoted by $\mathscr{U}_\alpha$, i.e.,

$$\mathscr{U}_\alpha = \{S \subset N \colon \alpha \in \mathrm{cl}_{\beta N} S\}.$$

For each free ultrafilter $\mathscr{U}_\alpha$ on N, P_α will denote the set

$$P_\alpha = \{f \in C(N^*): Z(f) - \{\omega\} \in \mathscr{U}_\alpha\}.$$

By [**1**, 14G.3] each P_α is a non-maximal prime z-ideal of $C(N^*)$ contained in M_ω, and every non-maximal prime z-ideal contained in M_ω is of the form P_α for some unique $\alpha \in \beta N - N$.

Now we construct our ideal I. We define a sequence, U_n, of subsets of N as follows:

$$U_1 = N,$$
$$U_2 = \{n \in N: n \text{ is odd}\},$$
$$U_3 = \{1, 5, 9, \ldots\},$$

and in general,

$$U_k = \{2^{k-1}(n-1) + 1: n \in N\}.$$

Since the intersection of a strictly decreasing sequence of open-and-closed subsets of $\beta N - N$ is never open in $\beta N - N$ [**1**, 6W.3], there is a point $\gamma \in \beta N - N$ such that

$$\gamma \in \cap_n \operatorname{cl}_{\beta N} U_n - \operatorname{int}_{\beta N - N}[\cap_n \operatorname{cl}_{\beta N} U_n].$$

Let

$$\Lambda = (\beta N - N) \cap [\cup_n \operatorname{cl}_{\beta N}(U_n - U_{n+1})].$$

Let j be the function $j(n) = 1/n$ for $n \in N$, $j(\omega) = 0$. For each positive integer k we define a function f_k by

$$f_k | U_n - U_{n+1} = j^{1+1/nk}, f_k(1) = 1, \text{ and } f_k(\omega) = 0.$$

We will denote by L the smallest absolutely convex ideal of $C(N^*)$ that contains the set $\{f_k: k \in N\}$. It is easy to verify that $f \in L$ if and only if $|f| \leqq Mf_k$ for some positive constant M and some k. Finally, we set

$$I = (P_\gamma + L) \cap (\bigcap_{\lambda \in \Lambda} P_\lambda|^j).$$

It is clear that I is absolutely convex. We show first that $j \notin I$. Assume, on the contrary, that $j \in I$. Then $j \in P_\gamma + L$, so there exists $U \in \mathscr{U}_\gamma$, a positive constant M, and a positive integer k such that $j(x) \leqq Mf_k(x)$ for all $x \in U$. Since $U \in \mathscr{U}_\gamma$, $\operatorname{cl}_{\beta N} U$ is a βN neighborhood of γ. Since $\gamma \notin \operatorname{int}_{\beta N - N}[\cap_n \operatorname{cl}_{\beta N} U_n]$, $\operatorname{cl}_{\beta N} U \cap (\beta N - N) \not\subset \cap_n \operatorname{cl}_{\beta N} U_n$. Therefore $\operatorname{cl}_{\beta N} U \cap (\beta N - N) \not\subset \operatorname{cl}_{\beta N} U_m$ for some m, and it follows that $U - U_m$ is infinite. For all $x \in U - U_m$, $j(x) \leqq Mf_k(x) < Mj^{1+1/mk}(x)$, so $j^{-(1/mk)}(x) < M$ for all $x \in U - U_m$. Since $U - U_m$ is infinite, this is clearly impossible. Hence $j \notin I$.

We next show that $j \in I^*$. Let Q be a primary ideal that contains I. Note that $Q \subset M_\omega$: for, since $f_1 \in I$ and $Z(f_1) = \{\omega\}$, M_ω is the only maximal ideal that contains I. Let P be a minimal prime ideal contained in Q. By [**1**, 14G.4] $P = P_\alpha$ for some $\alpha \in \beta N - N$. If $j \notin Q$, then $Q \subset P_\alpha|_j$, and so $I \subset P_\alpha|_j$. We show that this is impossible. There are two possibilities for the point α: either $\alpha \in \cap_n \operatorname{cl}_{\beta N} U_n$, or $\alpha \notin \bigcap_n \mathrm{cl}_{\beta N} U_n$. If $\alpha \in \bigcap_n \mathrm{cl}_{\beta N} U_n$, then $f_1 \in I - P_\alpha|_j$, so $I \not\subset P_\alpha|_j$. If $\alpha \notin \bigcap_n \mathrm{cl}_{\beta N} U_n$, then $\alpha \notin \mathrm{cl}_{\beta N} U_m$ for some m. Define a function g as follows: $g|N^* - U_m = j$, $g|U_m = f_1$. Since $0 \leqq g \leqq j$,

