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ON A COMPARISON BETWEEN DWORK AND RIGID COHOMOLOGIES OF PROJECTIVE COMPLEMENTS

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Abstract. For homogeneous polynomials $G_1,\ldots,G_k$ over a finite field, their Dwork complex is defined by Adolphson and Sperber, based on Dwork's theory. In this article, we will construct an explicit cochain map from the Dwork complex of $G_1,\ldots,G_k$ to the Monsky–Washnitzer complex associated with some affine bundle over the complement $\mathbb{P}^n \setminus X_G$ of the common zero X_G of $G_1,\ldots,G_k$, which computes the rigid cohomology of $\mathbb{P}^n \setminus X_G$. We verify that this cochain map realizes the rigid cohomology of $\mathbb{P}^n \setminus X_G$ as a direct summand of the Dwork cohomology of $G_1,\ldots,G_k$. We also verify that the comparison map is compatible with the Frobenius and the Dwork operator defined on both complexes, respectively. Consequently, we extend Katz's comparison results in [19] for projective hypersurface complements to arbitrary projective complements.

§1. Introduction

Let X be an algebraic variety over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. The zeta function of X is defined to be the following exponential sum:

$$Z(X/\mathbb{F}_q,t) := \exp\left(\sum_{s\geq 0} \frac{N_s}{s} t^s\right),$$

where N_s is the number of $\mathbb{F}_{q^s}$-rational points of X. This function is known to be a rational function in t with coefficients in $\mathbb{Z}$ by Dwork [13]. For a projective hypersurface X, Dwork expressed the zeta function of X as an alternating product of characteristic polynomials of a suitably chosen representative of a Frobenius action in a series of articles [14]–[17], following his proof of the rationality of zeta functions. Based on Dwork's theory, Adolphson and Sperber developed a cohomology theory and got an estimate for the zeta function when X is a closed subvariety of $\mathbb{A}^r \times \mathbb{G}_m^s$ in [1], and when X is a smooth projective complete intersection in [4], [5].

On the other hand, Monsky and Washnitzer developed rather an intrinsic cohomology theory in [29] when X is a smooth affine variety admitting a nice p-adic lift. Then Monsky proved the Lefschetz fixed-point theorem in [25], [27] to express the zeta function of X as an alternating product of characteristic polynomials of a Frobenius action on Monsky–Washnitzer cohomology. Later, van der Put [32] removed the technical condition on X assumed by Monsky and Washnitzer to make the theory work for every smooth affine variety X over $\mathbb{F}_q$. Berthelot [9] extended this theory to not necessarily affine varieties, and the resulting theory is called rigid cohomology theory.

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Since Dwork cohomology and rigid cohomology compute the same important invariant, one may ask whether there is a connection between the two theories. For smooth hypersurfaces in projective spaces, Katz answered this question in [19]. It is strongly believed that the corresponding comparison results hold for more general cases, but up to the best of author's knowledge, there is no written proof so the author hope that this article provides a detailed proof for general cases with several equations.

Let us briefly explain the contents of [19]. Let $\Bbbk/\mathbb{Q}_p$ be a finite extension with the valuation ring $\mathcal{O}_\Bbbk$. Given a homogeneous polynomial $G \in \mathcal{O}_\Bbbk[x_0,\dots,x_n]$ of degree $d \geq 1$, consider a $\Bbbk$-linear span of some monomials (cf. [19, p. 77]):

$$\mathscr{L}^{0,+} := \left\{ \sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus n+2}} a_{u,v} x^u y^v \in \mathcal{O}_\Bbbk[x,y] \;\middle|\; \begin{array}{c} vd = u_0+\cdots+u_n, \\ a_{u,0}=0 \end{array} \right\}.$$

For a fixed constant $\gamma \in \Bbbk$, there are differential operators on $\mathscr{L}^{0,+}$:

$$D_{x_i} := \exp(-\gamma y G)\circ x_i \frac{\partial}{\partial x_i} \circ \exp(\gamma y G) = x_i \frac{\partial}{\partial x_i} + \gamma y x_i \frac{\partial G}{\partial x_i},$$
$$D_y := \exp(-\gamma y G)\circ y \frac{\partial}{\partial y} \circ \exp(\gamma y G) = y \frac{\partial}{\partial y} + \gamma y G.$$

On the other hand, suppose that the hypersurface $X_G \subseteq \mathbb{P}^n_\Bbbk$ defined by G is smooth. If $H_i \subseteq \mathbb{P}^n_\Bbbk$ is the hyperplane defined by $x_i = 0$ for $i = 0,\dots,n$, and $X_G^\emptyset := X_G \setminus (H_0 \cup \cdots \cup H_n)$, then, by [19, Th. 1.16], there is an exact sequence

$$0 \longrightarrow \left(D_y \mathscr{L}^{0,+} + \sum_{i=0}^n D_{x_i} \mathscr{L}^{0,+} \right) \longrightarrow \mathscr{L}^{0,+} \xrightarrow{\Theta} H^n_{\mathrm{dR}}(X_G^\emptyset) \longrightarrow 0,$$

where the local description of Θ is given in [19, Th. I]. Here, $H^\bullet_{\mathrm{dR}}$ denotes algebraic de Rham cohomology. One way of getting a global description of Θ is using the complement of X_G. Namely, denote $\mathbb{T}^n_\Bbbk := \mathbb{P}^n_\Bbbk \setminus (H_0 \cup \cdots \cup H_n)$ with local coordinates $t_i := x_i/x_0$ for $i = 1,\dots,n$. Then there is a $\Bbbk$-linear map given by (cf. [19, p. 78])

$$\mathscr{R} : \mathscr{L}^{0,+} \longrightarrow H^n_{\mathrm{dR}}(\mathbb{T}^n_\Bbbk \setminus X_G^\emptyset) \qquad x^u y^v \longmapsto \frac{(-1)^{v-1}}{(v-1)!} \frac{x^u}{G^v} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

inducing an exact sequence (cf. [19, p. 79])

$$0 \longrightarrow \sum_{i=0}^n D_{x_i} \mathscr{L}^{0,+} \longrightarrow \mathscr{L}^{0,+} \xrightarrow{\mathscr{R}} H^n_{\mathrm{dR}}(\mathbb{T}^n_\Bbbk \setminus X_G^\emptyset) \longrightarrow 0\,.$$

Here, the map is defined via the inhomogeneous coordinates of $\mathbb{P}^n_\Bbbk \setminus H_0$. One gets a description in the homogeneous coordinates using the relation:

$$\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} = \sum_{i=0}^n (-1)^i \frac{dx_0}{x_0} \wedge \cdots \wedge \widehat{\frac{dx_i}{x_i}} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

To relate $\mathscr{R}$ and Θ, we use the canonical exact sequence:

$$0 \longrightarrow \Omega^\bullet_{\mathbb{P}^n_\Bbbk/\Bbbk} \longrightarrow \Omega^\bullet_{\mathbb{P}^n_\Bbbk/\Bbbk}(\log X_G) \xrightarrow{\mathrm{Res}_G} \Omega^\bullet_{X_G/\Bbbk}[-1] \longrightarrow 0,$$

where Res_G is the residue map uniquely characterized by the property

$$\mathrm{Res}_G\left(\frac{dG}{G}\wedge\omega\right)=\omega.$$

Then, by [19, Th. 1.18], $\mathrm{Res}_G\circ\mathscr{R}=\Theta$. The remaining part of [19] is dedicated to compute representatives of Frobenius actions. To achieve this, we need to develop a p-adic analytic theory. Then Θ and $\mathscr{R}$ extend by continuity, and they are compatible with the Frobenius actions in a suitable sense. Since we discuss the corresponding version of the p-adic analytic theory in this article, we do not explain the remaining part of [19].

Monsky's lecture note [26] gave a more detailed discussion of the algebraic version of the Dwork complex in p-adic setting together with its relations with algebraic de Rham cohomology and Monsky–Washnitzer cohomology. Then the complex algebraic analog of Dwork theory together with the connection of de Rham cohomology has been studied. Adolphson and Sperber dealt with the smooth complete intersections in affine varieties in [3]. Dimca, Maaref, Sabbah, and Saito studied the singular subvarieties embedded in smooth varieties in [12] using the theory of algebraic $\mathcal{D}$-modules. These results were again implemented in the rigid setting by Baldassarri and Berthelot for singular projective hypersurfaces in [7] using the theory of arithmetic $\mathcal{D}$-modules. On the other hand, Bourgeois [11] directly constructed a quasi-isomorphism between the Dwork complex used by Adolphson and Sperber in [1] and the complex of Monsky and Washnitzer in the smooth affine setting.

The goal of this article is to construct an explicit comparison between the Dwork cohomology of given homogeneous polynomials and the rigid cohomology of the complement of their common zero in a projective space, together with Frobenius actions defined on both sides. This generalizes the complement comparison result in [19] described above, but with a different choice of cochain complexes. Note that if the given homogeneous polynomials define a smooth complete intersection, then we can recover the essential information of the rigid cohomology of the common zero. The more detailed exposition will be given in the following two subsections.

As mentioned before, Adolphson and Sperber studied Dwork complexes in various settings, and it seems that the Dwork complex which appears in this article resembles the one in [5]. Our academic contribution is to find a correct version of the p-adic Dwork complex which is appropriate to construct the desired comparison map, and give a systematic treatment of getting a connection between the two theories via the Cayley trick[1] as the author did in [31, 22] to study the period integrals in the complex geometric setting.

1.1 The idea and motivation

One remarkable observation so far is that the comparison becomes more transparent when we consider the complement of the hypersurface X in the ambient projective space $\mathbb{P}^n_{\mathbb{F}_q}$. Moreover, we may extract information of $H^\bullet_{\mathrm{rig}}(X)$ from $H^\bullet_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}\setminus X)$, where $H^\bullet_{\mathrm{rig}}$ denotes rigid cohomology. Indeed, if $X\subseteq Y$ is a codimension k closed embedding of smooth varieties

[1] The Cayley trick gives an isomorphism between the cohomology of the open complement in the projective space and the cohomology of the hypersurface complement in a larger space. For the detail, see §2.

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over $\mathbb{F}_q$, then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^i_{X,\mathrm{rig}}(Y) & \longrightarrow & H^i_{Y,\mathrm{rig}}(Y) & \longrightarrow & H^i_{Y\setminus X,\mathrm{rig}}(Y\setminus X) & \longrightarrow \cdots \\ & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\ \cdots \longrightarrow & H^{i-2k}_{\mathrm{rig}}(X) & \longrightarrow & H^i_{\mathrm{rig}}(Y) & \longrightarrow & H^i_{\mathrm{rig}}(Y\setminus X) & \longrightarrow \cdots \end{array}$$

where the top row is a special case of the excision exact sequence [10, Prop. 2.5], and the isomorphisms in the columns come from the Gysin isomorphism [23, §9.3]. Therefore, if $X_{\overline{G}} \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ is a smooth projective complete intersection given by homogeneous polynomials $\overline{G}_1,\ldots,\overline{G}_k \in \mathbb{F}_q[x_0,\ldots,x_n]$, then there is a long exact sequence, called the Gysin exact sequence:

$$\begin{array}{ccccc} \cdots \longrightarrow H^i_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}) \longrightarrow & H^i_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}\setminus X_{\overline{G}}) & & & \\ & \downarrow & & & \\ & H^{i-2k+1}_{\mathrm{rig}}(X_{\overline{G}}) & \longrightarrow & H^{i+1}_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}) & \longrightarrow \cdots \end{array} \tag{1.1}$$

which is a rigid cohomology analog of the excision exact sequence of algebraic de Rham cohomology. As in the case of algebraic de Rham cohomology, this sequence induces an isomorphism

$$H^{n+k-1}_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}\setminus X_{\overline{G}}) \xrightarrow{\sim} H^{n-k}_{\mathrm{prim}}(X_{\overline{G}}),$$

where $H^{n-k}_{\mathrm{prim}}(X_{\overline{G}})$ is the primitive part of $H^{n-k}_{\mathrm{rig}}(X_{\overline{G}})$. Using the interpretation of the zeta function as the characteristic polynomial of the Frobenius action on the cohomology (see, e.g., [18]), one can deduce that the zeta function of $X_{\overline{G}}$ can be written as

$$Z(X_{\overline{G}}/\mathbb{F}_q,t) = \frac{P(t)^{(-1)^{n-k-1}}}{(1-t)(1-qt)\ldots(1-q^{n-k}t)}$$

and $P(t)$ is completely determined by the Frobenius action on the primitive part. Hence, the computation of the cohomology of the projective complement has its own importance. Once we decide to focus on the cohomology of the complement, we may forget about the regularity of $X_{\overline{G}} \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ because $\mathbb{P}^n_{\mathbb{F}_q}\setminus X_{\overline{G}}$ is always smooth, being an open subset of the smooth space $\mathbb{P}^n_{\mathbb{F}_q}$.

On the other hand, the Dwork complex can be defined for any homogeneous polynomials $\overline{G}_1,\ldots,\overline{G}_k \in \mathbb{F}_q[x_0,\ldots,x_n]$, regardless of the regularity of their common zero $X_{\overline{G}} \subseteq \mathbb{P}^n_{\mathbb{F}_q}$. Namely, taking the Teichmüller lifts of the coefficients of each $\overline{G}_i$, we get homogeneous polynomials G_i defined over some finite extension $\Bbbk/\mathbb{Q}_p$ with $\deg G_i = \deg \overline{G}_i$ such that the reduction of each G_i becomes $\overline{G}_i$. Then, we define the Dwork complex associated with $\overline{G}_1,\ldots,\overline{G}_k$ to be the twisted de Rham complex of the form

$$\left(\Bbbk\{x,\hbar y\} \otimes_{\Bbbk[x,y]} \Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, \hbar d + \hbar d(\gamma y_1 G_1 + \cdots + \gamma y_k G_k)\wedge -\right), \tag{1.2}$$

where $\Bbbk\{x,\hbar y\}$ is the Tate algebra over $\Bbbk$ (see Definition 4.8), and $\hbar,\gamma \in \Bbbk^\times$ are some parameters.

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Although the Dwork complex is defined for homogeneous polynomials, its cohomology would depend only on their common zero locus. For example, when we are working with one homogeneous polynomial $\overline{G} \in \mathbb{F}_q[x_0,\dots,x_n]$, there are comparison theorems between the Dwork cohomology of $\overline{G}$ and the rigid cohomology of $\mathbb{P}^n_{\mathbb{F}_q} \setminus X_{\overline{G}}$. In the existing results, [19] and [7], they remove the hyperplane divisors in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by $x_0,\dots,x_n$ to get an affine open subset, where one can write down a comparison map, and then use the log de Rham complex to recover the original situation. Consequently, their Dwork complexes are exactly the ones defined by Adolphson and Sperber in [2, §2].

Instead of removing hyperplane divisors in $\mathbb{P}^n_{\mathbb{F}_q}$, we use the Cayley trick to convert the computation involving polynomials to the computation involving a hypersurface contained in a larger space. With the above notation, the hypersurface is cut out by $y_1G_1+\cdots+y_kG_k$ in a projective bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n_{\Bbbk}$ for a suitably chosen locally free $\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}$-module $\mathcal{E}$ of finite rank, where $y_1,\dots,y_k$ play the role of fiber coordinates. Consequently, we get the Dwork complex as in (1.2) which resembles Adolphson and Sperber's Dwork complex defined in [5, §2]. The difference of our Dwork complex and the one in [5] comes from the different choice of Dwork's splitting functions (for a definition, see §5), which causes the different choice of the lift of $y_1\overline{G}_1+\cdots+y_k\overline{G}_k \in \mathbb{F}_q[x_0,\dots,x_n]$ over the p-adic field. Since the lift of Adolphson and Sperber, denoted by H in [5, eq. (2.10)], is a power series in $y_1,\dots,y_k$, it does not define a hyperplane in $\mathbb{P}(\mathcal{E})$. Hence, we cannot get the desired geometric object. However, our lift $y_1G_1+\cdots+y_kG_k$ is linear in $y_1,\dots,y_k$ so it indeed define a hypersurface in $\mathbb{P}(\mathcal{E})$. Although the two Dwork complexes are different, their reductions on the finite field are exactly the same so one may expect that both Dwork complexes have the same cohomology. This is true when $\overline{G}_1,\dots,\overline{G}_k$ defines a smooth projective complete intersection in $\mathbb{P}^n_{\mathbb{F}_q}$ (see Remark 4.13). Hence, the two Dwork complexes may be regarded as equivalent at least for this case.

1.2 The main results

Let $\Bbbk/\mathbb{Q}_p$ be a finite extension with the valuation ring $(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)$. Denote val_p the p-adic valuation such that $\mathrm{val}_p(p)=1$. Given homogeneous polynomials $G_1,\dots,G_k \in \mathcal{O}_\Bbbk[x_0,\dots,x_n]$ of positive degrees $d_1,\dots,d_k$ not divisible by the uniformizer of $\mathcal{O}_\Bbbk$, we introduce formal variables $y_1,\dots,y_k$ corresponding to $G_1,\dots,G_k$ so that the polynomial

$$S(x,y) := y_1G_1+\cdots+y_kG_k \in \mathcal{O}_\Bbbk[x,y]$$

defines a hypersurface in an affine space. Consider the twisted de Rham complex

$$\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S} := \hbar d + \hbar d(\gamma S)\right),$$

where $\hbar,\gamma \in \mathcal{O}_\Bbbk$ are regarded as parameters. Introduce gradings

$$\begin{cases} \deg_c x_i := 1 & i=0,\dots,n, \\ \deg_c y_j := -d_j & j=1,\dots,k, \\ \deg_c dx_i := 1 & i=0,\dots,n, \\ \deg_c dy_j := -d_j & j=1,\dots,k, \end{cases} \qquad \begin{cases} \deg_w x_i := 0 & i=0,\dots,n, \\ \deg_w y_j := 1 & j=1,\dots,k, \\ \deg_w dx_i := 0 & i=0,\dots,n, \\ \deg_w dy_j := 1 & j=1,\dots,k, \end{cases} \tag{1.3}$$

so that S and dS become homogeneous of bidegree $(\deg_c,\deg_w)=(0,1)$ and the twisted de Rham complex is graded with respect to $\deg_c$. Then the Dwork complex associated with $G_1,\dots,G_k$ will be defined by

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$$(\Omega^\bullet_\hbar, D_{\hbar,\gamma S}) := \left(\Bbbk\{x, \hbar y\} \otimes_{\Bbbk[x,y]} \Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right),$$

where $\Bbbk\{x, \hbar y\}$ will be a version of the Tate algebra (see Remark 4.9).

