LMS J. Comput. Math. 17 (1) (2014) 259–272
doi:10.1112/S1461157013000375

A symmetric C^3 non-stationary subdivision scheme

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Abstract

This paper proposes a new family of symmetric 4-point ternary non-stationary subdivision schemes that can generate the limit curves of C^3 continuity. The continuity of this scheme is higher than the existing 4-point ternary approximating schemes. The proposed scheme has been developed using trigonometric B-spline basis functions and analyzed using the theory of asymptotic equivalence. It has the ability to reproduce or regenerate the conic sections, trigonometric polynomials and trigonometric splines as well. Some graphical and numerical examples are being considered, by choosing an appropriate tension parameter $0 < \alpha < \pi/3$, to show the usefulness of the proposed scheme. Moreover, the Hölder regularity and the reproduction property are also being calculated.

1. *Introduction*

The subdivision scheme has become one of the most important ingredients for creating smooth curves (surfaces) in graphic modelling and computer geometric designing etc. Seeing that, it has a simple and attractive approach to producing curves (surfaces) from initial control polygons (mesh) by subdividing them according to some refining rules, recursively. These refining rules take the initial polygon (mesh) to produce a sequence of finer polygons (mesh) converging to a smooth limiting curve (surface).

One of the important capabilities of non-stationary schemes as compared to stationary schemes is the reproduction of trigonometric polynomials, trigonometric splines and conic sections like circles, ellipses etc. (see Figures 1, 2). A brief introduction of some non-stationary schemes is given, as follows. The first non-stationary scheme was presented by Jena *et al.* [**12**], in which they introduced a 4-point binary interpolatory non-stationary C^1 scheme, using trigonometric polynomials that can reproduce the elements of the linear space spanned by $1, \cos(\alpha x)$ and $\sin(\alpha x)$. It was considered as the generalization of the 4-point stationary subdivision scheme developed by Dyn *et al.* [**8**]. A novel non-stationary subdivision scheme, based on the subdivision generation analysis of B-spline curves and surfaces, was introduced by Chen *et al.* [**6**] .

In 2007, Beccari *et al.* presented a 4-point binary non-stationary interpolating subdivision scheme, using a tension parameter, that was capable of producing certain families of conics and cubic polynomials [**1**]. A 4-point ternary interpolating non-stationary subdivision scheme that generates C^2 continuous limit curves, showing considerable variation of shapes with a tension parameter, was presented by the same authors in the same year [**2**]. In 2009, Daniel and Shunmugaraj [**7**] developed a non-stationary 2-point approximating scheme that generates C^1 limiting curves and two 3-point binary schemes that generate C^2 and C^3 limiting curves. The masks of these schemes were defined in terms of trigonometric B-spline basis functions. Conti and Romani [**5**] presented a new family of 6-point interpolatory non-stationary subdivision schemes using cubic exponential B-spline symbol generating functions that can reproduce conic sections. In the recent past, Siddiqi and Younis [**16**] presented a new family of 5-point binary approximating non-stationary schemes using a laurent polynomial of fourth degree, which can

Received 13 July 2012; revised 3 October 2012.

2010 Mathematics Subject Classification 41A15, 65D07, 65D17, 68U07 (primary).

reproduce the functions of linear spaces spanned by $\{\cos(\alpha x), \sin(\alpha x)\}$. In the following, the structure of the 4-point ternary approximating subdivision scheme is presented.

As approximating subdivision schemes do not retain the points of stage k as the subset of the points of stage $k+1$, so a 4-point ternary approximating scheme in general form can be written as

$$\left.\begin{aligned} p_{3i}^{k+1} &= a_0^k p_{i-1}^k + a_1^k p_i^k + a_2^k p_{i+1}^k + a_3^k p_{i+2}^k \\ p_{3i+1}^{k+1} &= b_0^k p_{i-1}^k + b_1^k p_i^k + b_2^k p_{i+1}^k + b_3^k p_{i+2}^k \\ p_{3i+2}^{k+1} &= a_3^k p_{i-i}^k + a_2^k p_i^k + a_1^k p_{i+1}^k + a_0^k p_{i+2}^k \end{aligned}\right\}. \tag{1.1}$$

The paper is organized as follows. In § 2 the basic notions and definitions of ternary subdivision schemes are considered. The proposed ternary scheme is presented in § 3. The convergence analysis is discussed in § 4. In § 5, the symmetry of basic limit functions, the Hölder regularity and the reproduction property of the scheme are investigated. For comparison, some examples are considered in § 6. The conclusion is drawn in § 7.

2. *Preliminaries*

In a subdivision scheme, the set of control points $P^k = \{p_i^k \in R \mid i \in Z^m\}$ (where $m = 1$ in the curve case and $m = 2$ in the surface case) of a polygon at the kth level is mapped to a refined polygon to generate the new set of control points $P^{k+1} = \{p_i^{k+1} \in R \mid i \in Z^m\}$ at the $(k+1)$st level by applying the subdivision rule

$$P_i^{k+1} = \{S_{a^k} P^k\}_i = \sum_{j \in Z} a_{i-3j}^k p_j^k \quad \forall i \in Z, \tag{2.1}$$

where the set $\{a_i^k \mid i \in Z^m, a_i^k \neq 0\}$ is finite for every $k \in Z_+$. If the masks of the scheme are independent of k, then the scheme is called stationary $\{S_a\}$, otherwise it is called non-stationary $\{S_{a^k}\}$. Equation (2.1) is called the compact form of subdivision scheme (1.1).

Definition 1 [**10**]. If the mask of a non-stationary scheme $\{S_{a^k}\}$ at the kth level is represented by a^k, then the set $\{i \in Z \mid a_i^k \neq 0\}$ is called the support of the mask a^k.

