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ON SOME COMPLEX SUBMANIFOLDS IN KAEHLER MANIFOLDS

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1. Introduction. The purpose of this paper is to give some conditions for complex submanifolds in a Kaehler manifold of constant holomorphic sectional curvature to be Einstein.

For a complex hypersurface which is Einstein, Smyth [**8**] has obtained its classification and Chern [**2**] has proved the corresponding local result. Moreover, Takahashi [**9**] and Nomizu-Smyth [**3**] generalized this to a complex hypersurface with parallel Ricci tensor. We shall consider a condition weaker than the requirement that the Ricci tensor be parallel, that is we shall consider a complex submanifold with commuting curvature and Ricci operator, which condition was treated by Bishop-Goldberg [**1**]. For such a complex submanifold, we shall prove that it is Einstein if the Ricci operator commutes to the second fundamental form (Theorem 1). This condition is satisfied for a complex hypersurface automatically.

We shall also consider a complex submanifold with parallel second fundamental form in a Kaehler manifold of constant holomorphic sectional curvature by using Simons' type formula which was given by Simons [**7**] and studied by Ogiue [**4**] for a complex submanifold.

2. Preliminaries. Let $\bar{M}$ be a Kaehler manifold of complex dimension $n + p$ with the structure tensor field J and the Kaehler metric $\langle\ ,\ \rangle$, and let M be an n-dimensional complex submanifold of $\bar{M}$. The Riemannian metric induced on M is a Kaehler metric, which is denoted by the same $\langle\ ,\ \rangle$ and all metric properties of M refer to this metric. The Kaehler structure of M is written by J as in $\bar{M}$. By $\bar{\nabla}$, we denote the covariant differentiation in $\bar{M}$ and by ∇ the one in M determined by the induced metric. Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \qquad X, Y \in \mathscr{X}(M),$$
$$\bar{\nabla}_X N = -A^N(X) + D_X N, \qquad X \in \mathscr{X}(M), \qquad N \in \mathscr{X}(M)^\perp$$

where $\langle B(X, Y), N\rangle = \langle A^N(X), Y\rangle$ and D is the linear connection in the normal bundle $T(M)^\perp$. Both A and B are called the second fundamental form of M. The second fundamental form B is a vector valued bilinear form on each $T_m(M)$ taking values in $T_m(M)^\perp$ and the second fundamental form A is a cross-section of a vector bundle $\mathrm{Hom}(T(M)^\perp, S(M))$ where $S(M)$ is the

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bundle whose fibre at each point is the space of symmetric linear transformations of $T_m(M) \to T_m(M)$, i.e., for $w \in T_m(M)^\perp$, $A^w : T_m(M) \to T_m(M)$.

M is called a totally geodesic submanifold of $\bar{M}$ if its second fundamental form is identically zero. Since any complex submanifold of a Kaehler manifold is minimal, its mean curvature vanishes, i.e., $\sum B(e_i, e_i) = 0$ where $e_1, \ldots, e_{2n}$ is a frame for $T_m(M)$.

Let $\bar{R}$ and R denote the curvature tensors of $\bar{M}$ and M respectively. If we assume that $\bar{M}$ is of constant holomorphic sectional curvature c, then we have

$$(2.1)\quad \bar{R}_{X,Y}Z = \tfrac{1}{4}c(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle Z, JY\rangle JX - \langle Z, JX\rangle JY + 2\langle X, JY\rangle JZ),$$

$$(2.2)\quad \bar{R}_{X,Y}Z = R_{X,Y}Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y).$$

Let $v_1, \ldots, v_{2p}$ be a frame for $T_m(M)^\perp$, and let $x, y \in T_m(M)$. Then the Ricci tensor S of M is given by

$$(2.3)\quad S(x, y) = \tfrac{1}{2}(n + 1)c\langle x, y\rangle - \sum_{i=1}^{2p} \langle A^i A^i(x), y\rangle.$$

Here we write A^i instead of A^{v_i} to simplify the presentation. From (2.3), the scalar curvature K of M is represented by

$$(2.4)\quad K = n(n + 1)c - ||A||^2$$

where $||A||$ denotes the length of the second fundamental form.

On the other hand, we have the following relations between the second fundamental form A and the complex structure J:

$$(2.5)\quad A^N J + JA^N = 0 \quad \text{and} \quad A^{JN} - JA^N = 0 \text{ for any } N \in \mathscr{X}(M)^\perp.$$

We also have

$$(2.6)\quad S(Jx, Jy) = S(x, y) \quad \text{and} \quad JQ = QJ$$

where Q is the Ricci operator of M defined by $S(x, y) = \langle Qx, y\rangle$.