On the other hand, denote $C_S^\dagger := \mathcal{O}_\Bbbk\{x, y, S^{-1}\}^\dagger$ the weak completion (see Definition 4.1 or [29, Th. 2.3]) of $\mathcal{O}_\Bbbk[x, y, S^{-1}]$, and

$$\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} := \frac{\Omega^\bullet_{C_S^\dagger/\mathcal{O}_\Bbbk}}{\bigcap_{i\geq 0} \mathfrak{m}_\Bbbk^{i+1}\Omega^\bullet_{C_S^\dagger/\mathcal{O}_\Bbbk}}$$

the module of $\mathfrak{m}_\Bbbk$-separated differentials. Then the above gradings extend to

$$\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right),$$

where d is the de Rham differential. With the valuation conditions on γ and $\hbar$ for the convergence (see Theorem 4.11), we have the following comparison map.

THEOREM 1.1. *If* $\mathrm{val}_p\gamma \leq \dfrac{1}{p-1}$ *and* $\mathrm{val}_p\hbar > 0$, *then there is a cochain map*

$$\rho_S : (\Omega^\bullet_\hbar, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} \longrightarrow \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right)_{(\deg_c=0,\deg_w=0)}$$

defined by the formula

$$\rho_S\left(x^u y^v dx_\alpha \wedge dy_\beta\right) := (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)! \frac{x^u y^v}{\gamma^{|v|}S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}S^{|\beta|}}$$

together with the $\Bbbk$*-linearity. Here,* u, v, α, β *are multi-indexes with*

$$\begin{cases} x^u := x_0^{u_0}\cdots x_n^{u_n}, & |u| := u_0 + \cdots + u_n, \\ y^v := y_1^{v_1}\cdots y_k^{v_k}, & |v| := v_1 + \cdots + v_k, \\ dx_\alpha := dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_i}, & |\alpha| := i \\ dy_\beta := dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_j}, & |\beta| := j. \end{cases} \tag{1.4}$$

We will see later that the inclusion

$$\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right)_{(\deg_c=0,\deg_w=0)} \subseteq \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right)$$

is a quasi-isomorphism, and the inclusion

$$(\Omega^\bullet_\hbar, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} \subseteq (\Omega^\bullet_\hbar, D_{\hbar,\gamma S})$$

induces a surjection of cohomology spaces with one-dimensional kernel generated by the class $[dS]$. By Definition 4.6,

$$H^i\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \cong H^i_{\mathrm{MW}}\left(\mathbb{F}_q[x.y, \overline{S}^{-1}]/\Bbbk\right),$$

where $\mathbb{F}_q$ is the residue field of $\mathcal{O}_\Bbbk$, and $\overline{S} = y_1\overline{G}_1 + \cdots + y_k\overline{G}_k \in \mathbb{F}_q[x, y]$ is the reduction of S. Since Monsky–Washnitzer cohomology is canonically isomorphic to rigid cohomology for smooth affine schemes, the ρ_S in Theorem 1.1 is a comparison map from Dwork cohomology to rigid cohomology.

On the other hand, if $X_G \subseteq \mathbb{P}^n_{\Bbbk}$ is the common zero of $G_1,\ldots,G_k$, then we will see in §2 that there is a canonical map

$$\mathrm{Spec}\ \Bbbk[x,y,S^{-1}]_{(\deg_c=0,\deg_w=0)} \longrightarrow \mathbb{P}^n_{\Bbbk} \setminus X_G$$

inducing a quasi-isomorphism on rigid cohomology spaces. Moreover, by Corollary 4.7, the Monsky–Washnitzer cohomology associated with the bidegree (0,0)-subalgebra above is computed via the complex of $\mathfrak{m}_{\Bbbk}$-adically separated forms of $C_S^{\dagger}$. The corresponding statement for algebraic de Rham cohomology is Proposition 2.4. This is a direct generalization of [26, Th. 9.2] which covers the case of projective hypersurface complement. With the notations so far, we can say more about the comparison map ρ_S.

THEOREM 1.2. *ρ_S induces an isomorphism*

$$H^i(\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S}) \cong H^{i-1}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{G}}) \oplus H^{i-2}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{G}})$$

for every $i \geq 2$. On the other hand,

$$H^0(\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S}) = 0, \quad H^1(\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S}) = 0.$$

The q-power map induces an endomorphism, called the Frobenius endomorphism on $\mathbb{F}_q[x,y]$. This map lifts to endomorphisms

$$\Phi_{q,S} : (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S}) \longrightarrow (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})$$

$$\mathrm{Fr} : \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \longrightarrow \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right)$$

both act on the zero forms by sending x_i and y_j to its qth power x_i^q and y_j^q, respectively. These endomorphisms admit retractions, that is, endomorphisms

$$\Psi_{q,S} : (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S}) \longrightarrow (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})$$

$$\psi : \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \longrightarrow \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right)$$

such that $\Psi_{q,S} \circ \Phi_{q,S}$ and $\psi \circ \mathrm{Fr}$ are the identity maps, respectively. The detailed expositions will be given in §5. Now, we have the following comparison of the endomorphisms above.

THEOREM 1.3. *ρ_S is compatible with the Frobenius and the Dwork operators defined on the source and the target, respectively. More precisely, the diagrams*

$$\begin{array}{ccc}
(\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \\
\downarrow{\scriptstyle q\Phi_{q,S}} & & \downarrow{\scriptstyle \mathrm{Fr}} \\
(\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right)
\end{array}$$

$$\begin{array}{ccc} (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \\ {\scriptstyle q^{-1}\Psi_{q,S}}\downarrow & & \downarrow{\scriptstyle\psi} \\ (\Omega^\bullet_{\hbar}, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)} & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \end{array}$$

are commutative.

REMARK 1.4. In Theorem 1.3, we dropped the subscript $(\deg_c = 0, \deg_w = 0)$ in the target of ρ_S because we are not sure that an arbitrary lift Fr of the q-power map preserves the bidegree $(0,0)$-subcomplex. However, the particular choice such that

$$\mathrm{Fr}: C_S^\dagger \longrightarrow C_S^\dagger \qquad (x_i, y_j) \longmapsto (x_i^q, y_j^q)$$

and the ψ coming from this choice are compatible with the bidegrees so we can recover the subscript $(\deg_c = 0, \deg_w = 0)$ in Theorem 1.3. See §§5.1 and 5.2 for the details. We will see in Theorem 4.5 that any lifts of the q-power map define homotopic cochain maps so we can always make such choices.

We have the following remark concerning formal deformation theory of the Dwork operator, which is not covered in the rest of this article.

REMARK 1.5. Using the twisted de Rham complex in Theorem 1.1, we may directly construct a DGBV (differential Gerstenhaber–Batalin–Vilkovisky) algebra with the isomorphic underlying complex, as the authors of [20] did on the complex geometry setting, and we may develop the formal deformation theory as in [31]. Theorem 1.3 enables us to apply this type of formal deformation theory to the Dwork operator. For the detailed discussion in the DGBV aspects of Dwork theory, see [21].

1.3 Outline of the article

In §2, we explain the Cayley trick. In particular, Proposition 2.4 gives a direct sum decomposition of the algebraic de Rham complex of the affine cone and the corresponding decomposition of the algebraic de Rham cohomology. This identification is used in the rest of the article.

In §3, we explicitly write down a comparison map ρ_S (Definition 3.1) in a corresponding algebraic setting. The comparison for this algebraic ρ_S will be given in Propositions 3.10 and 3.16.

After establishing the algebraic theory, we will define the required p-adic analytic complexes in §4 and give a proof of Theorems 1.1 and 1.2. In §4.1, we recall the basics on Monsky–Washnitzer cohomology. In particular, Proposition 4.7 is the Monsky–Washnitzer version of Proposition 2.4. This gives the target complex of the ρ_S in Theorem 1.1. In §4.2, we recall the basics on Dwork complexes and introduce the source complex of the ρ_S in Theorem 1.1. Now, the main results of §3 yield Theorem 4.11 which is the combination of Theorems 1.1 and 1.2.

Finally, in §5, we will prove Theorem 1.3, following Katz's proof in [19, §III] with some appropriate changes. Namely, Propositions 5.2 and 5.3 together give Theorem 1.3.

The appendix A is an explanation of the computation of algebraic de Rham cohomology via the cosimplicial algebra coming from Čech covering of affine open subsets which is in the proof of Proposition 2.4. Up to the best of author's knowledge, the suitable reference

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for this simplest case is not available in the literature so the appendix is added for this article to be more self-contained.

§2. The Cayley trick

In this section, we give a detailed explanation of the Cayley trick and its consequences. We begin with motivation. Let $\Bbbk$ be a field and $X \subseteq \mathbb{P}^n_\Bbbk$ a smooth projective complete intersection of codimension k. For a "reasonable" cohomology theory $H^\bullet$ defined for quasiprojective schemes over $\Bbbk$, one may obtain the Gysin exact sequence of the following form:

$$\begin{array}{ccccc} \cdots \longrightarrow H^i(\mathbb{P}^n_\Bbbk) \longrightarrow H^i(\mathbb{P}^n_\Bbbk \setminus X) & & \\ \downarrow{\scriptstyle \mathrm{Res}_X} & & \\ H^{i-2k+1}(X) \longrightarrow H^{i+1}(\mathbb{P}^n_\Bbbk) \longrightarrow \cdots \end{array} \tag{2.1}$$

The particular cases we consider are:

(1) $\Bbbk$ is a field of characteristic zero, and $H^\bullet = H^\bullet_{\mathrm{dR}}$ is algebraic de Rham cohomology.
(2) $\Bbbk = \mathbb{F}_q$ is a finite field, and $H^\bullet = H^\bullet_{\mathrm{rig}}$ is rigid cohomology.

Case (2) is mentioned in the introduction (1.1) briefly. For (1), see [30, §3.1] for example. In particular, in the cases (1) and (2) above,

$$H^i(X) \cong H^i(\mathbb{P}_\Bbbk) \quad \text{for} \quad i \neq n-k, \quad 0 \leq i \leq 2(n-k)$$

by the weak Lefschetz property and Poincaré duality. Denote in this situation

$$H^{n-k}_{\mathrm{prim}}(X) := \ker\Big(H^{n-k}(X) \longrightarrow H^{n+k}(\mathbb{P}^n_\Bbbk) \Big).$$

Then Res_X induces an isomorphism:

$$\mathrm{Res}_X : H^{n+k-1}(\mathbb{P}^n_\Bbbk \setminus X) \longrightarrow H^{n-k}_{\mathrm{prim}}(X).$$

Therefore, we may focus on the cohomology of the complement $\mathbb{P}^n_\Bbbk \setminus X$. Because we decided to consider the complements, we may assume that $X = X_G \subseteq \mathbb{P}^n_\Bbbk$ is defined by any finite set of homogeneous polynomials $G_1, \dots, G_k \in \Bbbk[x_0, \dots, x_n]$ of positive degrees $d_1, \dots, d_k$, respectively. The Cayley trick is a method of translating the computation of $H^\bullet(\mathbb{P}^n_\Bbbk \setminus X_G)$ to a computation of the cohomology of the complement of a hypersurface living in a larger space. This larger space is simply given by the projective bundle

$$\mathbb{P}(\mathcal{E}) := \underline{\mathrm{Proj}}_{\mathbb{P}^n_\Bbbk} \mathrm{Sym}^\bullet_{\mathcal{O}_{\mathbb{P}^n_\Bbbk}} \mathcal{E} \longrightarrow \mathbb{P}^n_\Bbbk$$

associated with a locally free $\mathcal{O}_{\mathbb{P}^n_\Bbbk}$-module $\mathcal{E} := \mathcal{O}_{\mathbb{P}^n_\Bbbk}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_\Bbbk}(d_k)$. Another way of describing $\mathbb{P}(\mathcal{E})$ comes from the toric geometry, via the geometric quotient:

$$\mathbb{P}(\mathcal{E}) \cong \frac{(\mathbb{A}^k_\Bbbk \setminus 0) \times (\mathbb{A}^{n+1}_\Bbbk \setminus 0)}{\mathbb{G}_m \times \mathbb{G}_m},$$

where the $\mathbb{G}_m \times \mathbb{G}_m$-action is given by

$$(\alpha, \beta) \cdot (y, x) := (\alpha^{-d_1}\beta y_1, \dots, \alpha^{-d_k}\beta y_k, \alpha x_0, \dots, \alpha x_n).$$

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Here, the new variables $y_1,\dots,y_k$ correspond to $\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(d_1),\dots,\mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(d_k)$, and the action above explains the gradings (1.3). Moreover, $S := y_1G_1 + \cdots + y_kG_k$ being of $\deg_c S = 0$ is equivalent to saying that S is a $(\mathbb{G}_m \times 1)$-invariant element. Hence, it defines a hypersurface $X_S \subseteq \mathbb{P}(\mathcal{E})$ and

$$\operatorname{Spec}\Bbbk[x,y,S^{-1}]_{(\deg_c=0,\deg_w=0)} = \operatorname{Spec}\Bbbk[x,y,S^{-1}]^{\mathbb{G}_m\times\mathbb{G}_m} \cong \mathbb{P}(\mathcal{E})\setminus X_S$$

so that there is a commutative diagram:

$$\begin{array}{ccc}
\operatorname{Spec}\Bbbk[x,y,S^{-1}] & \longrightarrow & (\mathbb{A}^k_{\Bbbk}\setminus 0)\times(\mathbb{A}^{n+1}_{\Bbbk}\setminus 0) \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{E})\setminus X_S & \longrightarrow & \mathbb{P}(\mathcal{E}) \\
\varphi\downarrow & & \downarrow \\
\mathbb{P}^n_{\Bbbk}\setminus X_G & \longrightarrow & \mathbb{P}^n_{\Bbbk}.
\end{array} \tag{2.2}$$

Denote $X_{G_i} \subseteq \mathbb{P}^n_{\Bbbk}$ the hypersurface cut out by G_i so $\{\mathbb{P}^n_{\Bbbk}\setminus X_{G_i}\}_{i=1,\dots,k}$ is an open covering of $\mathbb{P}^n_{\Bbbk}\setminus X_G$. Since $\mathcal{E}|_{\mathbb{P}^n_{\Bbbk}\setminus X_{G_i}} \cong \mathcal{O}^{\oplus k-1}_{\mathbb{P}^n_{\Bbbk}\setminus X_{G_i}}$ is a trivial bundle of rank $k-1$, φ is an $\mathbb{A}^{k-1}_{\Bbbk}$-bundle. In this setting, if an abstract cohomology theory $H^\bullet$ satisfies the Künneth formula, and $H^\bullet(\mathbb{A}^{k-1}_{\Bbbk}) \cong \Bbbk$, then φ induces an isomorphism

$$\varphi^* : H^\bullet(\mathbb{P}^n_{\Bbbk}\setminus X_G) \xrightarrow{\sim} H^\bullet(\mathbb{P}(\mathcal{E})\setminus X_S)\,, \tag{2.3}$$

which is true for the cases (1) and (2) above. In this section, we focus on $H^\bullet = H^\bullet_{\mathrm{dR}}$ over a characteristic zero field $\Bbbk$. Then φ in (2.3) induces an isomorphism

$$\varphi^* : H^\bullet_{\mathrm{dR}}(\mathbb{P}^n_{\Bbbk}\setminus X_G) \xrightarrow{\sim} H^\bullet_{\mathrm{dR}}(\mathbb{P}(\mathcal{E})\setminus X_S)\,.$$

Since $\mathbb{P}(\mathcal{E})\setminus X_S$ is affine with coordinate ring $A := \Bbbk[x,y,S^{-1}]^{\mathbb{G}_m\times\mathbb{G}_m}$, we have

$$H^\bullet_{\mathrm{dR}}(\mathbb{P}^n_{\Bbbk}\setminus X_G) \cong H^\bullet_{\mathrm{dR}}(\mathbb{P}(\mathcal{E})\setminus X_S) \cong H^\bullet(\Omega^\bullet_{A/\Bbbk}, d),$$

where $(\Omega^\bullet_{A/\Bbbk}, d)$ is the algebraic de Rham complex of A.

Notation 2.1. In what follows, we denote

$$A := \Bbbk[x,y,S^{-1}]_{(\deg_c=0,\deg_w=0)}, \quad B := \Bbbk[x,y,S^{-1}]_{\deg_w=0}, \quad C_S := \Bbbk[x,y,S^{-1}]$$

so that $A \subseteq B \subseteq C_S$.