The notation and definitions for the convergence and smoothness of ternary subdivision C^m schemes, similar to that given by Dyn and Levin in [**9**] for a binary scheme, are given below.

Definition 2. A ternary non-stationary subdivision scheme $\{S_{a^k}\}$ is said to be convergent if for every initial data $P^0 \in l^\infty$ there exits a limit function $f \in C^m(R)$ such that

$$\lim_{k\to\infty} \sup_{i \in Z} |P_i^k - f(3^{-k} i)| = 0,$$

and f is not identically zero for some initial data P^0.

Definition 3. Two ternary subdivision schemes $\{S_{a^k}\}$ and $\{S_{b^k}\}$ are said to be asymptotically equivalent if

$$\sum_{k=1}^{\infty} \|S_{a^k} - S_{b^k}\|_\infty < \infty,$$

where

$$\|S_{a^k}\|_\infty = \max\left\{\sum_{i\in Z} |a_{3i}^k|, \sum_{i\in Z} |a_{3i+1}^k|, \sum_{i\in Z} |a_{3i+2}^k|\right\}.$$

https://doi.org/10.1112/S1461157013000375 Published online by Cambridge University Press

THEOREM 2.1 [9, p. 607]. *The non-stationary scheme $\{S_{a^k}\}$ and stationary scheme $\{S_a\}$ are said to be asymptotically equivalent subdivision schemes, if they have finite masks of the same support. If the stationary scheme $\{S_a\}$ is C^m and*

$$\sum_{k=0}^{\infty} 3^{mk}\|S_{a^k} - S_a\|_\infty < \infty,$$

then the non-stationary scheme $\{S_{a^k}\}$ is also said to be C^m.

DEFINITION 4. Koch *et al.* [14] provided the definition of trigonometric B-splines. For this, let $m > n > 0$ and $0 < \alpha < \pi/n$, then uniform trigonometric B-splines $\{T_j^n(x;\alpha)\}_{j=1}^m$ of order n associated with the knot sequence $\Delta := \{t_i = i\alpha \mid i = 0,1,2,\ldots,m+n\}$, with the mesh size α, are defined by the recurrence relation

$$T_0^1(x;\alpha) = \begin{cases} 1, & x \in [0,\alpha), \\ 0, & \text{otherwise,} \end{cases}$$

for $1 < r \leqslant n$,

$$T_0^r(x;\alpha) = \frac{1}{\sin((r-1)\alpha)}\{\sin(x)T_0^{r-1}(x;\alpha) + \sin(t_r - x)T_0^{r-1}(x-\alpha;\alpha)\} \tag{2.2}$$

and $T_j^r(x;\alpha) = T_0^r(x - j\alpha;\alpha)$, for $j = 1,2,\ldots,m$. The trigonometric B-spline $T_j^n(x;\alpha)$ is supported on $[t_j, t_{j+n}]$ and it is the interior of its support. Moreover, $\{T_j^n\}_{j=1}^m$ are linearly independent sets of the interval $[t_{n-1}, t_{m+1}]$. Hence, on $[t_{n-1}, t_{m+1}]$, any uniform trigonometric spline $S(x)$ has a unique representation of the form $S(x) = \sum_{j=0}^m p_j T_j^n(x;\alpha)$, $p_j \in R$; see also [7].

3. *The scheme*

In this section, a 4-point ternary approximating non-stationary subdivision scheme is presented. The masks $\gamma_i^k(\alpha) = \gamma_i^k$, $i = 0,1,2,3$ and $\omega_i^k(\alpha) = \omega_i^k$, $i = 0,1,2,3$ of the proposed ternary scheme can be calculated, for any value of k, using the relation

$$\eta_i^k(\alpha) = T_0^4\left((3-i)\frac{\alpha}{3^k} + \frac{\alpha}{2\cdot 3^{k+1}}; \frac{\alpha}{3^k}\right), \quad i = 0,1,2,3$$

and

$$\omega_i^k(\alpha) = T_0^4\left((3-i)\frac{\alpha}{3^k} + \frac{\alpha}{2\cdot 3^{k}}; \frac{\alpha}{3^k}\right), \quad i = 0,1,2,3$$

where $T_0^4(x;\alpha/3^k)$ with mesh size $(\alpha/3^k)$ is a cubic trigonometric B-spline basis function and can be calculated from (2.2). Thus, the proposed non-stationary scheme can be written, for some value of $\alpha \in]0,\pi/3[$, as

$$\left.\begin{aligned} p_{3i}^{k+1} &= \eta_0^k p_{i-1}^k + \eta_1^k p_i^k + \eta_2^k p_{i+1}^k + \eta_3^k p_{i+2}^k \\ p_{3i+1}^{k+1} &= \omega_0^k p_{i-1}^k + \omega_1^k p_i^k + \omega_2^k p_{i+1}^k + \omega_3^k p_{i+2}^k \\ p_{3i+2}^{k+1} &= \eta_3^k p_{i-i}^k + \eta_2^k p_i^k + \eta_1^k p_{i+1}^k + \eta_0^k p_{i+2}^k \end{aligned}\right\} \tag{3.1}$$

where

$$\eta_0^k = \frac{\sin^3\frac{5\alpha}{2\cdot 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}$$