Next we define operators which we later use. Simons [**7**] defined the following symmetric, positive semi-definite operators:

$$(2.7)\quad A^{\sim} = {}^tA \cdot A \quad \text{and} \quad A_{\sim} = \sum_{i=1}^{2p} \operatorname{ad}A^i \operatorname{ad}A^i.$$

And we define the operator A^* by setting

$$(2.8)\quad A^* = \sum_{i=1}^{2p} (A^i)^2.$$

Clearly A^* is symmetric, positive semi-definite operator. And we have $\operatorname{Tr} A^* = ||A||^2$ where Tr is the trace of a operator.

3. Complex submanifolds with certain Ricci tensor. Let $\bar{M}$ be a Kaehler manifold of complex dimension $n + p$ and constant holomorphic sectional curvature c, which will be denoted by $\bar{M}^{n+p}(c)$. Let M be a complex submanifold of $\bar{M}$ of complex dimension n. In this section we consider a condition weaker than the requirement that Q be parallel ($\nabla Q = 0$):

(P) $\quad R_{X,Y}(Q) = 0,$

which is equivalent to $R_{X,Y} \cdot Q = Q \cdot R_{X,Y}$ (cf. [**1**]). This condition is also equivalent to

(T) $\quad R_{X,Y}(S) = 0.$

We also consider a condition $QA^j = A^jQ$, that is, Q and A^j are commuting as operators. This condition is satisfied obviously when M is an Einstein manifold and we shall prove that this condition is satisfied for any complex hypersurface in a Kaehler manifold of constant holomorphic sectional curvature.

Let $e_1, \ldots, e_{2n}$ be a frame for $T_m(M)$ such that $e_{n+t} = Je_t$, and let $v_1, \ldots, v_{2p}$ be a frame for $T_m(M)^\perp$ such that $v_{p+s} = Jv_s$.

THEOREM 1. *Let $\bar{M}$ be a Kaehler manifold of complex dimension $n + p$ $(n > 1)$ and constant holomorphic sectional curvature c, and let M be an n-dimensional complex submanifold in $\bar{M}$ with a condition* (P). *If $c < 0$, then M is an Einstein manifold. If $c > 0$, then M is an Einstein manifold if and only if Q is commuting with A^j $(j = 1, \ldots, p)$.*

Proof. Let x, y $T_m(M)$. By the condition (T), we obtain

$$\sum_{i=1}^{2n} (S(R_{e_i,x}e_i, y) + S(e_i, R_{e_i,x}y)) = 0.$$

By equation (2.2), this becomes

$$\sum_{i=1}^{2n} \{S(\bar{R}_{e_i,x}e_i, y) + S(A^{B(x,e_i)}(e_i), y) + S(\bar{R}_{e_i,x}y, e_i) + S(A^{B(x,y)}(e_i), e_i) - S(A^{B(e_i,y)}(x), e_i)\} = 0.$$

In the following, we calculate this equation. First we have, by (2.1),

$$\sum_{i=1}^{2n} (S(\bar{R}_{e_i,x}e_i, y) + S(\bar{R}_{e_i,x}y, e_i)) = \tfrac{1}{2}nc\left(\frac{K}{2n}\langle x, y\rangle - S(x, y)\right).$$

We obtain $\sum S(A^{B(x,y)}(e_i), e_i) = 0$, by using (2.5) and (2.6), i.e.,

$$-S(A^{B(x,y)}(e_i), e_i) = -S(JA^{B(x,y)}(e_i), Je_i) = S(A^{B(x,y)}(Je_i), Je_i).$$

We have

$$\begin{aligned}\sum_{i=1}^{2n} &(S(A^{B(x,e_i)}(e_i), y) - S(A^{B(e_i,y)}(x), e_i)) \\ &= \sum_{i=1}^{2n}\sum_{j=1}^{2p} (\langle A^j(e_i), Qy\rangle\langle A^j(x), e_i\rangle - \langle A^j(x), Qe_i\rangle\langle A^j(y), e_i\rangle) \\ &= \sum_{j=1}^{2p} (\langle QA^jA^j(x), y\rangle - \langle A^jQA^j(x), y\rangle).\end{aligned}$$