In the rest of this section, we will describe the algebraic de Rham cohomology of A using the algebraic de Rham cohomology of C_S. Note that $\operatorname{Spec} A$ is smooth over $\Bbbk$, being an open subset of a smooth $\Bbbk$-scheme $\mathbb{P}(\mathcal{E})$, and $\operatorname{Spec} B$ is smooth over $\Bbbk$, being an open subset of $\mathbb{P}^{k-1}_{\Bbbk}\times\mathbb{A}^{n+1}_{\Bbbk}$. Since A, B, and C_S are smooth over $\Bbbk$, the inclusions $A \subseteq B \subseteq C_S$ induces embeddings

$$\left(\Omega^\bullet_{A/\Bbbk}, d\right) \hookrightarrow \left(\Omega^\bullet_{B/\Bbbk}, d\right) \hookrightarrow \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)\,.$$

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We will see in Proposition 2.4 that these induce split injections of cohomology spaces. Since the de Rham differential preserves the bidegree $(\deg_c, \deg_w)$, the inclusion from $\Omega^\bullet_{A/\Bbbk}$ to $\Omega^\bullet_{C_S/\Bbbk}$ factors through

$$\left(\Omega^\bullet_{A/\Bbbk}, d\right) \hookrightarrow \left(\Omega^\bullet_{B/\Bbbk}, d\right)_{\deg_c=0} \hookrightarrow \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)},$$

where the bidegree $(0,0)$ part of $\Omega^\bullet_{C_S/\Bbbk}$ is the $\Bbbk$-linear span of differential forms

$$\frac{x^u y^v}{S^{|v|}} dx_\alpha \wedge \frac{dy_\beta}{S^{|\beta|}},$$

where u, v, α, β are multi-indexes following convention (1.4) such that

$$|u| + i - (v_1 d_1 + \cdots + v_k d_k) - (d_{\beta_1} + \cdots + d_{\beta_j}) = 0.$$

This explains the gradings (1.3) in the introduction. Then each grading has the corresponding Euler vector field:

$$E_c := \sum_{i=0}^n \deg_c(x_i) x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k \deg_c(y_j) y_j \frac{\partial}{\partial y_j} = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k (-d_j y_j) \frac{\partial}{\partial y_j}$$

$$E_w := \sum_{i=0}^n \deg_w(x_i) x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k \deg_w(y_j) y_j \frac{\partial}{\partial y_j} = \sum_{j=1}^k y_j \frac{\partial}{\partial y_j},$$

respectively. Denote

$$\theta_c := \langle E_c, - \rangle, \quad \theta_w := \langle E_w, - \rangle$$

the contraction with each Euler vector field.

Lemma 2.2. *θ_c and θ_w above have the following properties.*

(1) *$\theta_c^2 = 0$, $\theta_w^2 = 0$, and $\theta_c \circ \theta_w + \theta_w \circ \theta_c = 0$.*
(2) *θ_c and θ_w are derivations of the wedge product, that is, if α is a differential ℓ-form, then*

$$\theta_c(\alpha \wedge \beta) = \theta_c \alpha \wedge \beta + (-1)^\ell \alpha \wedge \theta_c \beta,$$

$$\theta_w(\alpha \wedge \beta) = \theta_w \alpha \wedge \beta + (-1)^\ell \alpha \wedge \theta_w \beta.$$

(3) *For a homogeneous f and $\lambda \in \Bbbk$, if we denote*

$$D_{\lambda,f} := \lambda d + \lambda df \wedge -,$$

then for a homogeneous ξ,

$$(D_{\lambda,f}\theta_c + \theta_c D_{\lambda,f})\xi = (\lambda \deg_c \xi + (\lambda \deg_c f) f)\xi,$$

$$(D_{\lambda,f}\theta_w + \theta_w D_{\lambda,f})\xi = (\lambda \deg_w \xi + (\lambda \deg_w f) f)\xi,$$

where the $\lambda \in \Bbbk$ is regarded as degree zero elements.

Proof. The results follow from direct computations. ∎

There are several basic but important consequences of Lemma 2.2.

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LEMMA 2.3. *With the notations above, the following hold.*

(1) *All inclusions in the following commutative square are quasi-isomorphisms:*

$$\begin{array}{ccc} \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)} & \hookrightarrow & \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_c=0} \\ \downarrow & & \downarrow \\ \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0} & \hookrightarrow & \left(\Omega^\bullet_{C_S/\Bbbk}, d\right). \end{array}$$

(2) *There are cochain maps induced from θ_c and θ_w, respectively:*

$$\theta_c : \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_c=0} \longrightarrow \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_c=0}[-1],$$

$$\theta_w : \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0} \longrightarrow \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0}[-1].$$

(3) *We can identify*

$$\Omega^\bullet_{A/\Bbbk} = \ker\theta_c \cap \ker\theta_w \subseteq \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)}$$

and there is a cochain map

$$\theta_c\theta_w : \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)} \longrightarrow \left(\Omega^\bullet_{A/\Bbbk}, d\right)[-2].$$

Moreover, the following relations hold:

$$\begin{aligned} \left(\Omega^\bullet_{B/\Bbbk}, d\right) &= \ker\theta_w \subseteq \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0}, \\ \left(\Omega^\bullet_{A/\Bbbk}, d\right) &= \ker\theta_c \subseteq \left(\Omega^\bullet_{B/\Bbbk}, d\right)_{\deg_c=0} \subseteq \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)}. \end{aligned}$$

Proof. From Lemma 2.2(3), we get relations

$$(d\theta_c + \theta_c d)\xi = (\deg_c \xi)\xi, \quad (d\theta_w + \theta_w d)\xi = (\deg_w \xi)\xi.$$

(1) If $d\xi = 0$, then ξ is in the image of d unless $(\deg_c \xi, \deg_w \xi) = (0,0)$ so (1) follows.

(2) The above relations also show that each θ becomes a cochain map on the subcomplex of homogeneous elements of degree zero.

(3) By Lemma 2.2(1), the image of $\theta_c\theta_w$ is contained in $\ker\theta_c \cap \ker\theta_w$. The asserted identifications follow from general theory of toric varieties (see, e.g., [8, Lem. 8.2]). ∎

PROPOSITION 2.4. *With the notations above, there is a decomposition of complexes*

$$\left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0} = \left(\Omega^\bullet_{B/\Bbbk}, d\right) \oplus \frac{dS}{S} \wedge \left(\Omega^\bullet_{B/\Bbbk}, d\right)$$

and for every $i \in \mathbb{Z}$, an isomorphism

$$H^i(\Omega^\bullet_{B/\Bbbk}, d) \cong H^i_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S) \oplus H^{i-1}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S).$$

Consequently, there is an isomorphism for every $i \in \mathbb{Z}$:

$$H^i\left(\Omega^\bullet_{C_S/\Bbbk}, d\right) \cong H^i_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S) \oplus H^{i-1}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S)^{\oplus 2} \oplus H^{i-2}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S).$$

Proof. Each $\xi \in \Omega^\bullet_{C_S/\Bbbk}$ can be decomposed into

$$\xi = \theta_w\left(\frac{dS}{S} \wedge \xi\right) + \frac{dS}{S} \wedge \theta_w \xi$$

so we span $\Omega^\bullet_{C_S/\Bbbk}$ as follows:

$$\Omega^\bullet_{C_S/\Bbbk} = \ker\theta_w + \frac{dS}{S} \wedge \ker\theta_w. \tag{2.4}$$

If $\xi \in \Omega^\bullet_{C_S/\Bbbk}$ is contained in the intersection of summands, that is,

$$\xi \in \ker\theta_w \cap \frac{dS}{S} \wedge \ker\theta_w,$$

then we may rewrite ξ as

$$\xi = \frac{dS}{S} \wedge \omega, \quad \omega \in \ker\theta_w.$$

Since $\theta_w\xi = 0$ and $\theta_w\omega = 0$,

$$\omega = \theta_w\left(\frac{dS}{S} \wedge \omega\right) = \theta_w\xi = 0.$$

Therefore, (2.4) becomes a direct sum decomposition:

$$\Omega^\bullet_{C_S/\Bbbk} = \ker\theta_w \oplus \frac{dS}{S} \wedge \ker\theta_w.$$

Since the restriction of θ_w on the subspace of $\deg_w = 0$ induces a cochain map by Lemma 2.3, we get a direct sum as a complex:

$$\left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{\deg_w=0} = \left(\Omega^\bullet_{B/\Bbbk}, d\right) \oplus \frac{dS}{S} \wedge \left(\Omega^\bullet_{B/\Bbbk}, d\right). \tag{2.5}$$

To compute the direct summand, consider the open subsets for $j = 1, \dots, k$

$$U_j := \mathbb{P}(\mathcal{E}) \setminus \left(X_S \cup X_{y_jG_j}\right) \cong \mathrm{Spec}(B_j)_{\deg_c=0}, \quad B_j := B[(y_jG_j)^{-1}S],$$

where $X_{y_jG_j}$ is the zero locus of y_jG_j. These open subsets form an affine open covering of $\mathbb{P}(\mathcal{E}) \setminus X_S$. On each U_j, there is a section of θ_c given by

$$\frac{1}{d_j}\frac{dG_j}{G_j} \wedge - : \left(\Omega^\bullet_{U_j/\Bbbk}, d\right)[-1] \longrightarrow \left(\Omega^\bullet_{B_j/\Bbbk}, d\right)_{\deg_c=0}.$$

They combine to give a section of the associated Čech–de Rham complex by Proposition A.6 and Example A.7. Since $\left(\Omega^\bullet_{A/\Bbbk}, d\right) = \ker\theta_c$ by Lemma 2.3, we obtain

$$H^i(\Omega^\bullet_{B/\Bbbk}, d) \cong H^i_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S) \oplus H^{i-1}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E}) \setminus X_S).$$

Now, the rest part of the proposition follows from combining the two observations so far. ∎

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§3. Cayley trick and twisted de Rham complexes

In this section, we develop the algebraic de Rham version of Theorems 1.1 and 1.2. This section is a generalization of Monsky's lecture note [26, Ch. 9]. We continue with the notation of §2. For fixed $\hbar, \gamma \in \Bbbk^{\times}$ which we regard as formal parameters, consider the twisted de Rham complex over $\Bbbk[x,y]$:

$$\left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) := \left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, \hbar d + \hbar d(\gamma S) \wedge -\right)$$

equipped with the gradings as in (1.3). Then the corresponding comparison map is given as follows.

Definition 3.1. Define the map

$$\rho_S : \left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)_{(\deg_c=0,\deg_w>0)} \longrightarrow \left(\Omega^{\bullet}_{\Bbbk[x,y,S^{-1}]/\Bbbk}, d\right)_{(\deg_c=0,\deg_w=0)}$$

by the formula

$$\rho_S\left(x^u y^v dx_\alpha \wedge dy_\beta\right) := (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)! \frac{x^u y^v}{\gamma^{|v|} S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} S^{|\beta|}}$$

together with the $\Bbbk$-linearity.

Under this map, we will obtain comparison results Propositions 3.10 and 3.16. These will be properly completed to give Theorems 1.1 and 1.2. We saw in Lemma 2.3 that the target complex of ρ_S in Definition 3.1 computes the algebraic de Rham cohomology of $C_S = \Bbbk[x,y,S^{-1}]$. On the other hand, the following lemma explains the effect of taking the subcomplex.

Lemma 3.2. *The inclusion*

$$\left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)_{\deg_c=0} \hookrightarrow \left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)$$

is a quasi-isomorphism. On the other hand, the inclusion

$$\left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)_{(\deg_c=0,\deg_w>0)} \hookrightarrow \left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)_{\deg_c=0}$$

induces a surjection on cohomology spaces with one-dimensional kernel spanned by the class of dS.

Proof. Since $\deg_c dS = 0$, the differential $D_{\hbar,\gamma S} = \hbar d + \hbar d(\gamma S) \wedge -$ is compatible with $\deg_c$ so the subcomplex is well-defined. Moreover, by Lemma 2.2(3), each $\xi \in \Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}$ homogeneous with respect to $\deg_c$ satisfies the relation

$$(D_{\hbar,\gamma S}\theta_c + \theta_c D_{\hbar,\gamma S})\xi = (\hbar \deg_c \xi)\xi$$

so if $D_{\hbar,\gamma S}\xi = 0$, then ξ is in the image of $D_{\hbar,\gamma S}$ unless $\deg_c \xi = 0$. Hence, the first inclusion is a quasi-isomorphism.

Note that $1 \in \Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}$ is the only bidegree $(\deg_c, \deg_w) = (0,0)$ element up to scalar multiplication by $\Bbbk$. Since

$$D_{\hbar,\gamma S}(1) = \hbar\gamma dS, \quad D_{\hbar,\gamma S}(dS) = 0,$$

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$1 \in \Omega^\bullet_{\Bbbk[x,y]/\Bbbk}$ does not contribute to the cohomology and kills the class $[dS]$. On the other hand, if $\deg_w f > 0$, then the equation

$$D_{\hbar,\gamma S}(f) = \hbar\gamma dS \iff df = (1-f)\gamma dS$$

has no solutions in $\Bbbk[x,y]$. Hence, $[dS]$ defines a nontrivial class in the subcomplex with $(\deg_c = 0, \deg_w > 0)$. ∎

REMARK 3.3. Since $(-1)!$ is not a well-defined number, in order to extend ρ_S to the $\deg_c = 0$ complex, we have to choose the value manually. Since 1 is the only bidegree $(\deg_c, \deg_w) = (0,0)$ element up to scalar multiplication by $\Bbbk$, it suffices to consider $\rho_S(1)$ only. For ρ_S to be a cochain map, $\rho_S(1)$ must satisfy

$$d\rho_S(1) = \rho_S(D_{\hbar,\gamma S}(1)) = \rho_S(\hbar\gamma dS) = \frac{dS}{S}$$

so $\rho_S(1) = \log S$. However, this is impossible in the polynomial ring, and even in the corresponding overconvergent power series ring (Definition 4.1).

NOTATION 3.4. In what follows, we will often denote

$$\mathscr{L}^\bullet_{(0,+)} := \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}\right)_{(\deg_c=0,\deg_w>0)}$$

as a $\mathbb{Z}$-graded $\Bbbk$-vector space so that

$$\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) = \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)_{(\deg_c=0,\deg_w>0)}.$$

Also, we will often abbreviate the subscripts

$$(0,0) := (\deg_c = 0, \deg_w = 0), \quad (0,+) := (\deg_c = 0, \deg_w > 0)$$

to indicate the bidegree restrictions whenever it is clear from the context.

LEMMA 3.5. *Properties of ρ_S.*

(1) *ρ_S is a $\Bbbk$-linear cochain map.*
(2) *ρ_S commutes with θ_c and θ_w. Here, θ_w is regarded as a degree -1 map of $\mathbb{Z}$-graded $\Bbbk$-vector spaces.*
(3) *If ξ_1 and ξ_2 are $\deg_w$-homogeneous of positive degree, then*

$$\rho_S(\xi_1 \wedge \xi_2) = -\frac{(\deg_w \xi_1 + \deg_w \xi_2 - 1)!}{(\deg_w \xi_1 - 1)!(\deg_w \xi_2 - 1)!}\rho_S(\xi_1) \wedge \rho_S(\xi_2).$$

In particular, if ξ is $\deg_w$-homogeneous of positive degree, then

$$\rho_S(\gamma^i S^i \xi) = (-1)^i \frac{(\deg_w \xi + i - 1)!}{(\deg_w \xi - 1)!}\rho_S(\xi)$$

for every integer $i > 0$.

Proof. The results follow from direct computation. ∎

To describe the kernel of ρ_S, we introduce the following auxiliary map:

DEFINITION 3.6. Define the map

$$\epsilon_{w,S} := D_{\hbar,\gamma S}\theta_w + \theta_w D_{\hbar,\gamma S} : \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) \longrightarrow \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right).$$

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By Lemma 2.2, if ξ is homogeneous with respect to $\deg_w$, then

$$\epsilon_{w,S}(\xi) = (\hbar \deg_w \xi + \hbar\gamma S)\xi.$$

REMARK 3.7. This map corresponds to the one in [26, Lem. 9.1] which is defined via the congruence condition on the degree of a defining hypersurface. However, we are working reversely via the Cayley trick. In our context, [26, Lem. 9.1] becomes Definition 3.6, and Monsky's definition follows from Lemma 2.2.

REMARK 3.8. One may analogously consider $\epsilon_{c,S} := D_{\hbar,\gamma S}\theta_c + \theta_c D_{\hbar,\gamma S}$, but this vanishes on the subspace of $\deg_c = 0$ by Lemma 2.2.

LEMMA 3.9. *Properties of* $\epsilon_{w,S}$.

(1) $\epsilon_{w,S}$ *is a* $\Bbbk$*-linear cochain map.*
(2) $\epsilon_{w,S}$ *is injective.*
(3) $\epsilon_{w,S}$ *restricts to the bidegree* $(\deg_c = 0, \deg_w > 0)$*-subcomplex:*

$$\epsilon_{w,S} : \left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \longrightarrow \left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right).$$

If $\xi \in \mathscr{L}^\bullet_{(0,+)}$ *is a* $\deg_w$*-homogeneous element, then*

$$\gamma^i S^i \xi \equiv (-1)^i \frac{(\deg_w \xi + i - 1)!}{(\deg_w \xi - 1)!} \xi \bmod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)}.$$

Proof. (1) $\epsilon_{w,S}$ is $\Bbbk$-linear by construction and is a cochain map by

$$D_{\hbar,\gamma S}\epsilon_{w,S} = D_{\hbar,\gamma S}\theta_w D_{\hbar,\gamma S} = (\epsilon_{w,S} - \theta_w D_{\hbar,\gamma S})D_{\hbar,\gamma S} = \epsilon_{w,S} D_{\hbar,\gamma S}.$$

(2) If ξ is a nonzero $\deg_w$-homogeneous element, then

$$\epsilon_{w,S}(\xi) = (\hbar \deg_w \xi + \hbar\gamma S)\xi$$

is nonzero by our choice of S.