$$\eta_1^k = \frac{\sin^2\frac{5\alpha}{2\cdot 3^{k+1}}\sin\frac{13\alpha}{2\cdot 3^{k+1}} + \sin\frac{5\alpha}{2\cdot 3^{k+1}}\sin\frac{7\alpha}{2\cdot 3^{k+1}}\sin\frac{11\alpha}{2\cdot 3^{k+1}} + \sin\frac{\alpha}{2\cdot 3^{k+1}}\sin^2\frac{11\alpha}{2\cdot^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}$$
$$\eta_2^k = \frac{\sin\frac{5\alpha}{2\cdot 3^{k+1}}\sin^2\frac{7\alpha}{2\cdot 3^{k+1}} + \sin\frac{\alpha}{2\cdot 3^{k+1}}\sin\frac{7\alpha}{2\cdot 3^{k+1}}\sin\frac{11\alpha}{2\cdot 3^{k+1}} + \sin^2\frac{\alpha}{2\cdot 3^{k+1}}\sin\frac{17\alpha}{2\cdot^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}$$
$$\eta_3^k = \frac{\sin^3\frac{\alpha}{2\cdot 3^{k+1}}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}$$
$$\omega_0^k = \omega_3^k = \frac{\sin^3\frac{\alpha}{2\cdot 3^k}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}, \quad \omega_1^k = \omega_2^k = \frac{2\sin\frac{\alpha}{2\cdot 3^k}\sin^2\frac{3\alpha}{2\cdot 3^k} + \sin^2\frac{\alpha}{2\cdot 3^k}\sin\frac{5\alpha}{2\cdot 3^k}}{\sin\frac{\alpha}{3^k}\sin\frac{2\alpha}{3^k}\sin\frac{3\alpha}{3^k}}.$$

It is to be mentioned that the weights of the proposed scheme are bounded by the mask of the 4-point ternary approximating scheme. The subdivision rules to refine the control polygon are defined as

$$\left.\begin{aligned} p_{3i}^{k+1} &= \tfrac{125}{1296}p_{i-1}^k + \tfrac{831}{1296}p_i^k + \tfrac{339}{1296}p_{i+1}^k + \tfrac{1}{1296}p_{i+2}^k \\ p_{3i+1}^{k+1} &= \tfrac{1}{48}p_{i-1}^k + \tfrac{23}{48}p_i^k + \tfrac{23}{48}p_{i+1}^k + \tfrac{1}{48}p_{i+2}^k \\ p_{3i+2}^{k+1} &= \tfrac{1}{1296}p_{i-i}^k + \tfrac{339}{1296}p_i^k + \tfrac{831}{1296}p_{i+1}^k + \tfrac{125}{1296}p_{i+2}^k \end{aligned}\right\}. \tag{3.2}$$

REMARK 1. The stationary scheme defined in equation (3.2) can generate the limit curves of C^3 continuity (for proof, see Lemma 4.4), which is also higher than the existing approximating 4-point schemes presented by Ko *et al.* [**13**] and Siddiqi and Rehan [**15**]. Furthermore, the Hölder regularity of that scheme (3.2) has been computed in § 5 (see Theorem 5.2).

So, it follows that

$$\eta_0^k \to \tfrac{125}{1296}, \quad \eta_1^k \to \tfrac{831}{1296}, \quad \eta_2^k \to \tfrac{339}{1296}, \quad \eta_3^k \to \tfrac{1}{1296}, \quad \omega_0^k = \omega_3^k \to \tfrac{1}{48}, \quad \omega_1^k = \omega_2^k \to \tfrac{23}{48}.$$

The proof of $\eta_0^k \to \frac{125}{1296}$ can be obtained from the Lemma 4.1 and the proofs of

$$\eta_1^k \to \tfrac{831}{1296}, \quad \eta_2^k \to \tfrac{339}{1296}, \quad \eta_3^k \to \tfrac{1}{1296}, \quad \omega_0^k \to \tfrac{1}{48} \quad \text{and} \quad \omega_1^k \to \tfrac{23}{48}$$

can be calculated similarly.

4. *Convergence analysis*

The theory of asymptotic equivalence is used to investigate the convergence and smoothness of the scheme following [**9**]. Some estimations (of $\eta_i^k, i = 0, 1, 2, 3$ and $\omega_i^k, i = 0, 1$) are used in order to prove the convergence of the proposed scheme, and are given in the following lemmas. To prove the lemmas, the following three inequalities are used,

$$\frac{\sin a}{\sin b} \geqslant \frac{a}{b} \quad \text{for } 0 < a < b < \frac{\pi}{2},$$
$$\theta \csc\theta \leqslant t \csc t \quad \text{for } 0 < \theta < t < \frac{\pi}{2}$$

and

$$\cos x \leqslant \frac{\sin x}{x} \quad \text{for } 0 < x < \frac{\pi}{2}.$$

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LEMMA 4.1. *For $k \geqslant 0$:*

$$\text{(i)}\quad \frac{125}{1296} \leqslant \eta_0^k \leqslant \frac{125}{1296}\frac{1}{\cos^3(\frac{6\alpha}{3^k})}; \qquad \text{(ii)}\quad \frac{831}{1296} \leqslant \eta_1^k \leqslant \frac{831}{1296}\frac{1}{\cos^3(\frac{\alpha}{3^k})};$$

$$\text{(iii)}\quad \frac{339}{1296} \leqslant \eta_2^k \leqslant \frac{339}{1296}\frac{1}{\cos^3(\frac{\alpha}{3^k})}; \qquad \text{(iv)}\quad \frac{1}{1296} \leqslant \eta_3^k \leqslant \frac{1}{1296}\frac{1}{\cos^3(\frac{3\alpha}{3^k})};$$

$$\text{(v)}\quad \frac{1}{48} \leqslant \omega_0^k \leqslant \frac{1}{48}\frac{1}{\cos^3(\frac{\alpha}{3^k})}; \qquad \text{(vi)}\quad \frac{23}{48} \leqslant \omega_1^k \leqslant \frac{23}{48}\frac{1}{\cos^3(\frac{3\alpha}{3^k})}.$$