$$g \in \bigcap_{\gamma \in \Lambda} P_\lambda|^j.$$

And since $\gamma \in \mathrm{cl}_{\beta N} U_m$, $g \in P_\gamma + L$, so $g \in I$. But since $\alpha \notin \mathrm{cl}_{\beta N} U_m$, $\alpha \in \mathrm{cl}_{\beta N}(N - U_m)$, and so $P_\alpha(g) = P_\alpha(j)$. Hence $g \notin P_\alpha|_j$, and we have $I \not\subset P_\alpha|_j$. Since $I \subset P_\alpha|_j$ is impossible in either case, $j \in Q$. Since Q is arbitrary, $j \in I^*$. We have shown that $I < I^*$.

It remains for us to verify that

$$I^2 = I \cdot (I : I^{1/2}).$$

Set

$$J = \bigcap_{\lambda \in \Lambda} P_\lambda|^j.$$

We showed above that for all $\alpha \in \beta N - N$, $I \not\subset P_\alpha|_j$. Thus the ideals $P_\lambda|^j$, $\lambda \in \Lambda$, are minimal primary ideals of I. By Theorem 2.8, $I : I^{1/2} \subset J$, so it is sufficient for us to show that $J \cdot I \subset I^2$. Now

$$I^2 = (P_\gamma + L)^2 \cap \left(\bigcap_{\lambda \in \Lambda} (P_\lambda|^j)^2\right)$$

by Corollary 2.14. Clearly,

$$J \cdot I \subset \bigcap_{\lambda \in \Lambda} (P_\lambda|^j)^2,$$

so we need only to verify that

$$J \cdot I \subset (P_\gamma + L)^2.$$

Let $g \in J$ and $h \in I$ such that $g \geqq 0$ and $h \geqq 0$. (It is sufficient to deal with non-negative functions since all the ideals involved are absolutely convex.) Since $h \in P_\gamma + L$, there exists $U \in \mathscr{U}_\gamma$, a positive constant M, and a positive integer k such that $h(x) \leqq Mf_k(x)$ for all $x \in U$. For each n set

$$G_n = \{x \in U_n - U_{n+1}\colon g(x) \geqq j^{1-1/2nk}(x)\}.$$

Since

$$g \in \bigcap_{\lambda \in \Lambda} P_\lambda|^j,$$

G_n is finite for all n. Let $V = N - \bigcup_n G_n$. Since G_n is finite for all n,

$$\mathrm{cl}_{\beta N}(\bigcup_n G_n) \cap (\beta N - N) \subset \bigcap_n \mathrm{cl}_{\beta N} U_n.$$

Therefore since $\gamma \notin \mathrm{int}_{\beta N - N}[\bigcap_n \mathrm{cl}_{\beta N} U_n]$, $\gamma \notin \mathrm{cl}_{\beta N}(\bigcup_n G_n)$, so we must have $\gamma \in \mathrm{cl}_{\beta N} V$. Hence

$$\gamma \in \mathrm{cl}_{\beta N} U \cap \mathrm{cl}_{\beta N} V = \mathrm{cl}_{\beta N}(U \cap V).$$

For all $x \in U \cap V \cap (U_n - U_{n+1})$ we have

$$gh(x) \leqq j^{1-1/2nk}(x) M j^{1+1/nk}(x) = M j^{2+1/2nk}(x).$$

Thus

$$(gh)^{1/2}(x) \leqq M^{1/2} j^{1+1/4nk}(x)$$

for all $x \in U \cap V \cap (U_n - U_{n+1})$, so

$$(gh)^{1/2}(x) \leqq M^{1/2} f_{4k}(x)$$

for all $x \in U \cap V - \{1\}$. Since $U \cap V - \{1\} \in \mathscr{U}_\gamma$,

$$(gh)^{1/2} \in P_\gamma + L,$$

and therefore $gh \in (P_\gamma + L)^2$.

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