(3) We proceed by induction on the power i. The case $i = 1$ follows immediately from construction. If $i > 1$, then

$$\begin{aligned}
(\gamma S)^i \xi &= (\gamma S)(\gamma S)^{i-1}\xi \\
&\equiv -(\deg_w \xi + i - 1)(\gamma S)^{i-1}\xi \bmod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)} \\
&\equiv -(\deg_w \xi + i - 1)\cdot(-1)^{i-1}\frac{(\deg_w \xi + i - 2)!}{(\deg_w \xi - 1)!}\xi \bmod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)} \\
&\equiv (-1)^i \frac{(\deg_w \xi + i - 1)!}{(\deg_w \xi - 1)!}\xi \bmod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)},
\end{aligned}$$

where the third line follows from induction hypothesis. ∎

PROPOSITION 3.10. *With notation 3.4, there is an exact sequence of cochain complexes*

$$0 \longrightarrow \left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \xrightarrow{\epsilon_{w,S}} \left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \xrightarrow{\rho_S} \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(0,0)} \longrightarrow 0\,.$$

Consequently, there is an exact sequence

$$0 \longrightarrow H^i\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \xrightarrow{\rho_S} H^i\left(\Omega^\bullet_{C_S/\Bbbk}, d\right) \xrightarrow{\delta} H^{i+1}\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \longrightarrow 0$$

for every $i \in \mathbb{Z}$, *and in particular,*

$$H^0\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) = 0, \quad H^1\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) = \Bbbk \cdot [dS].$$

Proof. Since the target of ρ_S admits a $\Bbbk$-basis

$$\left\{ \frac{x^u y^v}{S^{|v|}} dx_\alpha \wedge \frac{dy_\beta}{S^{|\beta|}} \;\middle|\; |u| + |\alpha| = v_1 d_1 + \cdots + v_k d_k + d_{\beta_1} + \cdots + d_{\beta_{|\beta|}} \right\}$$

which is in the image of ρ_S, the surjectivity of ρ_S follows. The injectivity of $\epsilon_{w,S}$ follows from Lemma 3.9. If $\xi \in \mathscr{L}^\bullet_{(0,+)}$ is $\deg_w$-homogeneous, then

$$(\rho_S \circ \epsilon_{w,S})(\xi) = \rho_S\left((\hbar \deg_w \xi)\xi + \hbar\gamma S\xi\right) = 0$$

by Lemma 3.5. Hence, ρ_S induces

$$\overline{\rho}_S : \left(\frac{\mathscr{L}^\bullet_{(0,+)}}{\epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)}}, D_{\hbar,\gamma S} \right) \longrightarrow \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(0,0)}.$$

Define the map of graded $\Bbbk$-vector spaces

$$\sigma : \left(\Omega^\bullet_{C_S/\Bbbk}\right)_{(0,0)} \longrightarrow \frac{\mathscr{L}^\bullet_{(0,+)}}{\epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)}}$$

by the formula

$$\sigma\left(\frac{x^u y^v}{S^{|v|}} dx_\alpha \wedge \frac{dy_\beta}{S^{|\beta|}} \right) := (-1)^{|v|+|\beta|-1} \frac{\hbar^{|\alpha|+|\beta|}\gamma^{|v|+|\beta|}}{(|v|+|\beta|-1)!} x^u y^v dx_\alpha \wedge dy_\beta$$

for $|v| + |\beta| > 0$ together with the $\Bbbk$-linearity. From

$$\begin{aligned}
&\sigma\left(\frac{S^i}{S^i} \frac{x^u y^v}{S^{|v|}} dx_\alpha \wedge \frac{dy_\beta}{S^{|\beta|}} \right) \\
&= (-1)^{|v|+|\beta|+i-1} \frac{\hbar^{|\alpha|+|\beta|}\gamma^{|v|+|\beta|+i}}{(|v|+|\beta|+i-1)!} S^i x^u y^v dx_\alpha \wedge dy_\beta \\
&\equiv (-1)^{|v|+|\beta|-1} \frac{\hbar^{|\alpha|+|\beta|}\gamma^{|v|+|\beta|}}{(|v|+|\beta|-1)!} x^u y^v dx_\alpha \wedge dy_\beta \mod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)} \\
&= \sigma\left(\frac{x^u y^v}{S^{|v|}} dx_\alpha \wedge \frac{dy_\beta}{S^{|\beta|}} \right),
\end{aligned}$$

we see that σ is well-defined. Note that this forces $\sigma(1) = \gamma S$. By construction, $\overline{\rho}_S\sigma$ is the identity. Since $\operatorname{coker}\epsilon_{w,S}$ is spanned over $\Bbbk$ by $S^i x^u y^v dx_\alpha \wedge dy_\beta$ with $i \geq 0$ and $|v| + |\beta| > 0$, and

$$\gamma^i S^i x^u y^v dx_\alpha \wedge dy_\beta \equiv (-1)^i \frac{(|v|+|\beta|+i-1)!}{(|v|+|\beta|-1)!} x^u y^v dx_\alpha \wedge dy_\beta \mod \epsilon_{w,S}\mathscr{L}^\bullet_{(0,+)}$$

is in the image of σ, we conclude that σ is surjective. Hence, $\overline{\rho}_S$ and σ are mutually inverses. Therefore, we achieve the desired exactness.

For the second part, take the cohomology long exact sequence. Since $\epsilon_{w,S}$ is homotopic to zero by definition of $\epsilon_{w,S}$ and Lemma 3.9, we get the desired exact sequences. In particular, the long exact sequence begins with

$$\begin{array}{ccccc} 0 \longrightarrow H^0\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) & & & & \\ \downarrow 0 & & & & \\ H^0\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & H^0\left(\Omega^\bullet_{C_S/\Bbbk}, d\right) & \xrightarrow{\delta} & H^1\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \\ & & & & \downarrow 0 \\ & & & & H^1\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right). \end{array}$$

Hence, we get the desired vanishing and the δ becomes an isomorphism

$$\delta : H^0\left(\Omega^\bullet_{C_S/\Bbbk}, d\right) \xrightarrow{\sim} H^1\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right).$$

Since $\operatorname{Spec} C_S = \operatorname{Spec} \Bbbk[x,y,S^{-1}]$ is connected, the right-hand side is one-dimensional with a basis $[dS]$ coming from 3.2. ∎

Corollary 3.11. *The cohomology groups of the twisted de Rham complex*

$$H^i\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right)$$

are finite-dimensional $\Bbbk$-vector spaces for every $i \in \mathbb{Z}$. In particular,

$$H^0\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) = 0, \quad H^1\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) = 0.$$

Proof. By Lemma 3.2, we may compute the cohomology of the twisted de Rham complex by using $\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right)$. Hence, the results follow from Proposition 3.10 and the finiteness of the algebraic de Rham cohomology of smooth $\Bbbk$-algebras [28, Th. 3.1]. ∎

To describe the image of ρ_S on the cohomology spaces, we introduce the following auxiliary map.

Definition 3.12. Define the $\Bbbk$-linear map

$$\chi : \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}\right)_{(\deg_c = 0, \deg_w > 0)} \longrightarrow \left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}\right)_{(\deg_c = 0, \deg_w > 0)}$$

as follows: If ξ is a $\deg_w \xi$-homogeneous of positive degree, then denote

$$\chi_\xi := \begin{cases} 0, & \text{if} \quad \deg_w \xi = 1, \\ -\left(1 + \frac{1}{2} + \cdots + \frac{1}{\deg_w \xi - 1}\right), & \text{if} \quad \deg_w \xi > 1, \end{cases}$$

and define $\chi(\xi) := \chi_\xi \cdot \xi$ on $\deg_w \xi$-homogeneous elements.

Remark 3.13. This map corresponds to the one in [26, p. 110]. The additional grading $\deg_w$ coming from the Cayley trick replaces the role of the congruence condition on the degree of a defining hypersurface in Monsky's definition.

Lemma 3.14. *As a cochain map, the following hold.*

$$\rho_S \circ (\chi D_{\hbar,\gamma S} - D_{\hbar,\gamma S} \chi) = \frac{dS}{S} \wedge \rho_S.$$

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Proof. If ξ is a $\deg_w$-homogeneous element, then

$$\begin{aligned}(\chi D_{\hbar,\gamma S} - D_{\hbar,\gamma S}\chi)(\xi) &= \chi_\xi \hbar d\xi + \chi_{dS\wedge\xi}\hbar\gamma dS\wedge\xi - \chi_\xi\hbar d\xi - \chi_\xi\hbar\gamma dS\wedge\xi \\ &= -\frac{\hbar\gamma}{\deg_w \xi} dS\wedge\xi\end{aligned}$$

because $\deg_w(dS\wedge\xi) = \deg_w\xi + 1$. By Lemma 3.5, this gives

$$\begin{aligned}(\rho_S\circ(\chi D_{\hbar,\gamma S} - D_{\hbar,\gamma S}\chi))(\xi) &= -\frac{\hbar\gamma}{\deg_w\xi}\rho_S(dS\wedge\xi) \\ &= \frac{\hbar\gamma}{\deg_w\xi}\frac{(\deg_w\xi)!}{(\deg_w\xi - 1)!}\rho_S(dS)\wedge\rho_S(\xi) \\ &= \frac{dS}{S}\wedge\rho_S(\xi)\end{aligned}$$

so the lemma follows. ∎

LEMMA 3.15. *With Notations 2.1 and 3.4, the square*

$$\begin{array}{ccc}\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\frac{dS}{S}\wedge\theta_w\rho_S} & \frac{dS}{S}\wedge\left(\Omega^\bullet_{B/\Bbbk}, d\right)_{\deg_c = 0} \\ \| & & \big\downarrow \\ \left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow[\rho_S]{} & \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)_{(0,0)}\end{array}$$

commutes up to homotopy where we use the identification coming from the decomposition in Proposition 2.4.

Proof. Using Lemmas 3.5, 3.9, and 3.14, and Proposition 3.10, we get

$$\begin{aligned}d\rho_S\theta_w\chi + \rho_S\theta_w\chi D_{\hbar,\gamma S} &= \rho_S D_{\hbar,\gamma S}\theta_w\chi + \rho_S\theta_w\chi D_{\hbar,\gamma S} \\ &= \rho_S(D_{\hbar,\gamma S}\theta_w + \theta_w D_{\hbar,\gamma S})\chi + \rho_S\theta(\chi D_{\hbar,\gamma S} - D_{\hbar,\gamma S}\chi) \\ &= \rho_S\epsilon_{w,S}\chi + \theta_w\rho_S(\chi D_{\hbar,\gamma S} - D_{\hbar,\gamma S}\chi) \\ &= \theta_w\left(\frac{dS}{S}\wedge\rho_S\right) \\ &= \rho_S - \frac{dS}{S}\wedge\theta_w\rho_S,\end{aligned}$$

where $\theta_w\rho_S$ maps into $\Omega^\bullet_{B/\Bbbk}$ by Lemma 2.3. ∎

PROPOSITION 3.16. *In the exact sequence as in Proposition 3.10 for $i\in\mathbb{Z}$:*

$$0\longrightarrow H^i\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right)\xrightarrow{\rho_S} H^i\left(\Omega^\bullet_{C_S/\Bbbk}, d\right)\xrightarrow{\delta} H^{i+1}\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right)\longrightarrow 0$$

ρ_S and δ above induce isomorphisms

$$\delta : H^i\left(\Omega^\bullet_{B/\Bbbk}, d\right)\xrightarrow{\sim} H^{i+1}\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right),$$

$$\rho_S : H^i\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right)\xrightarrow{\sim}\frac{dS}{S}\wedge H^{i-1}\left(\Omega^\bullet_{B/\Bbbk}, d\right).$$

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Here, we use the identification of Proposition 2.4. Consequently, ρ_S induces

$$H^i\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) \cong H^{i-1}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E})\setminus X_S) \oplus H^{i-2}_{\mathrm{dR}}(\mathbb{P}(\mathcal{E})\setminus X_S)$$

for every $i \geq 2$.

Proof. Suppose that $\xi \in \Omega^{i+1}_{\Bbbk[x,y]/\Bbbk}$ is a $D_{\hbar,\gamma S}$-closed form. If we take

$$\omega := \theta_w \rho_S \xi = \rho_S \theta_w \xi,$$

then $\omega \in \Omega^i_{B/\Bbbk}$ and

$$d\omega = d\theta_w \rho_S \xi = \theta_w \rho_S D_{\hbar,\gamma S}\xi = 0.$$

Since ω is in the image of ρ_S, it represents a class in $H^i(\Omega^\bullet_{B/\Bbbk}, d)$. Now, $\theta_w\xi$ is a lift of ω along ρ_S and, since $D_{\hbar,\gamma S}\xi = 0$, we have

$$D_{\hbar,\gamma S}\theta_w \xi = (D_{\hbar,\gamma S}\theta_w + \theta_w D_{\hbar,\gamma S})\xi = \epsilon_{w,S}(\xi).$$

Therefore, by the construction of connecting map δ,

$$\delta[\omega] = \left[\epsilon^{-1}_{w,S} D_{\hbar,\gamma S}\theta_w \xi\right] = [\xi]$$

so δ restricted to $H^i(\Omega^\bullet_{B/\Bbbk}, d)$ is surjective.

On the other hand, ρ_S defines an injection into $H^{i-1}(\Omega^\bullet_{B/\Bbbk}, d)$ by Proposition 3.10 and Lemma 3.15. If $\xi \in H^{i-1}(\Omega^\bullet_{B/\Bbbk}, d)$, then there is $\widetilde{\xi} \in H^i(\Omega^\bullet_{B/\Bbbk}, d)$ with

$$\delta\widetilde{\xi} = \delta\left(\frac{dS}{S}\wedge \xi\right)$$

by the surjectivity of δ observed above. Hence,

$$\widetilde{\xi} - \frac{dS}{S}\wedge\xi \in \rho_S H^i\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right)$$

but this implies $\widetilde{\xi} = 0$ by Lemma 3.15. Hence, ρ_S is surjective as well, that is, it is an isomorphism. By the identification of Proposition 2.4, this implies that δ is an isomorphism as well. The last assertion follows from Proposition 2.4 together with Lemma 3.2. ∎

§4. *p*-adic cohomology and Cayley trick

In this section, we will prove Theorems 1.1 and 1.2, by constructing p-adic models of the complexes studied in §2 and §3, respectively. From now on, $\Bbbk$ will be a finite extension of $\mathbb{Q}_p$ with the valuation ring $(\mathcal{O}_\Bbbk, \mathfrak{m}_\Bbbk)$ and the residue field $\mathbb{F}_q$. Also, we keep the notation in §2 and §3, but we assume that $G_1,\ldots,G_k$ belong to $\mathcal{O}_\Bbbk[x_0,\ldots,x_n]$ and their reductions $\overline{G}_1,\ldots,\overline{G}_k$ are nonzero in $\mathbb{F}_q[x_0,\ldots,x_n]$.

4.1 Monsky–Washnitzer cohomology

In this subsection, we briefly review the theory of Monsky–Washnitzer cohomology, which gives a p-adic model of algebraic de Rham cohomology studied in §2. Using this, we translate Proposition 2.4 into Monsky–Washnitzer setting and get the corresponding results in Proposition 4.7.

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DEFINITION 4.1. Denote the ring of overconvergent power series over $\mathcal{O}_{\Bbbk}$ by

$$\mathcal{O}_{\Bbbk}\{t_1,\ldots,t_n\}^{\dagger} = \left\{ \sum_{u\in\mathbb{Z}_{\geq 0}^{\oplus n}} c_u t^u \in \mathcal{O}_{\Bbbk}[[t_1,\ldots,t_n]] \;\middle|\; \begin{array}{c}\text{there is } r>1 \text{ such that}\\ \lim_{|u|\to\infty} c_u r^u = 0\end{array} \right\}.$$

Then a *weakly complete finitely generated* algebra over $\mathcal{O}_{\Bbbk}$ is a homomorphic image of some overconvergent power series ring.

PROPOSITION 4.2. *$\mathcal{O}_{\Bbbk}\{t_1,\ldots,t_m\}^{\dagger}$ satisfies Weierstrass' preparation and division. Consequently,*

(1) *$\mathcal{O}_{\Bbbk}\{t_1,\ldots,t_n\}^{\dagger}$ is Noetherian, and*
(2) *the inclusion $\mathcal{O}_{\Bbbk}[t_1,\ldots,t_n] \subseteq \mathcal{O}_{\Bbbk}\{t_1,\ldots,t_n\}^{\dagger}$ is flat.*

Proof. This is [32, Prop. 2.2]. ∎

DEFINITION 4.3. Given an $\mathcal{O}_{\Bbbk}$-algebra A, denote

$$\Omega^{\bullet}_{A/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} := \frac{\Omega^{\bullet}_{A/\mathcal{O}_{\Bbbk}}}{\bigcap_{\lambda\geq 0} \mathfrak{m}_{\Bbbk}^{\lambda+1}\Omega^{\bullet}_{A/\mathcal{O}_{\Bbbk}}} \cong \operatorname{im}\left(\Omega^{\bullet}_{A/\mathcal{O}_{\Bbbk}} \longrightarrow \varprojlim_{\lambda\geq 0} \frac{\Omega^{\bullet}_{A/\mathcal{O}_{\Bbbk}}}{\mathfrak{m}_{\Bbbk}^{\lambda+1}\Omega^{\bullet}_{A/\mathcal{O}_{\Bbbk}}} \right),$$

which is called the $\mathfrak{m}_{\Bbbk}$-*separated* (or $\mathfrak{m}_{\Bbbk}$-*continuous*) differentials on A.

DEFINITION 4.4. Given an (usually smooth) $\mathbb{F}_q$-algebra $\overline{A}$, a w.c.f.g. $\mathcal{O}_{\Bbbk}$-algebra A is called a *lift* if A is flat over $\mathcal{O}_{\Bbbk}$ and $A/\mathfrak{m}_{\Bbbk}A \cong \overline{A}$.

THEOREM 4.5. *Given a smooth $\mathbb{F}_q$-algebra $\overline{A}$, there is always a lift A of $\overline{A}$. Moreover, the following hold.*

(1) *Every lift of $\overline{A}$ is isomorphic to A as an $\mathcal{O}_{\Bbbk}$-algebra.*
(2) *Let $\overline{B}$ be a smooth $\mathbb{F}_q$-algebra with a lift B. If $\overline{\varphi}:\overline{A}\to\overline{B}$ is an $\mathbb{F}_q$-algebra map, then there is an $\mathcal{O}_{\Bbbk}$-algebra map $\varphi: A\to B$ such that*

$$\varphi \bmod \mathfrak{m}_{\Bbbk} = \overline{\varphi}.$$

(3) *If $\varphi,\psi: A\to C$ are two maps into a w.c.f.g. $\mathcal{O}_{\Bbbk}$-algebra such that*

$$\varphi \bmod \mathfrak{m}_{\Bbbk} = \psi \bmod \mathfrak{m}_{\Bbbk},$$

then the induced maps

$$\varphi,\psi: \Omega^{\bullet}_{A/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}\otimes_{\mathcal{O}_{\Bbbk}}\Bbbk \longrightarrow \Omega^{\bullet}_{C/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}\otimes_{\mathcal{O}_{\Bbbk}}\Bbbk$$

are homotopic.

Proof. This is [32, Th. 2.4.4]. ∎

DEFINITION 4.6. Let $\overline{A}$ be a smooth $\mathbb{F}_q$-algebra. Define

$$H^i_{\mathrm{MW}}(\overline{A}/\Bbbk) := H^i\left(\Omega^{\bullet}_{A/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}\otimes_{\mathcal{O}_{\Bbbk}}\Bbbk, d\right),$$

where A is any lift of $\overline{A}$ given by Theorem 4.5.