Proof. To prove the inequality (i)

$$\eta_0^k = \frac{\sin^3(\frac{5\alpha}{2\cdot 3^{k+1}})}{\sin(\frac{\alpha}{3^k})\sin(\frac{2\alpha}{3^k})\sin(\frac{3\alpha}{3^k})} \geqslant \frac{(\frac{5\alpha}{2\cdot 3^{k+1}})^3}{(\frac{\alpha}{3^k})(\frac{2\alpha}{3^k})(\frac{3\alpha}{3^k})} = \frac{125}{1296}$$

and

$$\begin{aligned}\eta_0^k &\leqslant \frac{125\alpha^3}{8\cdot 3^{3k+3}}\csc\left(\frac{\alpha}{3^k}\right)\csc\left(\frac{2\alpha}{3^k}\right)\csc\left(\frac{3\alpha}{3^k}\right) \leqslant \frac{125\alpha^3}{8\cdot 3^{3k+3}}\quad 36\csc^3\left(\frac{6\alpha}{3^k}\right)\\ &\leqslant \frac{125\alpha^3}{8\cdot 3^{3k+3}}36\frac{1}{(\frac{6\alpha}{3^k})^3\cos^3(\frac{6\alpha}{3^k})} \leqslant \frac{125}{1296}\frac{1}{\cos^3(\frac{6\alpha}{3^k})}.\end{aligned}$$

The proofs of (ii)–(vi) can be obtained similarly. □

LEMMA 4.2. *For some constants K_0, K_1, K_2, K_3, K_4 and K_5 independent of k, we have:*

$$\text{(i)}\quad \left|\eta_0^k - \frac{125}{1296}\right| \leqslant K_0\frac{1}{3^{2k}}; \qquad \text{(ii)}\quad \left|\eta_1^k - \frac{831}{1296}\right| \leqslant K_1\frac{1}{3^{2k}};$$

$$\text{(iii)}\quad \left|\eta_2^k - \frac{339}{1296}\right| \leqslant K_2\frac{1}{3^{2k}}; \qquad \text{(iv)}\quad \left|\eta_3^k - \frac{1}{1296}\right| \leqslant K_3\frac{1}{3^{2k}};$$

$$\text{(v)}\quad \left|\omega_0^k - \frac{1}{48}\right| \leqslant K_4\frac{1}{3^{2k}}; \qquad \text{(vi)}\quad \left|\omega_1^k - \frac{23}{48}\right| \leqslant K_5\frac{1}{3^{2k}}.$$

Proof. To prove the inequality (i) use Lemma 4.1,

$$\begin{aligned}\text{(i)}\quad |\eta_0^k - \frac{125}{1296}| &\leqslant \frac{125}{1296}\left(\frac{1-\cos^3(\frac{6\alpha}{3^k})}{\cos^3\alpha}\right)\\ &\leqslant \frac{125}{1296}\left(\frac{3\sin^2(\frac{6\alpha}{3^k})}{2\cos^3\alpha}\right)\\ &\leqslant \frac{125\alpha^2}{24\cos^3(\alpha)}\frac{1}{3^{2k}}.\end{aligned}$$

The proofs of (ii)–(vi) can be obtained similarly. □

LEMMA 4.3. *The laurent polynomial $c^k(z)$ of the scheme $\{S_{c^k}\}$ at the kth level can be written as $c_1^k(z) = ((1+z+z^2)/3)c^k(z)$, where*

$$\begin{aligned}c^k(z) = 3\{&\eta_3^k z^{-6} + (\omega_0^k - \eta_3^k)z^{-5} + (\eta_0^k - \omega_0^k)z^{-4} + (-\eta_0^k + \eta_2^k + \eta_3^k)z^{-3}\\ &+ (\eta_0^k + \eta_1^k - \omega_2^k - \omega_3^k)z^{-2} + (\omega_2^k + \omega_3^k - \eta_2^k - \eta_3^k)z^{-1} + (-\eta_0^k + \eta_2^k + \eta_3^k)\\ &+ (\eta_0^k - \omega_0^k)z + (\omega_0^k - \eta_3^k)z^2 + \eta_3^k z^3\}.\end{aligned}$$

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Proof. Since

$$
\begin{aligned}
c_1^k(z) = \eta_3^k z^{-6} + \omega_0^k z^{-5} + \eta_0^k z^{-4} + \eta_2^k z^{-3} + \omega_1^k z^{-2} + \eta_1^k z^{-1} + \eta_1^k \\
+ \omega_2^k z + \eta_2^k z^2 + \eta_0^k z^3 + \omega_3^k z^4 + \eta_3^k z^5,
\end{aligned}
$$

$d^k(z) = ((1+z+z^2)/3)b^k(z)$ can be proved using $\eta_0^k + \eta_1^k + \eta_2^k + \eta_3^k = 1$ and $\omega_0^k + \omega_1^k = \frac{1}{2}$. □

LEMMA 4.4. *The laurent polynomial $c(z)$ of the scheme $\{S_c\}$ can be written as $c_1(z) = ((1+z+z^2)/3)c(z)$, where*

$$
c(z) = \tfrac{1}{432}\{z^{-6} + 26z^{-5} + 98z^{-4} + 215z^{-3} + 308z^{-2} + 308z^{-1} + 215 + 98z + 26z^2 + z^3\}
$$

and the subdivision scheme $\{S_c\}$ corresponding to the symbol $c(z)$ is C^2.