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Return to the situation of §2, but with the assumption that $G_1,\dots,G_k$ belong to $\mathcal{O}_{\Bbbk}[x_0,\dots,x_n]$ and their reductions $\overline{G}_1,\dots,\overline{G}_k$ in $\mathbb{F}_q[x_0,\dots,x_n]$ are nonzero. As we observed in §2, there is an isomorphism

$$\varphi^*: H^\bullet_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q}\setminus X_{\underline{G}}) \xrightarrow{\sim} H^\bullet_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}})$$

coming from (2.3). Following Notations 2.1 and 3.4, we denote

$$\mathcal{O}_A := \mathcal{O}_{\Bbbk}[x,y,S^{-1}]_{(0,0)}, \quad \mathcal{O}_B := \mathcal{O}_{\Bbbk}[x,y,S^{-1}]_{\deg_w=0}, \quad \mathcal{O}_{C_S} := \mathcal{O}_{\Bbbk}[x,y,S^{-1}].$$

Then the w.c.f.g. $\mathcal{O}_{\Bbbk}$-algebra

$$C_S^\dagger = \mathcal{O}_{\Bbbk}\{x,y,S^{-1}\}^\dagger \cong \frac{\mathcal{O}_{\Bbbk}\{x,y,t\}^\dagger}{(tS-1)}$$

satisfies

$$\frac{C_S^\dagger}{\mathfrak{m}_{\Bbbk}C_S^\dagger} \cong \frac{\mathbb{F}_q[x,y,t]}{(t\overline{S}-1)} \cong \mathbb{F}_q[x,y,\overline{S}^{-1}].$$

Moreover, its subalgebras

$$A^\dagger := (C_S^\dagger)_{(0,0)}, \quad B^\dagger := (C_S^\dagger)_{\deg_w=0}$$

are still w.c.f.g. $\mathcal{O}_{\Bbbk}$-algebras such that

$$\frac{A^\dagger}{\mathfrak{m}_{\Bbbk}A^\dagger} \cong \mathbb{F}_q[x,y,\overline{S}^{-1}]_{(0,0)} \cong \overline{\mathcal{O}}_A, \quad \frac{B^\dagger}{\mathfrak{m}_{\Bbbk}B^\dagger} \cong \mathbb{F}_q[x,y,\overline{S}^{-1}]_{\deg_w=0} \cong \overline{\mathcal{O}}_B.$$

Hence, $A^\dagger$, $B^\dagger$, and $C_S^\dagger$ compute the Monsky–Washnitzer of $\overline{\mathcal{O}}_A$, $\overline{\mathcal{O}}_B$, and $\overline{\mathcal{O}}_{C_S}$, respectively: for $R = A$, B, or C_S,

$$H^\bullet_{\mathrm{MW}}(\overline{\mathcal{O}}_R/\Bbbk) \cong H^\bullet\left(\Omega^\bullet_{R^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}\otimes_{\mathcal{O}_{\Bbbk}}\Bbbk, d\right).$$

Proposition 4.7. *With the notations above, there is a decomposition of complexes*

$$\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right)_{\deg_w=0} = \left(\Omega^\bullet_{B^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right) \oplus \frac{dS}{S}\wedge\left(\Omega^\bullet_{B^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right)$$

and for every $i\in\mathbb{Z}$, an isomorphism

$$H^i_{\mathrm{MW}}(\overline{\mathcal{O}}_B/\Bbbk) \cong H^i_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}}) \oplus H^{i-1}_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}}).$$

Consequently, there is an isomorphism for every $i\in\mathbb{Z}$:

$$H^i_{\mathrm{MW}}(\overline{\mathcal{O}}_{C_S}/\Bbbk) \cong H^i_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}}) \oplus H^{i-1}_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}})^{\oplus 2} \oplus H^{i-2}_{\mathrm{rig}}(\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}}).$$

Proof. Note that $\Omega^1_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}$ is generated over $C_S^\dagger$ by $dx_0,\dots,x_n,dy_1,\dots,dy_k$ and similarly for $\Omega^1_{B^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}$ and $\Omega^1_{A^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}$ (cf. [29, Th. 4.5]). Since θ_c and θ_w in §2 acts only on dx and dy, the proof of Proposition 2.4 works for overconvergent algebras to give the desired decomposition:

$$\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right)_{\deg_w=0} = \left(\Omega^\bullet_{B^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right) \oplus \frac{dS}{S}\wedge\left(\Omega^\bullet_{B^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})}, d\right).$$

To get the second assertion, consider the affine weak formal scheme $(\mathbb{P}(\mathcal{E})\setminus X_S)^\dagger$ in the sense of [24, Def. 15], that is, the topological space $\mathbb{P}(\mathcal{E})\setminus X_{\overline{S}}$ endowed with the structure sheaf associated with $A^\dagger$. Then the open subsets for $j = 1, \ldots, k$

$$\overline{U}_j := \mathbb{P}(\mathcal{E}) \setminus \left(X_{\overline{S}} \cup X_{y_j \overline{G}_j} \right) \cong \operatorname{Spec}(\overline{\mathcal{O}}_{B_j})_{\deg_c = 0}, \quad \mathcal{O}_{B_j} := \mathcal{O}_B[(y_j G_j)^{-1} S]$$

give a covering $\{U_j^\dagger\}_{j=1,\ldots,k}$ of $(\mathbb{P}(\mathcal{E}) \setminus X_S)^\dagger$ by principal open subsets associated with the w.c.f.g. $\mathcal{O}_\Bbbk$-algebras $(B_j^\dagger)_{\deg_c=0}$ where $B_j^\dagger := \mathcal{O}_{B_j}^\dagger$. From the vanishing of higher cohomology [24, Th. 14] of finitely generated modules on affine weak formal schemes, we deduce that the Čech–de Rham complexes of $\mathfrak{m}_\Bbbk$-separated differentials compute the Monsky–Washnitzer cohomology of the corresponding reduction. On the other hand, the section of θ_c on each $U_j^\dagger$,

$$\frac{1}{d_i} \frac{dG_i}{G_i} \wedge - : \left(\Omega^\bullet_{U_j^\dagger/(\mathcal{O}_\Bbbk, \mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d \right) [-1] \longrightarrow \left(\Omega^\bullet_{B_j^\dagger/(\mathcal{O}_\Bbbk, \mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d \right)_{\deg_c = 0}$$

as in the proof of Proposition 2.4 still works. Since restriction to a principal open subset is given by tensoring with weakly completed principal localizations (cf. [24, p. 4]), the Čech–de Rham cosimplicial algebra for $\mathfrak{m}_\Bbbk$-separated differentials is 0-coskeletal as in algebraic de Rham case (Example A.7). Hence, we obtain

$$H^i \left(\Omega^\bullet_{B^\dagger/(\mathcal{O}_\Bbbk, \mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d \right) \cong H^i_{\mathrm{rig}}(\mathbb{P}(\mathcal{E}) \setminus X_{\overline{S}}) \oplus H^{i-1}_{\mathrm{rig}}(\mathbb{P}(\mathcal{E}) \setminus X_{\overline{S}}).$$

Now, the rest part of the proposition follows from combining the two observations so far. ∎

4.2 Dwork cohomology

In this subsection, we introduce the Dwork complex associated with $G_1, \ldots, G_k$, which gives a p-adic model of twisted de Rham complexes studied in §3. Then, we extend the ρ_S in Definition 3.1 to the Dwork complex in Proposition 4.11, which proves Theorems 1.1 and 1.2.

Definition 4.8. Denote

$$\mathcal{O}_\Bbbk\{z_1, \ldots, z_N\} := \varprojlim_{\lambda \geq 0} \frac{\mathcal{O}_\Bbbk[z_1, \ldots, z_N]}{\mathfrak{m}_\Bbbk^{\lambda+1} \mathcal{O}_\Bbbk[z_1, \ldots, z_N]}$$

the ring of *restricted power series* over $\mathcal{O}_\Bbbk$ (in N variables), and

$$\Bbbk\{z_1, \ldots, z_N\} := \Bbbk \otimes_{\mathcal{O}_\Bbbk} \mathcal{O}_\Bbbk\{z_1, \ldots, z_N\}$$

the *Tate algebra* over $\Bbbk$ (in N variables).

Remark 4.9. Tate algebra can be written as

$$\Bbbk\{z_1, \ldots, z_N\} = \left\{ \sum_{w \in \mathbb{Z}_{\geq 0}^{\oplus N}} a_w z^w \in \Bbbk[[z_1, \ldots, z_N]] \;\middle|\; \lim_{|w| \to \infty} a_w = 0 \right\}.$$

Hence, given an N-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ of positive real numbers, we denote

$$\Bbbk\{\epsilon_1^{-1} z_1, \ldots, \epsilon_N^{-1} z_N\} := \left\{ \sum_{w \in \mathbb{Z}_{\geq 0}^{\oplus N}} a_w z^w \in \Bbbk[[z_1, \ldots, z_N]] \;\middle|\; \lim_{|w| \to \infty} a_w \epsilon^w = 0 \right\}.$$

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We sometimes use notation $\Bbbk\{\epsilon^{-1}z\}$. In terms of rigid geometry, this algebra corresponds to the closed polydisk of radius ϵ. If $\epsilon = |c|$ for some $c \in \Bbbk^{\oplus N}$, then

$$\Bbbk\{\epsilon^{-1}z\} = \left\{ \sum_{w\in\mathbb{Z}_{\geq 0}^{\oplus N}} a_w c^{-w} z^w \in \Bbbk[[z]] \;\middle|\; \lim_{|w|\to\infty} = 0 \right\} = \Bbbk\{c^{-1}z\},$$

where $\Bbbk\{c^{-1}z\}$ is the Tate algebra with respect to the variables $c_1^{-1}z_1,\dots,c_N^{-1}z_N$.

Denote $N := n+k+1$, and denote for $\hbar \in \Bbbk$ with $\mathrm{val}_p\hbar > 0$

$$C(\hbar) := \left\{ \sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus N}} a_{u,v}\hbar^{|v|}x^u y^v \in \Bbbk[[x,y]] \;\middle|\; \lim_{|(u,v)|\to\infty} a_{u,v} = 0 \right\}$$

so that $C(\hbar) \cong \Bbbk\{x,\hbar y\}$. Then the twisted de Rham complex of the form

$$(\Omega_\hbar^\bullet, D_{\hbar,\gamma S}) := \left(C(\hbar) \otimes_{\Bbbk[x,y]} \Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S} \right)$$

will be called the *Dwork complex* associated with $G_1,\dots,G_k$, or to $\overline{G}_1,\dots,\overline{G}_k$. The gradings (1.3) is valid on our Dwork complex.

NOTATION 4.10. We will often denote

$$\mathscr{L}^\bullet_{\hbar,(0,+)} := C(\hbar) \otimes_{\Bbbk[x,y]} \mathscr{L}^\bullet_{(0,+)}$$

as a $\mathbb{Z}$-graded $\Bbbk$-vector space so that

$$\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) = (\Omega_\hbar^\bullet, D_{\hbar,\gamma S})_{(\deg_c=0,\deg_w>0)}.$$

Now, Theorems 1.1 and 1.2 follow from the following theorem.

THEOREM 4.11. *If* $\mathrm{val}_p\gamma \leq \dfrac{1}{p-1}$ *and* $\mathrm{val}_p\hbar > 0$, *then the* ρ_S *in Definition 3.1 extends continuously to p-adic analytic complexes, that is, there is a commutative square*

$$\begin{array}{ccc}
\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S/\Bbbk}, d\right)^\bullet_{(0,0)} \\
\big\downarrow & & \big\downarrow \\
\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right)_{(0,0)}
\end{array}.$$

Moreover, the extended ρ_S *induces an isomorphism*

$$\rho_S : H^i\left(\Omega_\hbar^\bullet, D_{\hbar,\gamma S}\right) \xrightarrow{\sim} H^{i-1}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{G}}) \oplus H^{i-2}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{G}})$$

for every $i \geq 2$. *On the other hand,*

$$H^0\left(\Omega_\hbar^\bullet, D_{\hbar,\gamma S}\right) = 0, \quad H^1\left(\Omega_\hbar^\bullet, D_{\hbar,\gamma S}\right) = 0.$$

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Proof. To extend ρ_S, we need to check the overconvergence of the expression. To see this, it suffices to show that there is some $r>1$ such that

$$\lim_{|v|\to\infty}\left(\mathrm{val}_p\left((|v|+|\beta|-1)!\frac{\hbar^{|v|-|\alpha|-|\beta|}}{\gamma^{|v|+|\beta|}}\right)-(|u|+2|v|+|\beta|)\log_p r\right)=\infty.$$

From the degree condition, we have

$$|u|+|\alpha|=v_1d_1+\cdots+v_kd_k+d_{\beta_1}+\cdots+d_{\beta_j}\leq(|v|+k)d_{\max}.$$

Since $|\alpha|\leq n+1$ and $|\beta|\leq k$ are bounded by constants, we may ignore them so roughly $|u|\sim|v|d_{\max}$ for large $|u|$. On the other hand, we have

$$\begin{aligned}&\mathrm{val}_p\left((|v|+|\beta|-1)!\frac{\hbar^{|v|-|\alpha|-|\beta|}}{\gamma^{|v|+|\beta|}}\right)-(|u|+2|v|+|\beta|)\log_p r\\ \geq\ &\frac{|v|+|\beta|-1}{p-1}-\log_p(|v|+|\beta|)+\\ &\quad+(|v|-|\alpha|-|\beta|)\mathrm{val}_p\hbar-(|v|+|\beta|)\mathrm{val}_p\gamma-(|u|+2|v|+|\beta|)\log_p r.\end{aligned}$$

Consequently, it suffices to take r such that

$$0<\log_p r<\frac{1}{2+d_{\max}}\left(\frac{1}{p-1}+\mathrm{val}_p\hbar-\mathrm{val}_p\gamma\right).$$

Next, since $\epsilon_{w,S}$ acts only on $dx_0,\ldots,dx_n,dy_1,\ldots,dy_k$, it extends to Dwork complexes. Then, we get a commutative diagram

$$\begin{array}{ccccccccc}0\longrightarrow&\left(\mathscr{L}^\bullet_{(0,+)},D_{\hbar,\gamma S}\right)&\xrightarrow{\epsilon_{w,S}}&\left(\mathscr{L}^\bullet_{(0,+)},D_{\hbar,\gamma S}\right)&\xrightarrow{\rho_S}&\left(\Omega^\bullet_{C_S/\Bbbk},d\right)_{(0,0)}&\longrightarrow 0\\ &\big\downarrow&&\big\downarrow&&\big\downarrow&\\ 0\longrightarrow&\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)&\xrightarrow{\epsilon_{w,S}}&\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)&\xrightarrow{\rho_S}&\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{\mathcal{O}_\Bbbk}\Bbbk,d\right)_{(0,0)}&\longrightarrow 0,\end{array}$$

where the top row is exact by Proposition 3.10. Since the polynomial complexes are dense and the maps are all $\mathfrak{m}_\Bbbk$-adically continuous, the bottom row is exact as well. Since the relation in Lemma 3.9 holds on $\mathscr{L}^\bullet_{\hbar,(0,+)}$ by continuity, $\epsilon_{w,S}$ becomes the zero map on the Dwork cohomology. Therefore, we get an exact sequence

$$0\longrightarrow H^i\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)\xrightarrow{\rho_S}H^i_{\mathrm{MW}}\left(\overline{\mathcal{O}}_{C_S}/\Bbbk\right)\xrightarrow{\delta}H^{i+1}\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)\longrightarrow 0$$

together with (since $\overline{\mathcal{O}}_{C_S}=\mathbb{F}_q[x,y,S^{-1}]$ is geometrically connected)

$$H^0\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)=0,\quad H^1\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)=\Bbbk\cdot[dS].$$

Moreover, since Lemma 3.2 applies to the inclusions

$$\left(\mathscr{L}^\bullet_{\hbar,(0,+)},D_{\hbar,\gamma S}\right)\hookrightarrow\left(\Omega^\bullet_\hbar,D_{\hbar,\gamma S}\right)_{\deg_c=0}\hookrightarrow\left(\Omega^\bullet_\hbar,D_{\hbar,\gamma S}\right),$$

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we get the desired vanishing of $H^1(\Omega^\bullet_\hbar, D_{\hbar,\gamma S})$. On the other hand, we have

$$\operatorname{val}_p\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right) \geq -\log_p m$$

for every positive integer m so $\rho_S\chi$, $\rho D_{\hbar,\gamma S}\chi$, and $\rho_S\theta_w\chi$ in Lemmas 3.14 and 3.15, all converge as maps from $\mathscr{L}^\bullet_{\hbar,(0,+)}$ to $\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk$. Therefore, the argument as in Proposition 3.16 works in p-adic setting to give isomorphisms

$$\delta : H^i_{\mathrm{MW}}\left(\overline{\mathcal{O}}_B/\Bbbk\right) \xrightarrow{\sim} H^{i+1}\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right)$$

$$\rho_S : H^i\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) \xrightarrow{\sim} \frac{dS}{S} \wedge H^{i-1}_{\mathrm{MW}}\left(\overline{\mathcal{O}}_B/\Bbbk\right).$$

Here, we use the identification of Proposition 4.7. Then ρ_S induces an isomorphism

$$\rho_S : H^i\left(\Omega^\bullet_\hbar, D_{\hbar,\gamma S}\right) \xrightarrow{\sim} H^{i-1}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{S}}) \oplus H^{i-2}_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{S}})$$

for every $i \geq 2$. Note that here we use the canonical isomorphism

$$H^\bullet_{\mathrm{rig}}(\mathbb{P}^n \setminus X_{\overline{S}}) \cong H^\bullet_{\mathrm{MW}}(\mathbb{P}^n \setminus X_{\overline{S}})$$

which exists because $\mathbb{P}(\mathcal{E}) \setminus X_{\overline{S}}$ is smooth affine. Finally, there is an isomorphism

$$\varphi^* : H^\bullet_{\mathrm{rig}}(\mathbb{P}^n_{\mathbb{F}_q} \setminus X_{\overline{G}}) \xrightarrow{\sim} H^\bullet_{\mathrm{rig}}(\mathbb{P}(\mathcal{E}) \setminus X_{\overline{S}})$$

as we have observed in (2.3). Therefore, the proposition follows. ∎

The following corollary is a generalization of Monsky's remark in [26, p. 115].