Proof. To prove that the subdivision scheme $\{S_c\}$ corresponding to the symbol $c(z)$ is C^2, we have

$$
\begin{aligned}
d(z) &= \frac{3c(z)}{(1+z+z^2)^3} \\
&= \frac{1}{144}\{z^{-6} + 23z^{-5} + 23z^{-4} + z^{-3}\}.
\end{aligned}
$$

Since the norm of the subdivision scheme $\{S_d\}$ is

$$
\begin{aligned}
\|S_d\|_\infty &= \max\left\{\sum_{i\in Z}|d_{3i}^k|, \sum_{i\in Z}|d_{3i+1}^k|, \sum_{i\in Z}|d_{3i+2}^k|\right\} \\
&= \max\left\{\frac{1}{24}, \frac{23}{48}, \frac{23}{48}\right\} < 1,
\end{aligned}
$$

in view of Dyn [**10**], the stationary scheme $\{S_c\}$ is C^2, and hence the scheme (3.2) is C^3. □

THEOREM 4.5. *The ternary 4-point non-stationary scheme defined in equation (3.1) converges and has the smoothness C^3 for the range $\alpha \in]0, \frac{\pi}{3}[$.*

Proof. To prove the proposed scheme to be C^3, it is sufficient to show that the scheme corresponding to the symbol $c^k(z)$ is C^2, as we know that $\{S_c\}$ is C^2 by Lemma 4.4. So, in view of [**9**], for the convergence and smoothness of proposed scheme (3.1) it is sufficient to show that

$$
\sum_{k=0}^{\infty} 3^{2k}\|S_{c^k} - S_c\|_\infty < \infty
$$

where

$$
\|S_{c^k} - S_c\|_\infty = \max\left\{\sum_{j=Z}^{\infty}|c_{i+3j}^k - c_{i+3j}| : i = 0, 1, 2\right\}.
$$

Using Lemmas 4.3 and 4.4, it can be written as

$$
\begin{aligned}
\sum_{j\in Z}^{\infty}\left|c_{3j}^k - c_{3j}\right| &= 2\left|3\eta_3^k - \frac{1}{432}\right| + 2\left|-3\eta_0^k + 3\eta_2^k + 3\eta_3^k - \frac{215}{432}\right| \\
&= 6\left|\eta_0^k - \frac{125}{1296}\right| + 6\left|\eta_2^k - \frac{339}{1296}\right| + 12\left|\eta_3^k - \frac{1}{1296}\right|
\end{aligned}
$$

and similarly,

$$\begin{aligned}\sum_{j\in Z}^{\infty} |c^k_{3j+1} - c_{3j+1}| &= \left|3\omega^k_0 - 3\eta^k_3 - \frac{26}{432}\right| + \left|-3\omega^k_2 - 3\omega^k_3 + 3\eta^k_0 + 3\eta^k_1 - \frac{308}{432}\right| \\ &\quad + \left|3\eta^k_0 - 3\omega^k_0 - \frac{98}{432}\right| \\ &= 6\left|\eta^k_0 - \frac{125}{1296}\right| + 3\left|\eta^k_1 - \frac{831}{1296}\right| + 3\left|\eta^k_3 - \frac{1}{1296}\right| \\ &\quad + 9\left|\omega^k_0 - \frac{27}{1296}\right| + 3\left|\omega^k_1 - \frac{621}{1296}\right| \\ &= \sum_{j\in Z}^{\infty} |c^k_{3j+2} - c_{3j+2}|\,.\end{aligned}$$

From (i)–(vi) of Lemma 4.2, it can be followed by

$$\sum_{k=0}^{\infty} 3^{2k}\left|\eta^k_0 - \frac{125}{1296}\right| < \infty, \quad \sum_{k=0}^{\infty} 3^{2k}\left|\eta^k_1 - \frac{831}{432}\right| < \infty, \quad \sum_{k=0}^{\infty} 3^{2k}\left|\eta^k_2 - \frac{339}{1296}\right| < \infty,$$
$$\sum_{k=0}^{\infty} 3^{2k}\left|\eta^k_3 - \frac{1}{1296}\right| < \infty, \quad \sum_{k=0}^{\infty} 3^{2k}\left|\omega^k_0 - \frac{1}{48}\right| < \infty$$

and

$$\sum_{k=0}^{\infty} 3^{2k}\left|\omega^k_1 - \frac{23}{48}\right| < \infty,$$

so it can be written as

$$\sum_{k=0}^{\infty} 3^{2k}\|S_{c^k} - S_c\|_\infty < \infty.$$

Thus by Theorem 2.1, $c^k(z)$ is C^2 as the associated scheme $c(z)$ is C^2. Hence, the proposed scheme (3.1) is C^3. □

5. *Properties of the scheme*

In this section some properties of the proposed scheme are discussed, such as the symmetry of the basic limit function, Hölder regularity and the polynomial reproduction property etc.

The basic limit function of the scheme is the limit function for the data

$$p^0_i = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

The basic limit function is symmetric about the Y-axis (see Theorem 5.1). Moreover, the support size of the proposed non-stationary approximating scheme is same as the stationary schemes presented in [**13**] and [**15**]. The limit curves (continuous lines) of the basic limit function of the proposed ternary scheme have been shown in Figure 3(c), for different values of α, while the limit curves of basic functions of the schemes introduced in [**13**] and [**15**] have been depicted in Figures 3(a) and (b), respectively. In the following theorem, the symmetry of the basic limit function about the Y-axis is presented following [**2**].

THEOREM 5.1. *The basic limit function F is symmetric about the Y-axis.*

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Proof. Let F denote the basic limit function and define $D_n := \{i/3^n \mid i \in Z\}$ such that restriction of F to D_n satisfies $F(i/3^n) = p_i^n$ for all $i \in Z$. The symmetry of the basic function is proved using induction on n.

It can be observed that $F(i) = F(-i)$ for all $i \in Z$, thus $F(i/3^n) = F(-i/3^n)$ for all $i \in Z$ and $n = 0$. Assume that $F(i/3^n) = F(-i/3^n)$ for all $i \in Z$ and then $p_i^n = p_{-i}^n$ for all $i \in Z$. It may be observed that

$$\begin{aligned} F\left(\frac{3i}{3^{n+1}}\right) &= p_{3i}^{n+1} \\ &= \eta_0^n p_{i-1}^n + \eta_1^n p_i^n + \eta_2^n p_{i+1}^n + \eta_3^n p_{i+2}^n \\ &= \eta_3^n p_{-i-2}^n + \eta_2^n p_{-i-1}^n + \eta_1^n p_{-i}^n + \eta_0^n p_{-i+1}^n \\ &= p_{-3i}^{n+1} = F\left(-\frac{3i}{3^{n+1}}\right). \end{aligned}$$

Similarly

$$F\left(\frac{3i+1}{3^{n+1}}\right) = p_{3i+1}^{n+1} = p_{-3i-1}^{n+1} = F\left(-\frac{3i+1}{3^{n+1}}\right)$$

and

$$F\left(\frac{3i+2}{3^{n+1}}\right) = p_{3i+2}^{n+1} = p_{-3i-2}^{n+1} = F\left(-\frac{3i+2}{3^{n+1}}\right).$$

Hence, $F(i/3^n) = F(-i/3^n)$ for all $i \in Z$ and $n = Z_+$, thus from the continuity of F, $F(x) = F(-x)$ holds for all $x \in R$, which completes the required result. □

THEOREM 5.2. *The Hölder regularity of the scheme* (3.2) *is* $3 - \log_2(\frac{23}{48})$.

Proof. The proof follows directly from [**11**]; the Hölder regularity of a subdivision scheme (3.2) can be computed in the following way. Since $c(z) = \big((1+z+z^2)/3\big)^3 e(z)$ where $e(z) = \{\frac{1}{48} + \frac{23}{48}z^{-1} + \frac{23}{48}z^{-2} + \frac{1}{48}z^{-3}\}$ is the symbol $c(z)$ of the scheme (3.2), the non-zero coefficients e_0, e_1, e_2 and e_3 can be written from $e(z)$ that are $\frac{1}{48}, \frac{23}{48}, \frac{23}{48}$ and $\frac{1}{48}$ respectively. Consider two matrices A_0 and A_1 of order 3×3 taking elements $(A_0)_{ij} = e_{3+i-2j}$ and $(A_1)_{ij} = e_{4+i-2j}$ for $i, j = 1, 2, 3$. The matrices take the form

$$A_0 = \begin{pmatrix} \frac{23}{48} & \frac{1}{48} & 0 \\ \frac{1}{48} & \frac{23}{48} & 0 \\ 0 & \frac{23}{48} & \frac{1}{48} \end{pmatrix} \quad A_1 = \begin{pmatrix} \frac{1}{48} & \frac{23}{48} & 0 \\ 0 & \frac{23}{48} & \frac{1}{48} \\ 0 & \frac{1}{48} & \frac{23}{48} \end{pmatrix}.$$

The largest eigen values of the matrices A_0, A_1 and $\max\{\|A_0\|_\infty, \|A_1\|_\infty\}$ are $\frac{23}{48}$. Hence the Hölder regularity is $3 - \log_2(\frac{23}{48})$. □

The polynomial reproduction property has its own importance, as the reproduction property of the polynomials up to a certain degree d implies that the scheme has $d+1$ approximation order. For this, polynomial reproduction can be made from the initial data which has been sampled from some polynomial function. In view of [**3**] and [**4**], the polynomial reproduction property of the proposed scheme can be obtained after having the parametrization τ and definitions in the following manner.

DEFINITION 5. For a ternary subdivision scheme the parametrization $\tau = a'(1)/3$ the corresponding parametric shift attaches the data f_i^k for $i \in Z$, $k \in N$ to the parameter

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values

$$t_i^k = t_0^k + \frac{i}{3^k} \quad \text{with } t_0^k = t_0^{k-1} - \frac{\tau}{3^k}. \tag{5.1}$$

DEFINITION 6. A ternary subdivision scheme reproduces a polynomial of degree d if it is convergent and its continuous *limit function* (for any polynomial $p \in \pi_d$) is equal to p and the initial data $f_i^0 = p(t_i^0)$, $i \in Z$.

THEOREM 5.3. *A convergent ternary subdivision scheme reproduces polynomials of degree d with respect to the parametrization defined in (5.1) if and only if*

$$a^{(k)}(1) = 3\prod_{l=0}^{k-1}(\tau - l) \quad \text{and} \quad a^{(k)}(e^{2i\pi/3}) = a^{(k)}(e^{4i\pi/3}) = 0 \quad \text{for } k = 0, 1, \ldots, d.$$

Proof. The induction over d can be performed to prove this theorem following [3]. □

In view of [3], the following proposition helps to find the necessary conditions defined in (5.2).

PROPOSITION 5.4. *Let $d \in N$ and $\tau \in R$. Then a subdivision symbol $a(z)$ satisfies*

$$a^{(k)}(1) = 3\prod_{l=0}^{k-1}(\tau - l) \quad \text{for } k = 0, 1, \ldots, d \tag{5.2}$$

if and only if $b(z) = a(z^3)z^{-3\tau}$ satisfies

$$b(1) = 3 \quad \text{and} \quad b^{(k)}(1) = 0 \quad \text{for } k = 0, 1, \ldots, d, \tag{5.3}$$

which in turn is equivalent to requiring that $b(z) = (1 - z)^{d+1}c(z) + 2$ for some $c(z)$.