Corollary 4.12. *With the assumptions in Proposition 4.11, if $X_{\overline{G}} \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ is a smooth complete intersection, then the inclusion*

$$\left(\Omega^\bullet_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) \hookrightarrow \left(\Omega^\bullet_\hbar, D_{\hbar,\gamma S}\right)$$

is a quasi-isomorphism.

Proof. By Lemma 3.2, it suffices to show that the inclusion

$$\left(\mathscr{L}^\bullet_{(0,+)}, D_{\hbar,\gamma S}\right) \hookrightarrow \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right)$$

is a quasi-isomorphism. By Propositions 3.16 and 4.11 together with its proof, this follows if we show that the inclusion

$$\left(\Omega^\bullet_{C_S/\Bbbk}, d\right) \hookrightarrow \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right)$$

is a quasi-isomorphism. Since $X_{\underline{G}}$ is smooth and proper,

$$H^\bullet_{\mathrm{dR}}(X_G) \cong H^\bullet_{\mathrm{cris}}(X_{\overline{G}}) \cong H^\bullet_{\mathrm{rig}}(X_{\overline{G}})$$

and these isomorphisms are compatible with the Gysin sequences for $H^\bullet_{\mathrm{dR}}$ and $H^\bullet_{\mathrm{rig}}$, we conclude that the above inclusion of algebraic de Rham complexes coming from $\mathcal{O}_\Bbbk[x,y,S^{-1}] \hookrightarrow C_S^\dagger$ is a quasi-isomorphism. ∎

REMARK 4.13. The condition on γ given in Proposition 4.11:

$$0 \leq \mathrm{val}_p \gamma \leq \frac{1}{p-1}$$

guarantees that

$$C(\hbar\gamma) := \Bbbk\{x_0, \ldots, x_n, \hbar\gamma y_1, \ldots, \hbar\gamma y_k\}$$

is a subring of $C(\hbar)$. Then

$$\Omega^{\bullet}_{\hbar\gamma} := C(\hbar\gamma) \otimes_{\Bbbk[x,y]} \Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}$$

is an $C(\hbar\gamma)$-submodule of $\Omega^{\bullet}_{\hbar}$. Arguing as in the proof of Proposition 4.11, we see that all inclusions

$$\left(\Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}, D_{\hbar,\gamma S}\right) \subseteq \left(\Omega^{\bullet}_{\hbar\gamma}, D_{\hbar,\gamma S}\right) \subseteq \left(\Omega^{\bullet}_{\hbar}, D_{\hbar,\gamma S}\right)$$

are quasi-isomorphisms. On the other hand, $C(\hbar\gamma)$ admits a filtration

$$F^e C(\hbar\gamma) := \left\{ \sum_{(u,v)\in\mathbb{Z}^{\oplus N}_{\geq 0}} a_{u,v} (\hbar\gamma)^{|v|} x^u y^v \in C(\hbar) \;\middle|\; a_{u,v} \in \pi^e \mathcal{O}_{\Bbbk} \right\}$$

which induces a ring isomorphism

$$\mathbb{F}_q[x,y] \cong \frac{\mathcal{O}_{\Bbbk}\{x,y\}}{\pi\mathcal{O}_{\Bbbk}\{x,y\}} \cong \frac{F^0 C(\hbar\gamma)}{F^1 C(\hbar\gamma)}.$$

The filtration on $C(\hbar\gamma)$ extends to

$$\Omega^{\bullet}_{\hbar\gamma} := C(\hbar\gamma) \otimes_{\Bbbk[x,y]} \Omega^{\bullet}_{\Bbbk[x,y]/\Bbbk}$$

given as follows:

$$F^e \Omega^m_{\hbar\gamma} := \bigoplus_{i+j=m} \bigoplus_{\substack{0\leq\alpha_1<\cdots<\alpha_i\leq n \\ 1\leq\beta_1<\cdots<\beta_j\leq k}} (\hbar\gamma)^j F^e C(\hbar\gamma) dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_i} \wedge dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_j}.$$

Then the above ring isomorphism extends to the isomorphism

$$\left(\frac{F^0 \Omega^{\bullet}_{\hbar\gamma}}{F^1 \Omega^{\bullet}_{\hbar\gamma}}, D_{\hbar,\gamma S} \right) \xrightarrow{\sim} \left(\Omega^{\bullet}_{\mathbb{F}_q[x,y]/\mathbb{F}_q}, d\overline{S} \wedge - \right),$$

where we denote $\overline{S} \in \mathbb{F}_q[x,y]$ the reduction of S. By this observation, we may apply [5, Prop. A.2] to lift a basis for the cohomology of $\left(\Omega^{\bullet}_{\mathbb{F}_q[x,y]/\mathbb{F}_q}, d\overline{S} \wedge -\right)$ to get a basis for the cohomology of $(\Omega^{\bullet}_{\hbar}, D_{\hbar,\gamma S})$ whenever the cohomology over the residue field is finite-dimensional. For the detailed computation over the residue field when $\overline{G}_1, \ldots, \overline{G}_k$ define a smooth projective complete intersection, see [4].

§5. Operators on p-adic analytic cohomologies

In this section, we will give more precise statement of Theorem 1.3 together with its detailed proof. This section is a generalization of [19, §III]. We begin with reviewing some

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necessary constructions. For each $i \geq 1$, the equation

$$t+\frac{t^p}{p}+\frac{t^{p^2}}{p^2}+\cdots+\frac{t^{p^i}}{p^i}=0$$

has a solution γ_i with

$$\mathrm{val}_p \gamma_i = \frac{1}{p-1}.$$

For each choice of γ_i, the corresponding *Dwork's splitting functions* is defined to be

$$\theta_i(t) := \exp\left(\gamma_i t + \frac{(\gamma_i t)^p}{p} + \frac{(\gamma_i t)^{p^2}}{p^2} + \cdots + \frac{(\gamma_i t)^{p^i}}{p^i}\right).$$

Each θ_i has integral coefficients and converges for

$$\mathrm{val}_p t > -\frac{1}{p-1} + \frac{1}{p^{i+1}}\left(i+1+\frac{1}{p-1}\right).$$

In this section, we will take $\gamma = \gamma_1$ so that $\gamma^{p-1} = -p$ and

$$\theta_1(t) = \exp\left(\gamma t + \frac{\gamma^p t^p}{p}\right) = \exp(\gamma t - \gamma t^p).$$

If $q = p^a$, then

$$\exp(\gamma t - \gamma t^q) = \theta_1(t)\theta_1(t^p)\cdots\theta(t^{p^{a-1}})$$

converges for

$$\mathrm{val}_p t > \frac{1-p}{pq}.$$

In this section, we denote for a nonzero $F \in \mathcal{O}_{\Bbbk}[x,y]$ by $C_F^\dagger := \mathcal{O}_{\Bbbk}\{x,y,F^{-1}\}^\dagger$, the corresponding weakly complete finitely generated algebra over $\mathcal{O}_{\Bbbk}$. Still we mainly consider $S := y_1G_1 + \cdots + y_kG_k$, where each G_i is not divisible by the uniformizer $\pi \in \mathfrak{m}_{\Bbbk}$, in which case, we denote ρ_S the cochain map as in Proposition 4.11. The following lemma is an analog of [19, Lem. 2.13].

LEMMA 5.1. *Let $S, T \in \mathcal{O}_{\Bbbk}[x,y]$ be homogeneous with respect to $\deg_w$. If S and T are of $\deg_w = 1$, and $S - T \equiv 0 \bmod \pi$, then*

$$\begin{array}{ccc} \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \\ {\scriptstyle \exp(\gamma S - \gamma T)}\downarrow & & \downarrow\wr \\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma T}\right) & \xrightarrow[\rho_T]{} & \left(\Omega^\bullet_{C_T^\dagger/(\mathcal{O}_{\Bbbk},\mathfrak{m}_{\Bbbk})} \otimes_{\mathcal{O}_{\Bbbk}} \Bbbk, d\right) \end{array}$$

is commutative.

Proof. Note that if $S - T \equiv 0 \bmod \pi$, then

$$\frac{1}{T} = \frac{1}{S\left(1-\dfrac{S-T}{S}\right)} = \sum_{m\geq 0} \frac{(S-T)^m}{S^{m+1}}$$

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converges in $C_S^\dagger$ so $C_S^\dagger \cong C_T^\dagger$ are canonically identified. Then the commutativity follows from direct computation:

$$\begin{aligned}
&\rho_T\left(\exp(\gamma S-\gamma T)x^u(\hbar y)^v dx_\alpha\wedge dy_\beta\right)=\rho_T\left(\sum_{m\geq 0}\frac{\gamma^m}{m!}(S-T)^m x^u(\hbar y)^v dx_\alpha\wedge dy_\beta\right)\\
&=\sum_{m\geq 0}\frac{\gamma^m}{m!}(-1)^{m+|v|+|\beta|-1}(m+|v|+|\beta|-1)!\frac{(S-T)^m}{\gamma^m T^m}\frac{\hbar^{|v|}x^u y^v}{\gamma^{|v|}T^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}T^{|\beta|}}\\
&=(-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\sum_{m\geq 0}\binom{-|v|-|\beta|}{m}\left(\frac{S-T}{T}\right)^m\frac{\hbar^{|v|}x^u y^v}{\gamma^{|v|}T^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}T^{|\beta|}}\\
&=(-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\left(1+\frac{S-T}{T}\right)^{-|v|-|\beta|}\frac{\hbar^{|v|}x^u y^v}{\gamma^{|v|}T^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}T^{|\beta|}}\\
&=(-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{T^{|v|+|\beta|}}{S^{|v|+|\beta|}}\frac{\hbar^{|v|}x^u y^v}{\gamma^{|v|}T^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}T^{|\beta|}}\\
&=(-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{\hbar^{|v|}x^u y^v}{\gamma^{|v|}S^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}S^{|\beta|}}\\
&=\rho_S\left(x^u(\hbar y)^v dx_\alpha\wedge dy_\beta\right).
\end{aligned}$$

∎

5.1 The Frobenius operator

Denote Fr the endomorphism on $C_S^\dagger$ lifting the qth power endomorphism over the residue field such that

$$\mathrm{Fr}: C_S^\dagger \longrightarrow C_S^\dagger \qquad (x_i,y_j)\longmapsto (x_i^q,y_j^q)\,.$$

This map is injective and extends to a cochain map

$$\mathrm{Fr}:\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)\longrightarrow\left(\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right).$$

Note that the above Fr sends the bidegree (c,w)-subspace to the bidegree (qc,qw)-subspace. Hence, our Fr restricts to the bidegree $(0,0)$-subcomplex:

$$\mathrm{Fr}:\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)_{(0,0)}\longrightarrow\left(\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)_{(0,0)}.$$

On the other hand, denote

$$\Phi_q: C(\hbar)\longrightarrow C(\hbar)\qquad \sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus N}}a_{u,v}\hbar^{|v|}x^u y^v\longmapsto\sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus N}}a_{u,v}\hbar^{|v|}x^{qu}y^{qv},$$

then Φ_q satisfies

$$z_i\frac{\partial}{\partial z_i}\circ\Phi_q=q\Phi_q\circ z_i\frac{\partial}{\partial z_i}.$$

Using this, we may extend Φ_q to the cochain map as follows:

$$\Phi_{q,\mathbb{P}^n_\Bbbk}:(\Omega^\bullet_\hbar,D_{\hbar,0})\longrightarrow(\Omega^\bullet_\hbar,D_{\hbar,0})\qquad f\,dx_\alpha\wedge dy_\beta\longmapsto\Phi_q(f)dx_\alpha^q\wedge dy_\beta^q$$

with the $\Bbbk$-linearity. We may write

$$\begin{aligned}\Phi_{q,\mathbb{P}^n_\Bbbk}(f\,dx_\alpha\wedge dy_\beta) &= \Phi_q(f)q^{|\alpha|}x_\alpha^{q-1}dx_\alpha\wedge q^{|\beta|}y_\beta^{q-1}dy_\beta\\ &= \frac{1}{x_\alpha y_\alpha}\Phi_q(x_\alpha y_\beta f)q^{|\alpha|}dx_\alpha\wedge q^{|\beta|}dy_\beta,\end{aligned}$$

where we extend convention (1.4) to monomials:

$$\begin{aligned}x_\alpha &:= x_{\alpha_1}\dots x_{\alpha_i}, & dx_\alpha &:= dx_{\alpha_1}\wedge\dots\wedge dx_{\alpha_i},\\ y_\beta &:= y_{\beta_1}\dots y_{\beta_j}, & dy_\beta &:= dy_{\beta_1}\wedge\dots\wedge dy_{\beta_j}.\end{aligned}$$

For $S := y_1G_1+\cdots+y_kG_k$ as before, define

$$\Phi_{q,S} : (\Omega^\bullet_\hbar, D_{\hbar,\gamma S}) \longrightarrow (\Omega^\bullet_\hbar, D_{\hbar,\gamma S})$$

by the formal identity

$$\Phi_{q,S} := \exp\left(-\frac{\gamma}{\hbar}S(x,\hbar y)\right)\circ\Phi_{q,\mathbb{P}^n_\Bbbk}\circ\exp\left(\frac{\gamma}{\hbar}S(x,\hbar y)\right),$$

which converges because we can rewrite

$$\begin{aligned}\Phi_{q,S} &= \exp(\gamma\mathrm{Fr}(S)-\gamma S)\circ\Phi_{q,\mathbb{P}^n_\Bbbk}\\ &= \left(\prod_{i=1}^k \exp\left(\gamma y_i^q\mathrm{Fr}_x G_i-\gamma y_i^q G_i^q\right)\exp\left(\gamma y_i^q G_i^q-\gamma y_i G_i\right)\right)\circ\Phi_{q,\mathbb{P}^n_\Bbbk}\end{aligned}$$

and the final expression converges. Since we may formally write

$$D_{\hbar,\gamma S} = \exp\left(-\frac{\gamma}{\hbar}S(x,\hbar y)\right)\circ D_{\hbar,0}\circ\exp\left(\frac{\gamma}{\hbar}S(x,\hbar y)\right)$$

$\Phi_{q,S}$ is still a cochain map. Now, we may compare Fr and $\Phi_{q,S}$ via ρ_S.

PROPOSITION 5.2. *There is a commutative diagram*

$$\begin{array}{ccc}\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{\mathcal{O}_\Bbbk}\Bbbk, d\right)\\ \downarrow{\scriptstyle q\Phi_{q,S}} & & \downarrow{\scriptstyle \mathrm{Fr}}\\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{\mathcal{O}_\Bbbk}\Bbbk, d\right).\end{array}$$

Proof. We will follow Katz's computation in the proof of [19, Th. 2.14] and [19, Th. 2.8]. Since the Frobenius on $\mathbb{F}_q[x,y]$ can be decomposed into

$$\begin{array}{ccccc}\mathbb{F}_q[x]\otimes_{\mathbb{F}_q}\mathbb{F}_q[y] & \xrightarrow{(\cdot)^q\otimes_{\mathbb{F}_q}1} & \mathbb{F}_q[x]^{(q)}\otimes_{\mathbb{F}_q}\mathbb{F}_q[y] & \xrightarrow{1\otimes_{\mathbb{F}_q}(\cdot)^q} & \mathbb{F}_q[x]^{(q)}\otimes_{\mathbb{F}_q}\mathbb{F}_q[y]^{(q)}\\ \downarrow\wr & & & & \downarrow\wr\\ \mathbb{F}_q[x,y] & & \xrightarrow[(\cdot)^q]{} & & \mathbb{F}_q[x,y]^{(q)},\end{array}$$

where the superscript (q) on each ring means that $\mathbb{F}_q$ acts by qth power. Denote the lifting of each factor by

$$\mathrm{Fr}_x : C_S^\dagger \longrightarrow C_S^\dagger \qquad \mathrm{Fr}_y : C_S^\dagger \longrightarrow C_S^\dagger.$$

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We abuse notation to denote $\mathrm{Fr}_G S := y_1 G_1^q + \cdots + y_k G_k^q$. Then there is a diagram

$$\begin{array}{ccc}
\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\
\downarrow {\scriptstyle \Phi^x_{q,\mathbb{P}^n_\Bbbk}} & & \downarrow {\scriptstyle \mathrm{Fr}_x} \\
\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma \mathrm{Fr}_x S}\right) & \xrightarrow{\rho_{\mathrm{Fr}_x S}} & \left(\Omega^\bullet_{C^\dagger_{\mathrm{Fr}_x S}/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\
\downarrow {\scriptstyle \exp(\gamma \mathrm{Fr}_x S - \gamma \mathrm{Fr}_G S)} & & \downarrow \wr \\
\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma \mathrm{Fr}_G S}\right) & \xrightarrow{\rho_{\mathrm{Fr}_G S}} & \left(\Omega^\bullet_{C^\dagger_{\mathrm{Fr}_G S}/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\
\downarrow {\scriptstyle q\exp(\gamma \mathrm{Fr}_y \mathrm{Fr}_G S - \gamma S)\circ \Phi^y_{q,\mathbb{P}^n_\Bbbk}} & & \downarrow {\scriptstyle \mathrm{Fr}_y} \\
\left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right),
\end{array}$$

where in the second and the third rows, we set $\deg_c y_i := -qd_i$. The top square is commutative because

$$\begin{aligned}
&(\mathrm{Fr}_x \circ \rho_S)(x^u y^v dx_\alpha \wedge dy_\beta) \\
&\quad = (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\,\mathrm{Fr}_x\left(\frac{x^u y^v}{\gamma^{|v|} S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} S^{|\beta|}}\right) \\
&\quad = (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\,\frac{x^u y^v}{\gamma^{|v|} \mathrm{Fr}_x S^{|v|}} \frac{q^{|\alpha|} x_\alpha^{q-1} dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} \mathrm{Fr}_x S^{|\beta|}} \\
&\quad = \rho_{\mathrm{Fr}_x S}\left(q^{|\alpha|} x_\alpha^{q-1} x^u y^v dx_\alpha \wedge dy_\beta\right) \\
&\quad = \left(\rho_{\mathrm{Fr}_x S} \circ \Phi^x_{q,\mathbb{P}^n_\Bbbk}\right)(x^u y^v dx_\alpha \wedge dy_\beta).
\end{aligned}$$