PROPOSITION 5.5. *Consider a ternary subdivision scheme that reproduces polynomials up to degree d. Then the smoothed scheme S_b with the symbol $b(z) = ((1 + z + z^2)/3)a(z)$ satisfies the conditions*

$$b(1) = 3 \quad \text{and} \quad b^{(k)}(e^{2i\pi/3}) = b^{(k)}(e^{4i\pi/3}) = 0, \quad k = 0, 1, \ldots, d + 1$$

(kth derivative of the symbol) and hence generates polynomials of degree $d+1$, but it has only linear reproduction.

Proof. Following [3], for some laurent polynomial $b(z)$ with $b(1) = 1/3^d$, we have

$$a(z) = (1 + z + z^2)^{d+1}b(z) = \left(\frac{1 - z^3}{1 - z}\right)b(z)$$

and the fact $b(1) = a(1)$. Thus, the first derivative of $b(z)$ is

$$b'(z) = \frac{1 + z + z^2}{3}a'(z) + \frac{1 + 2z}{3}a(z)$$

and the correct parametric shift for S_b is

$$\tau_b = \frac{b'(z)}{3} = \frac{a'(1) + a(1)}{3} = \tau_a + 1.$$

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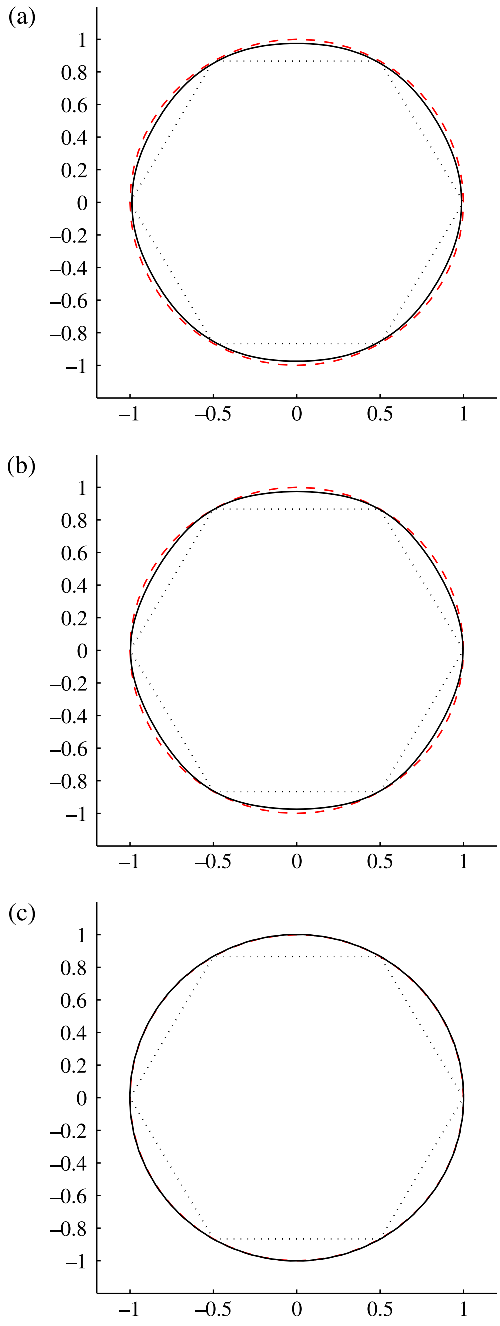

FIGURE 1. *The continuous lines represent the limit curves of the schemes:* [**13**] *in* (a), [**15**] *in* (b) *and the proposed scheme* (3.1), *taking* $\alpha = \pi/6$ *in* (c) *after three subdivision steps. The broken and dotted lines represent the unit circles and control polygons, respectively.*

The second derivative of $b(z)$ is

$$b''(z) = \frac{1 + z + z^2}{3} a''(z) + \frac{1 + 2z}{3} a'(z) + \frac{2}{3} a(z)$$

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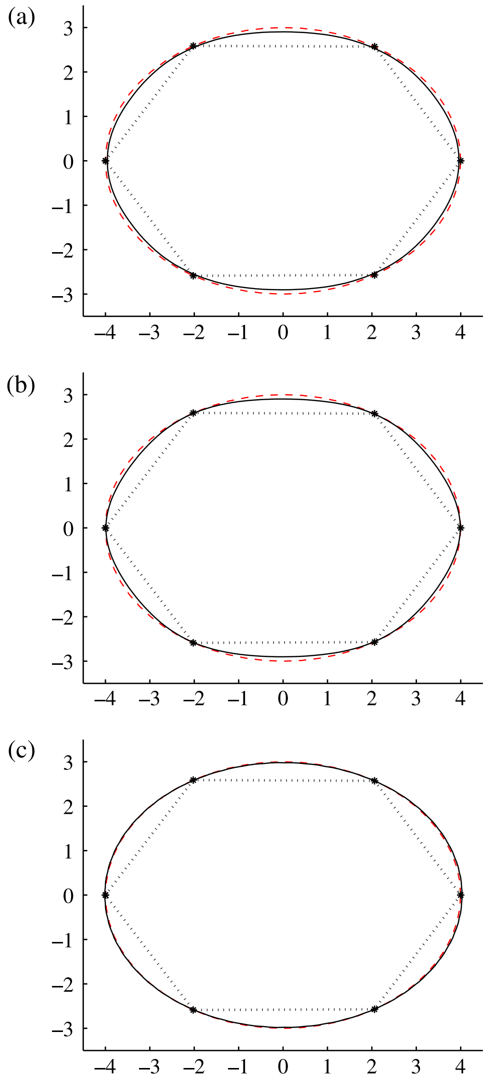

FIGURE 2. *The continuous lines represent the limit curves of the schemes:* [**13**] *in* (a), [**15**] *in* (b) *and the proposed scheme* (3.1), *taking* $\alpha = \pi/6$ *in* (c) *after three subdivision steps. The broken and dotted lines represent the ellipses* (*with parametric equations* $x = 4\cos t$ *and* $y = 3\sin t$) *and control polygons, respectively.*

which produces

$$\begin{aligned} b''(1) &= a''(1) + 2a'(z) + \tfrac{2}{3}a(1) \\ &= 3\tau_a(\tau_a - 1) + 6\tau_a + 2. \end{aligned}$$

After simplification, it can be yielded that

$$b''(1) - 3\tau_b(\tau_b - 1) = 2 \neq 0.$$

Hence, it does not reproduce polynomials of degree $d > 1$. □

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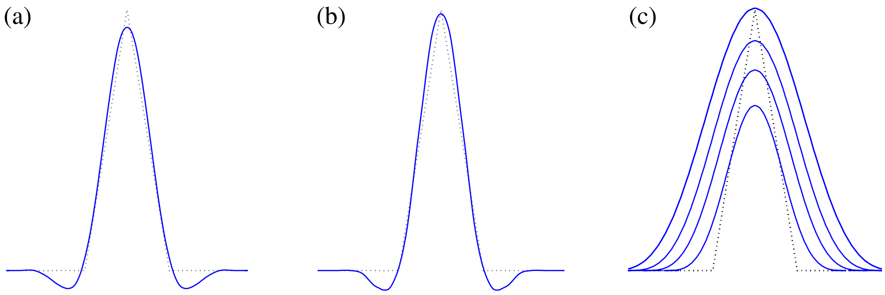

FIGURE 3. *The continuous lines show the different behavior of the basic limit function of the schemes:* [**13**] *in* (a), [**15**] *in* (b) *and the proposed scheme* (3.1) *taking* $\alpha = \pi/180$, $\alpha = \pi/9$, $\alpha = \pi/7$ *and* $\alpha = \pi/6$, *from inner to outer in* (c), *after three subdivision steps. The dotted line represents the control polygon.*

TABLE 1. *Comparison of the 4-point approximating subdivision scheme.*

Scheme	Type	Continuity	Support	Range
Ternary [**13**]	Stationary	C^2	$[-6, 5]$	For some particular value
Ternary [**15**]	Stationary	C^2	$[-6, 5]$	For some particular value
Ternary (Proposed)	Non-stationary	C^3	$[-6, 5]$	$0 < \alpha < \frac{\pi}{3}$

In the following, a comparison of the proposed scheme with the existing 4-point ternary approximating schemes has been shown.

6. *Comparison*

In this section, the comparison of the proposed scheme has been shown with the existing 4-point ternary approximating schemes.

In Figure 1, the regular hexagon is taken as the control polygon to reproduce the unit circle and red broken lines are the limit curves of the unit circle. The limit curves of the schemes [**13**] and [**15**] are drawn by continuous lines in Figures 1(a) and 1(b), respectively, while the limit curve of the proposed scheme has been depicted by a continuous line in Figure 1(c), taking $\alpha = \pi/6$. The comparison shows that the limit curve of the proposed scheme is closer to the unit circle as compared with the other schemes defined in [**13**] and [**15**] (see also Table 2). The control polygons are represented by dotted line and all limit curves are taken after the third iteration. In Figure 2, similar results can be obtained for the ellipse case.

Table 1 shows the comparison of the proposed scheme with [**13**] and [**15**]. It tells that the proposed scheme has the higher order of derivative continuity. Table 1 also shows that the proposed scheme can generate the family of limit curves whereas the existing 4-point approximating schemes can not generate the families of limit curves.

7. *Conclusion*

A 4-point ternary approximating non-stationary subdivision scheme has been developed which generates a family of C^3 limiting curves for $0 < \alpha < \pi/3$ and its limiting function has support $[-6, 5]$, which is similar to the schemes presented in [**13**] and [**15**]. The proposed scheme has a higher order derivative continuity as compared to schemes presented in [**13**] and [**15**].

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TABLE 2. *Comparison of the schemes with the unit circle.*

x	4-point stationary [13]	4-point stationary [15]	4-point (proposed) non-stationary
0.0	$2.52e-2$	$2.57e-2$	$0.01e-2$
0.1	$2.36e-2$	$2.39e-2$	$0.07e-2$
0.2	$1.97e-2$	$1.88e-2$	$0.15e-2$
0.3	$1.55e-2$	$1.17e-2$	$0.20e-2$
0.4	$1.29e-2$	$0.56e-2$	$0.18e-2$
0.5	$1.33e-2$	$0.33e-2$	$0.15e-2$
0.6	$1.48e-2$	$0.68e-2$	$0.25e-2$
0.7	$2.29e-2$	$1.97e-2$	$0.40e-2$
0.8	$4.02e-2$	$4.07e-2$	$0.12e-2$
0.9	$5.31e-2$	$5.35e-2$	$0.57e-2$
1.0			$0.64e-2$

The construction of the proposed scheme is associated with the trigonometric B-spline basis function. It is evident from examples that the proposed scheme behaves more pleasantly and gives better results. It is also mentioned that the limit curves of the proposed scheme are much closer to the unit circle and ellipse.

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