The middle square is commutative by Lemma 5.1. For the bottom square, we first compute

$$\begin{aligned}
&(\mathrm{Fr}_y \circ \rho_{\mathrm{Fr}_G S})(x^u y^v dx_\alpha \wedge dy_\beta) \\
&\quad = (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\,\mathrm{Fr}_y\left(\frac{x^u y^v}{\gamma^{|v|} \mathrm{Fr}_G S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} \mathrm{Fr}_G S^{|\beta|}}\right) \\
&\quad = (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\,\frac{x^u y^{qv}}{\gamma^{|v|} \mathrm{Fr}_y \mathrm{Fr}_G S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{q^{|\beta|} y_\beta^{q-1} dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} \mathrm{Fr}_y \mathrm{Fr}_G S^{|\beta|}}.
\end{aligned}$$

If we write

$$\exp(\gamma t^q - \gamma t) = \sum_{m \geq 0} a_m t^m,$$

then

$$\begin{aligned}
&\rho_S\left(q\exp(\gamma \mathrm{Fr}_y \mathrm{Fr}_G S - \gamma S)\Phi^y_{q,\mathbb{P}^n_\Bbbk}(x^u y^v dx_\alpha \wedge dy_\beta)\right) \\
&\quad = \rho_S\left(q \sum_{m \in \mathbb{Z}_{\geq 0}^{\oplus k}} a_{m_1} \ldots a_{m_k} (y_1 G_1)^{m_1} \ldots (y_k G_k)^{m_k} x^u y^{qv} dx_\alpha \wedge q^{|\beta|} y_\beta^{q-1} dy_\beta\right)
\end{aligned}$$

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$$= q \sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} a_m (-1)^{|m|+q|v|+q|\beta|-1}(|m|+q|v|+q|\beta|-1)!\frac{(yG)^m}{\gamma^{|m|}S^{|m|}}\frac{x^u y^{q|v|}}{\gamma^{q|v|}S^{q|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{q^{|\beta|}y_\beta^{q-1}dy_\beta}{\hbar^{|\beta|}\gamma^{q|\beta|}S^{q|\beta|}}$$

$$= q \sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} a_m (-1)^{|m|+q|v|+q|\beta|-1}(|m|+q|v|+q|\beta|-1)!\frac{(yG)^m}{\gamma^{|m|}S^{|m|}}\frac{\gamma^{|v|+|\beta|}\mathrm{Fr}_y\mathrm{Fr}_G S^{|v|+|\beta|}}{\gamma^{q|v|+q|\beta|}S^{q|v|+q|\beta|}}\times$$
$$\times\,\mathrm{Fr}_y\left(\frac{x^u y^v}{\gamma^{|v|}\mathrm{Fr}_G S^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}\mathrm{Fr}_G S^{|\beta|}}\right).$$

Hence, the commutativity follows if we show that

$$\begin{aligned} &q \sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} a_m (-1)^{|m|+q|v|+q|\beta|-1}(|m|+q|v|+q|\beta|-1)!\frac{(yG)^m}{\gamma^{|m|}S^{|m|}} \\ &\quad = (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{\gamma^{q|v|+q|\beta|}S^{q|v|+q|\beta|}}{\gamma^{|v|+|\beta|}\mathrm{Fr}_y\mathrm{Fr}_G S^{|v|+|\beta|}}. \end{aligned} \tag{5.1}$$

To do this, consider

$$\begin{aligned} f(t) &:= t^{qw-1}\exp\left(\frac{\mathrm{Fr}_y\mathrm{Fr}_G S\cdot t^q}{\gamma^{q-1}S^q}-t\right) \\ &= t^{qw-1}\prod_{i=1}^k \exp\left(\frac{(y_iG_i)^q t^q}{\gamma^{q-1}S^q}-\frac{y_iG_it}{S}\right) \\ &= t^{qw-1}\prod_{i=1}^k\sum_{m\geq 0} a_m\frac{(y_iG_i)^m t^m}{S^m} = \sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} a_m\frac{(yG)^m t^{|m|+qw-1}}{S^{|m|}} \end{aligned}$$

and

$$g(t) := \sum_{\ell\geq 0}(-1)^\ell\frac{d^\ell f}{dt^\ell}.$$

Note that g satisfies

$$g(0) = \sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} a_m(-1)^{|m|+qw-1}(|m|+qw-1)!\frac{(yG)^m}{S^{|m|}}.$$

Moreover, g is by definition a formal solution of the differential equation

$$g+\frac{dg}{dt} = f$$

which converges for $\mathrm{val}_p t > \dfrac{1}{p-1}-\dfrac{p-1}{pq}$. Since the only solution of

$$g+\frac{dg}{dt} = 0$$

is a constant multiple of $\exp(-t)$ which converges only for $\mathrm{val}_p t > \dfrac{1}{p-1}$, g is the unique power series solution. On the other hand, there is a solution of the form

$$h(t^q)\exp\left(\frac{\mathrm{Fr}_y\mathrm{Fr}_G S\cdot t^q}{\gamma^{q-1}S^q}-t\right).$$

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After substituting and dividing by the exponential, we get

$$h(t^q)+\frac{d}{dt}h(t^q)+\left(\frac{q\mathrm{Fr}_y\mathrm{Fr}_G S\cdot t^{q-1}}{\gamma^{q-1}S^q}-1\right)h(t^q)=t^{qw-1},$$

which is equivalent to

$$\frac{q\mathrm{Fr}_y\mathrm{Fr}_G S}{\gamma^{q-1}S^q}t^q h(t^q)+t\frac{d}{dt}h(t^q)=t^{qw}.$$

By change of variables, it is equivalent to

$$\frac{q\mathrm{Fr}_y\mathrm{Fr}_G S}{\gamma^{q-1}S^q}th(t)+qt\frac{d}{dt}h(t)=t^{w}.$$

Hence, we may write

$$h(t)=\left(1+\frac{\gamma^{q-1}S^q}{\mathrm{Fr}_y\mathrm{Fr}_G S}\frac{d}{dt}\right)^{-1}\left(\frac{\gamma^{q-1}S^q}{q\mathrm{Fr}_y\mathrm{Fr}_G S}t^{w-1}\right)$$

with the condition

$$h(0)=(-1)^{w-1}(w-1)!\frac{\gamma^{qw}S^{qw}}{q\gamma^w\mathrm{Fr}_y\mathrm{Fr}_G S^w}.$$

Therefore, since $g(0)=h(0)$ by the observation so far,

$$\sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}}a_m(-1)^{|m|+qw-1}(|m|+qw-1)!\frac{(yG)^m}{S^{|m|}}=(-1)^{w-1}(w-1)!\frac{\gamma^{qw}S^{qw}}{q\gamma^w\mathrm{Fr}_y\mathrm{Fr}_G S^w}.$$

Substituting $w=|v|+|\beta|$, we get equality (5.1), and the proof is completed. ∎

5.2 The Dwork operator

Since $\mathrm{Fr}(C_S^\dagger)\subseteq C_S^\dagger$ is a finite locally free ring extension of integral domains, there is a cochain map

$$\mathrm{Tr}:\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)\longrightarrow\left(\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)$$

as in [32, Prop. 3.1]. Denote ψ the composite

$$\psi:\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)\xrightarrow{\mathrm{Tr}}\left(\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)\xrightarrow{\mathrm{Fr}^{-1}}\left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)},d\right)$$

that is, the unique map satisfying $\mathrm{Fr}\circ\psi=\mathrm{Tr}$. By the description of [32, Prop. 3.1], Tr on the differential forms fits into the commutative diagram

$$\begin{array}{ccccc}
\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} & \longrightarrow & \Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{C_S^\dagger}K(C_S^\dagger) & \xrightarrow{\sim} & \Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{\mathrm{Fr}(C_S^\dagger)}K(C_S^\dagger) \\
\Big\downarrow{\scriptstyle\mathrm{Tr}} & & & & \Big\downarrow{\scriptstyle 1\otimes_{\mathrm{Fr}(C_S^\dagger)}\mathrm{Tr}} \\
\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} & & \longrightarrow & & \Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}\otimes_{\mathrm{Fr}(C_S^\dagger)}K(\mathrm{Fr}(C_S^\dagger)),
\end{array}$$

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where the isomorphism comes from the inclusion $\Omega^\bullet_{\mathrm{Fr}(C_S^\dagger)/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \subseteq \Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}$, and $\mathrm{Tr}: K(C_S^\dagger) \to K(\mathrm{Fr}(C_S^\dagger))$ is the usual trace map for fields extends of finite degree. This gives a description of ψ:

$$\begin{aligned}\psi(f dx_\alpha \wedge dy_\beta) &= \psi\left(\frac{f}{x_\alpha^{q-1} y_\beta^{q-1}} \frac{dx_\alpha^q}{q^{|\alpha|}} \wedge \frac{dy_\beta^q}{q^{|\beta|}}\right) \\ &= \mathrm{Fr}^{-1}\left(\mathrm{Tr}\left(\frac{f}{x_\alpha^{q-1} y_\beta^{q-1}}\right) \frac{dx_\alpha^q}{q^{|\alpha|}} \wedge \frac{dy_\beta^q}{q^{|\beta|}}\right) \\ &= \mathrm{Fr}^{-1}\left(\mathrm{Tr}\left(\frac{f}{x_\alpha^{q-1} y_\beta^{q-1}}\right)\right) \frac{dx_\alpha}{q^{|\alpha|}} \wedge \frac{dy_\beta}{q^{|\beta|}},\end{aligned}$$

where, following (1.4), we denote

$$\begin{aligned} x_\alpha^q &:= x_{\alpha_1}^q \cdots x_{\alpha_i}^q, & dx_\alpha^q &:= dx_{\alpha_1}^q \wedge \cdots \wedge dx_{\alpha_i}^q, \\ y_\beta^q &:= y_{\beta_1}^q \cdots y_{\beta_j}^q, & dy_\beta^q &:= dy_{\beta_1}^q \wedge \cdots \wedge dy_{\beta_j}^q. \end{aligned}$$

By our choice of Fr in §5.1, Fr^{-1} sends the bidegree (qc,qw)-subspace to the bidegree (c,w)-subspace. Hence, the corresponding Tr restricts to the bidegree $(0,0)$-subcomplex; and hence, our ψ restricts to the bidegree $(0,0)$-subcomplex:

$$\psi : \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}, d\right)_{(0,0)} \longrightarrow \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)}, d\right)_{(0,0)}.$$

On the other hand, denote

$$\Psi_q : C(\hbar) \longrightarrow C(\hbar) \qquad \sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus N}} a_{u,v}\hbar^{|v|}x^u y^v \longmapsto \sum_{(u,v)\in\mathbb{Z}_{\geq 0}^{\oplus N}} a_{qu,qv}\hbar^{|v|}x^u y^v,$$

then Ψ_q satisfies

$$\Psi_q \circ z_i \frac{\partial}{\partial z_i} = q z_i \frac{\partial}{\partial z_i} \circ \Psi_q.$$

Using this, we may extend Ψ_q to the cochain map analogously to $\Phi_{q,\mathbb{P}^n_\Bbbk}$:

$$\Psi_{q,\mathbb{P}^n_\Bbbk} : (\Omega^\bullet_\hbar, D_{\hbar,0}) \longrightarrow (\Omega^\bullet_\hbar, D_{\hbar,0}) \qquad f dx_\alpha \wedge dy_\beta \longmapsto \frac{q^{n+k+1}}{x_\alpha y_\beta} \Psi_q(x_\alpha y_\beta f) \frac{dx_\alpha}{q^{|\alpha|}} \wedge \frac{dy_\beta}{q^{|\beta|}}.$$

For $S := y_1 G_1 + \cdots + y_k G_k$ as before, define

$$\Psi_{q,S} : (\Omega^\bullet_\hbar, D_{\hbar,\gamma S}) \longrightarrow (\Omega^\bullet_\hbar, D_{\hbar,\gamma S})$$

by the formal identity

$$\Psi_{q,S} := \exp\left(-\frac{\gamma}{\hbar} S(x,\hbar y)\right) \circ \Psi_{q,\mathbb{P}^n_\Bbbk} \circ \exp\left(\frac{\gamma}{\hbar} S(x,\hbar y)\right),$$

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which converges because we can rewrite

$$\begin{aligned}\Psi_{q,S} &= \Psi_{q,\mathbb{P}^n_{\Bbbk}} \circ \exp(\gamma S - \gamma \mathrm{Fr}(S)) \\ &= \Psi_{q,\mathbb{P}^n_{\Bbbk}} \circ \left(\prod_{i=1}^{k} \exp\left(\gamma y_i G_i - \gamma y_i^q G_i^q\right) \exp\left(\gamma y_i^q G_i^q - \gamma y_i^q \mathrm{Fr}_x G_i\right) \right)\end{aligned}$$

and the final expression converges. Hence, $\Psi_{q,S}$ is still a cochain map by the same reason as in §5.1. Now, we may compare ψ and $\Psi_{q,S}$ via ρ_S.

PROPOSITION 5.3. *There is a commutative diagram*

$$\begin{array}{ccc} \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\ {\scriptstyle q^{-1}\Psi_{q,S}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right). \end{array}$$

Proof. We will follow Katz's trick in the proof of [19, Th. 2.15]. We may decompose $\psi = \psi_x \circ \psi_y$ as in the case of $\mathrm{Fr} = \mathrm{Fr}_y \circ \mathrm{Fr}_x$. Hence, there is a diagram

$$\begin{array}{ccc} \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\ {\scriptstyle q^{-1}\Psi^y_{q,\mathbb{P}^n_\Bbbk} \circ \exp(\gamma S - \gamma \mathrm{Fr}_y \mathrm{Fr}_G S)}\downarrow & & \downarrow{\scriptstyle \psi_y} \\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma \mathrm{Fr}_G S}\right) & \xrightarrow{\rho_{\mathrm{Fr}_G S}} & \left(\Omega^\bullet_{C_{\mathrm{Fr}_G S}^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\ {\scriptstyle \exp(\gamma \mathrm{Fr}_G S - \gamma \mathrm{Fr}_x S)}\downarrow & & \downarrow{\scriptstyle \wr} \\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma \mathrm{Fr}_x S}\right) & \xrightarrow{\rho_{\mathrm{Fr}_x S}} & \left(\Omega^\bullet_{C_{\mathrm{Fr}_x S}^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right) \\ {\scriptstyle \Psi^x_{q,\mathbb{P}^n_\Bbbk}}\downarrow & & \downarrow{\scriptstyle \psi_x} \\ \left(\mathscr{L}^\bullet_{\hbar,(0,+)}, D_{\hbar,\gamma S}\right) & \xrightarrow{\rho_S} & \left(\Omega^\bullet_{C_S^\dagger/(\mathcal{O}_\Bbbk,\mathfrak{m}_\Bbbk)} \otimes_{\mathcal{O}_\Bbbk} \Bbbk, d\right), \end{array}$$

where in the second and the third rows, we set $\deg_c y_i = -q d_i$. The bottom square is commutative because

$$\begin{aligned} &(\psi_x \circ \rho_{\mathrm{Fr}_x S})(x^u y^v dx_\alpha \wedge dy_\beta) \\ &= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\psi_x \left(\frac{x^u y^v}{\gamma^{|v|} \mathrm{Fr}_x S^{|v|}} \frac{dx_\alpha}{\hbar^{|\alpha|}} \wedge \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}\mathrm{Fr}_x S^{|\beta|}} \right) \\ &= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\mathrm{Fr}_x^{-1} \left(\mathrm{Tr}_x \left(\frac{x^u}{x_\alpha^{q-1}} \right) \right) \frac{dx_\alpha}{\hbar^{|\alpha|} q^{|\alpha|}} \wedge \frac{y^v}{\gamma^{|v|} S^{|v|}} \frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|} S^{|\beta|}} \\ &= \rho_S \left(\mathrm{Fr}_x^{-1} \left(\mathrm{Tr}_x \left(\frac{x^u}{x_\alpha^{q-1}} \right) \right) \frac{dx_\alpha}{q^{|\alpha|}} \wedge y^v dy_\beta \right) \\ &= \rho_S \left(\frac{1}{x_\alpha} \mathrm{Fr}_x^{-1} (\mathrm{Tr}_x(x_\alpha x^u)) \frac{dx_\alpha}{q^{|\alpha|}} \wedge y^v dy_\beta \right) \end{aligned}$$

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$$
\begin{aligned}
&= \rho_S\left(\frac{q^{n+1}}{x_\alpha}\Psi_q(x_\alpha x^u)\frac{dx_\alpha}{q^{|\alpha|}}\wedge y^v dy_\beta\right)\\
&= \left(\rho_S\circ\Psi^x_{q,\mathbb{P}^n_{\Bbbk}}\right)(x^u y^v dx_\alpha\wedge dy_\beta).
\end{aligned}
$$

The middle square is commutative by Lemma 5.1. For the top square, we first compute

$$
\begin{aligned}
&(\psi_y\circ\rho_S)(x^u y^v dx_\alpha\wedge dy_\beta)\\
&\quad= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\psi_y\left(\frac{x^u y^v}{\gamma^{|v|}S^{|v|}}\frac{dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}\gamma^{|\beta|}S^{|\beta|}}\right)\\
&\quad= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\mathrm{Fr}_y^{-1}\left(\mathrm{Tr}_y\left(\frac{y^v}{y_\beta^{q-1}\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}\right)\right)\frac{x^u dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}q^{|\beta|}}\\
&\quad= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{1}{y_\beta}\mathrm{Fr}_y^{-1}\left(\mathrm{Tr}_y\left(\frac{y_\beta y^v}{\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}\right)\right)\frac{x^u dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}q^{|\beta|}}\\
&\quad= (-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{q^k}{y_\beta}\Psi^y_q\left(\frac{y_\beta y^v}{\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}\right)\frac{x^u dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}q^{|\beta|}}.
\end{aligned}
$$

If we write

$$
\exp(\gamma t-\gamma t^q)=\sum_{m\geq 0} b_m t^m,
$$

then, denoting $e_\beta := e_{\beta_1}+\cdots+e_{\beta_j}$ where $j=|\beta|$ and e_{β_i} is the β_ith standard basis for $\mathbb{Z}^{\oplus k}$,

$$
\begin{aligned}
&\rho_{\mathrm{Fr}_G S}\left(\frac{1}{q}\Psi^y_{q,\mathbb{P}^n_{\Bbbk}}(\exp(\gamma S-\gamma\mathrm{Fr}_y\mathrm{Fr}_G S)x^u y^v dx_\alpha\wedge dy_\beta)\right)\\
&\quad= \rho_{\mathrm{Fr}_G S}\left(\frac{1}{q}\Psi^y_{q,\mathbb{P}^n_{\Bbbk}}\left(\sum_{m\in\mathbb{Z}^{\oplus k}_{\geq 0}} b_m (yG)^m x^u y^v dx_\alpha\wedge dy_\beta\right)\right)\\
&\quad= \rho_{\mathrm{Fr}_G S}\left(\frac{1}{q}\frac{q^k}{y_\beta}\Psi^y_q\left(\sum_{m\in\mathbb{Z}^{\oplus k}_{\geq 0}} b_m (yG)^m y_\beta y^v\right)x^u dx_\alpha\wedge\frac{dy_\beta}{q^{|\beta|}}\right)\\
&\quad= \rho_{\mathrm{Fr}_G S}\left(\frac{1}{q}\frac{q^k}{y_\beta}\sum_{m\in\mathbb{Z}^{\oplus k}_{\geq 0}} b_{qm-v-e_\beta} y^m G^{qm-v-e_\beta} x^u dx_\alpha\wedge\frac{dy_\beta}{q^{|\beta|}}\right)\\
&\quad= \frac{q^k}{y_\beta}\frac{1}{q}\sum_{m\in\mathbb{Z}^{\oplus k}_{\geq 0}} b_{qm-v-e_\beta}(-1)^{|m|-1}(|m|-1)!\frac{y^m G^{qm-v-e_\beta}}{\gamma^{|m|}\mathrm{Fr}_G S^{|m|}}\frac{x^u dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}q^{|\beta|}}.
\end{aligned}
$$

Hence, the commutativity follows if we show that

$$
\begin{aligned}
&(-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\Psi^y_q\left(\frac{y_\beta y^v}{\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}\right)\frac{x^u dx_\alpha}{\hbar^{|\alpha|}}\wedge\frac{dy_\beta}{\hbar^{|\beta|}q^{|\beta|}}\\
&\quad= \frac{1}{q}\sum_{m\in\mathbb{Z}^{\oplus k}_{\geq 0}} b_{qm-v-e_\beta}(-1)^{|m|-1}(|m|-1)!\frac{y^m G^{qm-v-e_\beta}}{\gamma^{|m|}\mathrm{Fr}_G S^{|m|}}.
\end{aligned}
\tag{5.2}
$$

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For this, consider the space of power series in t with the "usual" growth condition

$$L := \left\{ f = \sum_{m \geq 0} f_m t^m \;\middle|\; \begin{array}{c} f_m \in \Bbbk[x,y] \\ \mathrm{val}_p(f_m) \geq bm + c \end{array} \right\}$$

for some fixed $b > 0$ and $c \in \mathbb{R}$, depending on each f, and denote $L^0 := tL$. Define

$$t\frac{\partial}{\partial t} : L \longrightarrow L^0 \qquad \Psi_q^{y,t} : L \longrightarrow L$$

in the usual way so that

$$\Psi_q^{y,t} \circ t\frac{\partial}{\partial t} = qt\frac{\partial}{\partial t} \circ \Psi_q^{y,t}.$$

Now, define for $f, g \in \Bbbk[x,y]$,

$$\Psi_{q,f,g}^{y,t} := \exp(-\gamma g t) \circ \Psi_q^{y,t} \circ \exp(\gamma f t), \quad \Psi_{q,f}^{y,t} := \Psi_{q,f,f}^{y,t}$$

$$D_f := \exp(-\gamma f t) \circ t\frac{\partial}{\partial t} \circ \exp(\gamma f t) = t\frac{\partial}{\partial t} + \gamma f t$$

so that there is a commutative diagram

$$\begin{array}{ccc} (L^0, D_S) & \xrightarrow{\Psi_{q,S}^{y,t}} & (L^0, D_S) \\ {\scriptstyle \exp(-\gamma \mathrm{Fr}_G S \cdot t + \gamma S t)} \downarrow & \searrow^{\Psi_{q,S,\mathrm{Fr}_G S}^{y,t}} & \downarrow {\scriptstyle \exp(-\gamma \mathrm{Fr}_G S \cdot t + \gamma S t)} \\ (L^0, D_{\mathrm{Fr}_G S}) & \xrightarrow{\Psi_{q,\mathrm{Fr}_G S}^{y,t}} & (L^0, D_{\mathrm{Fr}_G S}), \end{array} \tag{5.3}$$

where each (L^0, D_f) is regarded as a two term complex. On $L^0/D_f L^0$, we have

$$(\gamma f)^m t^{m+1} = -m(\gamma f)^{m-1} t^m = \cdots = (-1)^m m! t.$$

At this point, we use the growth condition on L^0. The differential equation

$$t\frac{\partial P}{\partial t} + \gamma f t P = t$$

with the condition $P \in t\Bbbk[x,y][[t]]$ has a unique power series solution

$$P = \frac{1 - \exp(-\gamma f)}{\gamma f}.$$

However, this does not belong to L^0 by the growth condition. Hence, $t \neq 0$ in $L^0/D_f L^0$ so it is a free $\Bbbk[x,y]$-module of rank 1 with basis $\{t\}$. For $\xi \in \Bbbk[x,y]$,

$$\begin{aligned} \Psi_{q,f}^{y,t}(\xi) &= \Psi_q^{y,t}\left(\sum_{m \geq 0} \frac{\gamma^m}{m!} (ft - f(x,y^q) \cdot t^q)^m \xi \right) \\ &= \Psi_q^{y,t}\left(\sum_{m \geq 0} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} (f(x,y^q) t^q)^{m-\ell} (ft)^\ell \xi \right) \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{m\geq 0}\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^{m-\ell}(ft)^{m-\ell}\Psi_q^{y,t}\left((ft)^{\ell}\xi\right)\\
&\equiv \Psi_q^y(\xi) \bmod t
\end{aligned}
$$

so we deduce that

$$
\Psi_{q,f}^{y,t}(D_f(1)) = qD_f\left(\Psi_{q,f}^{y,t}(1)\right) = qD_f(1) + qD_f(\omega)
$$

for some $\omega \in L^0$. Hence, $\Psi_{q,f}^{y,t}$ on L^0/D_fL^0 is merely

$$
L^0/D_fL^0 \longrightarrow L^0/D_fL^0 \qquad \xi t \longmapsto q\Psi_q^y(\xi)t.
$$

Therefore,

$$
\begin{aligned}
&\Psi_q^y\left((-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{y_\beta y^v}{\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}\right)t\\
&\quad= \frac{1}{q}\Psi_{q,S}^{y,t}\left((-1)^{|v|+|\beta|-1}(|v|+|\beta|-1)!\frac{y_\beta y^v}{\gamma^{|v|+|\beta|}S^{|v|+|\beta|}}t\right)\\
&\quad= \frac{1}{q}\Psi_{q,S,\mathrm{Fr}_G S}^{y,t}\left(y_\beta y^v t^{|v|+|\beta|}\right)\\
&\quad= \frac{1}{q}\Psi_q^{y,t}\left(\exp(\gamma St - \gamma \mathrm{Fr}_y \mathrm{Fr}_G S t^q) y_\beta y^v t^{|v|+|\beta|}\right)\\
&\quad= \frac{1}{q}\Psi_q^{y,t}\left(\sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} b_m (yG)^m y_\beta y^v t^{|m|+|v|+|\beta|}\right)\\
&\quad= \frac{1}{q}\sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} b_{qm-v-e_\beta} y^m G^{m-v-e_\beta} t^{|m|}\\
&\quad= \frac{1}{q}\sum_{m\in\mathbb{Z}_{\geq 0}^{\oplus k}} b_{qm-v-e_\beta}(-1)^{|m|-1}(|m|-1)!\frac{y^m G^{m-v-e_\beta}}{\gamma^{|m|}\mathrm{Fr}_G S^{|m|}}t,
\end{aligned}
$$

that is, (5.2) holds, and the proof is completed. ∎

§A Remarks on algebraic de Rham cohomology

Definition A.1. Given a map of schemes $X \to S$, define its *(relative) de Rham cohomology* to be

$$
H^n_{\mathrm{dR}}(X/S) := H^n\left(\mathbb{R}\Gamma(X, \Omega^\bullet_{X/S})\right),
$$

where $\Omega^\bullet_{X/S}$ is the algebraic de Rham complex.

Lemma A.2. *Given a map of affine schemes $X \to S$, if $A = \Gamma(S, \mathcal{O}_S)$, $X = \Gamma(X, \mathcal{O}_X)$, and $A \to B$ the corresponding ring map, then*

$$
\mathbb{R}\Gamma(X, \Omega^\bullet_{X/S}) \cong \Omega^\bullet_{B/A}
$$

in $\mathrm{D}(A)$*, the derived category of* A*. Consequently,*

$$H^n_{\mathrm{dR}}(X/S) \cong H^n(\Omega^\bullet_{B/A}, d),$$

where d *is the usual de Rham differential.*

Proof. Since $\Omega^p_{X/S}$ is quasicoherent $\mathcal{O}_X$-module and X is affine,

$$H^q(X, \Omega^p_{X/S}) = \begin{cases} \Gamma(X, \Omega^p_{X/S}), & \text{if} \quad q = 0, \\ 0, & \text{if} \quad q > 0. \end{cases}$$

In other words, $\Omega^\bullet_{X/S}$ is a bounded below complex of $\Gamma(X,-)$-acyclic objects so the canonical map

$$\Omega^\bullet_{B/A} \xrightarrow{\sim} \Gamma(X, \Omega^\bullet_{X/S}) \longrightarrow \mathbb{R}\Gamma(X, \Omega^\bullet_{X/S})$$

is an isomorphism. ∎

Remark A.3. To show the second assertion of Lemma A.2, one may argue with the Čech spectral sequence for a chosen covering $\mathcal{U}$ of X:

$$E_2^{p,q} = H^p\left(\mathrm{Tot}\left(\check{C}^\bullet(\mathcal{U}, \underline{H}^q(\Omega^\bullet_{X/S}))\right)\right) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/S}),$$

where $\underline{H}^q(\Omega^\bullet_{X/S})$ is the presheaf associate with U, a complex of abelian groups

$$H^q(U, \Omega^0_{X/S}) \longrightarrow H^q(U, \Omega^1_{X/S}) \longrightarrow \cdots$$

and Tot takes the total complex of a double complex. For this, one may even use the covering $\{\mathrm{Id}_X : X \to X\}$ to get the desired vanishing because X is affine.

In computing algebraic de Rham cohomology of affine schemes, one may rely on cosimplicial de Rham algebras. For this, we introduce some terminologies on (co-)simplicial objects. Let Δ be the simplex category and $\mathscr{C}$ a finitely bicomplete category, that is, $\mathscr{C}$ has finite limits and finite colimits. For $n \in \mathbb{N}$, denote $\Delta_{\leq n}$ the full subcategory of Δ consisting of $[0], \dots, [n]$ and the obvious inclusion

$$i_n : \Delta_{\leq n} \hookrightarrow \Delta \,.$$

Since $\mathscr{C}$ has finite limits and finite colimits, there are adjoint pairs

$$\begin{array}{c} \left[\Delta^{\mathrm{op}}_{\leq n}, \mathscr{C}\right] \\ i_{n!} \big\downarrow \ \big\uparrow i_n^* \ \big\downarrow i_{n*} \\ [\Delta^{\mathrm{op}}, \mathscr{C}] \end{array}$$

given as in [6, V. 7.1]. Using these, we introduce the following terminologies.

Definition A.4. Let Δ be the simplex category.

(1) The *nth truncation* is

$$\mathrm{tr}_n := i_n^* : [\Delta^{\mathrm{op}}, \mathscr{C}] \longrightarrow \left[\Delta^{\mathrm{op}}_{\leq n}, \mathscr{C}\right].$$

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(2) The *nth skeleton* is

$$\mathrm{sk}_n := i_{n!}i_n^* : [\Delta^{\mathrm{op}}, \mathscr{C}] \longrightarrow [\Delta^{\mathrm{op}}, \mathscr{C}].$$

(3) The *nth coskeleton* is

$$\mathrm{cosk}_n := i_{n*}i_n^* : [\Delta^{\mathrm{op}}, \mathscr{C}] \longrightarrow [\Delta^{\mathrm{op}}, \mathscr{C}].$$

A simplicial object X in $\mathscr{C}$ is said to be *n-skeletal* (resp. *n-coskeletal*) if X is isomorphic to its nth skeleton (resp. nth coskeleton).

In our context, we will only work with 0-coskeletal simplicial objects. Namely, given a map of schemes $X \to S$, that is, an object in $\mathrm{Sch}_{/S}$, the category of S-schemes, we may regard it as a constant simplicial object in $\mathrm{Sch}_{/S}$. Since $\mathrm{Sch}_{/S}$ has finite limits, we may take its 0th coskeleton, which will be given by

$$\Delta^{\mathrm{op}} \longrightarrow \mathrm{Sch}_{/S} \qquad [n] \longmapsto X \times_S X \times_S \cdots \times_S X \quad (n+1 \text{ times}),$$

where the ith boundary map forgets ith factor and the ith degeneracy map duplicates the ith factor, counted from 0. When $X \to S$ is a map of affine schemes corresponding to a ring map $A \to B$, the above simplicial object defines a cosimplicial A-algebra:

$$(B/A)^\bullet : \Delta \longrightarrow \mathrm{CAlg}_A \qquad [n] \longmapsto B \otimes_A B \otimes_A \cdots \otimes_A B \quad (n+1 \text{ times}).$$

If, furthermore, $A \to B$ is a $\Bbbk$-algebra map over a ground ring $\Bbbk$, then we may take the de Rham complex degreewisely to get a cosimplicial de Rham algebra:

$$\Omega^\bullet_{(B/A)^\bullet/\Bbbk} : \Delta \longrightarrow \mathrm{CDGA}_{\Bbbk}^{\geq 0} \qquad [n] \longmapsto \Omega^\bullet_{(B/A)^n/\Bbbk}.$$

LEMMA A.5. *If $A \to B$ is an étale $\Bbbk$-algebra map, then the associated cosimplicial de Rham algebra $\Omega^\bullet_{(B/A)^\bullet/\Bbbk}$ is 0-coskeletal (in the opposite category).*

Proof. Recall that for $n \geq 0$,

$$(B/A)^n = B \otimes_A B \otimes_A \cdots \otimes_A B \quad (n+1 \text{ times})$$

so $A \to (B/A)^n$ remains étale for every $n \geq 0$. Hence, the exact sequence

$$0 \longrightarrow (B/A)^n \otimes_A \Omega^1_{A/\Bbbk} \longrightarrow \Omega^1_{(B/A)^n/\Bbbk} \longrightarrow \Omega^1_{(B/A)^n/A} \longrightarrow 0$$

together with $\Omega^1_{(B/A)^n/A} = 0$ shows that

$$\Omega^1_{(B/A)^n/\Bbbk} \cong (B/A)^n \otimes_A \Omega^1_{A/\Bbbk}.$$

This, together with the flatness of $A \to (B/A)^n$, gives

$$\Omega^\bullet_{(B/A)^\bullet/\Bbbk} \cong (B/A)^\bullet \otimes_A \Omega^\bullet_{A/\Bbbk}$$

so the assertion follows. ∎

PROPOSITION A.6. *Let $\Bbbk$ be a ring. If $A \to B$ is a faithfully flat étale $\Bbbk$-algebra, then*

$$\Omega^\bullet_{A/\Bbbk} \longrightarrow \Omega^\bullet_{(B/A)^\bullet/\Bbbk}$$

is a 0-coskeletal cosimplicial resolution.

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Proof. Being a 0-coskeletal object follows immediately from Lemma A.5. Being a cosimplicial resolution means that the given map induces a quasi-isomorphism of cochain complexes:

$$\Omega^\bullet_{A/\Bbbk} \longrightarrow \mathrm{Tot}\left(C^\bullet \Omega^\bullet_{(B/A)^\bullet/\Bbbk}\right),$$

where $C^\bullet$ takes the unnormalized complex (see, e.g., [33, Def. 8.2.1]) in the cosimplicial direction, and Tot takes its total complex. In fact, the total complex will be a Čech–de Rham complex of $\Omega^\bullet_{A/\Bbbk}$ with respect to the covering $\{\mathrm{Spec}\, B \to \mathrm{Spec}\, A\}$ which is surjective as $A \to B$ is faithfully flat. Therefore, the total complex computes the algebraic de Rham cohomology of A over $\Bbbk$, and the above map becomes the augmentation map. ∎

Example A.7. Given an affine scheme X and a finite affine open covering $\{U_i\}_{i\in I}$ of X, the induced map

$$U := \coprod_{i\in I} U_i \longrightarrow X$$

is a faithfully flat étale map of affine schemes. Here, the finiteness of I is necessary for the coproduct to be affine. Then the total complex induced from the cosimplicial de Rham algebra $\Omega^\bullet_{(U/X)_\bullet/\Bbbk}$ will be the Čech–de Rham complex with respect to the Zariski cover $\{U_i\}_{i\in I}$ of X. However, being 0-coskeletal cosimplicial objects, maps of such de Rham algebras are determined at the level of 0th truncation:

$$\prod_{i\in I} \Omega^\bullet_{U_i/\Bbbk}.$$

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