Consequently, we obtain

$$\tfrac{1}{2}nc\left(\frac{K}{2n}\langle x, y\rangle - S(x, y)\right) + \sum_{j=1}^{2p} (\langle QA^jA^j(x), y\rangle - \langle A^jQA^j(x), y\rangle) = 0.$$

From this, we can see that

$$nc\left(\frac{K^2}{2n} - ||Q||^2\right) = 2\sum_{j=1}^{2p} Tr(QA^jQA^j - QQA^jA^j) \tag{3.1}$$
$$= -\sum_{j=1}^{2p} ||[Q, A^j]||^2 \leqslant 0,$$

where $[Q, A^j] = QA^j - A^jQ$.
On the other hand, we always have $K^2 \leqq 2n||Q||^2$, and equality holding if and only if M is Einstein. If $c < 0$, then M is an Einstein manifold by (3.1). Let $c > 0$. If $QA^j = A^jQ$, then $QJA^j = JQA^j = JA^jQ$ by using (2.6). Therefore if $QA^j = A^jQ$ $(j = 1, \ldots, p)$, (3.1) implies that M is Einstein.

COROLLARY 1. *Let M be a complex hypersurface in $\bar{M}^{n+1}(c)$, $c > 0$, $n > 1$. If M satisfies* (P), *then either M is totally geodesic, or M is a locally symmetric Einstein manifold with scalar curvature $K = n^2c$.*

Proof. Let v, Jv be a frame for $T_m(M)^\perp$. Then we have

$$S(x, y) = \tfrac{1}{2}(n + 1)c\langle x, y\rangle - 2\langle A^vA^v(x), y\rangle.$$

Hence we have $S(A^v(x), y) = S(x, A^v(y))$, which shows that $QA^v = A^vQ$. By the above theorem, M is an Einstein manifold and we have our assertion by Theorem C of Takahashi [**9**].

COROLLARY 2. *Let M be a complex hypersurface in $\bar{M}^{n+1}(c)$, $c < 0$, $n > 1$. If M has the property* (P), *then M is totally geodesic.*

Proof. By Theorem 1, M is Einstein and we have our result by the theorem of Chern [**2**], or Takahashi [**9**].

Remark 1. Let M be a complex hypersurface of $\bar{M}(c)$. If the Ricci tensor of M is parallel, then M is Einstein [**3**; **9**]. Our results are the partial generalization of these results.

4. Simons' type formula of complex submanifolds. Let $\bar{M}$ be a Kaehler manifold of complex dimension $n + p$, and let M be an n-dimensional complex submanifold of $\bar{M}$. We can take a frame $e_1, \ldots, e_{2n}$ for $T_m(M)$ such that $e_{n+t} = Je_t$ and a frame $v_1, \ldots, v_{2p}$ for $T_m(M)^\perp$ such that $v_{p+s} = Jv_s$.

From (2.5), we obtain $JA^jA^iJA^j = -A^jA^iA^j$, hence we have (see [**7**, p. 94])

$$\langle A_\sim \cdot A, A\rangle = \sum_{i,j=1}^{2p} \|[A^i, A^j]\|^2 = 2\sum_{i,j=1}^{2p}\sum_{t=1}^{2n} \langle A^iA^iA^j(e_t), A^j(e_t)\rangle$$
$$= 2\sum_{t=1}^{2n} \langle (A^*)^2(e_t), e_t\rangle.$$

By (2.5), we can see easily $JA^* = A^*J$. Since A^* is symmetric, positive semi-definite, using a suitable basis, A^* is represented by the matrix form

$$A^* \equiv \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_{2n} \end{bmatrix}, \qquad \lambda_{n+t} = \lambda_t, \quad \lambda_t \geqq 0.$$

Then we have

$$\langle A_\sim \cdot A, A\rangle = 2\sum_{i=1}^{2n} \lambda_i^2 \geqslant \frac{1}{n}\left(\sum_{i=1}^{2n} \lambda_i\right)^2,$$
$$\langle A_\sim \cdot A, A\rangle = 2\left(\left(\sum_{i=1}^{2n} \lambda_i\right)^2 - \sum_{i\neq j}^{2n} \lambda_i\lambda_j\right)$$
$$= 2\left(\sum_{i=1}^{2n} \lambda_i\right)^2 - 8\sum_{i\neq j}^{n} \lambda_i\lambda_j - \langle A_\sim \cdot A, A\rangle.$$

On the other hand, we obtain $(\sum_{i=1}^{2n}\lambda_i)^2 = \|A\|^4$. Consequently, we get the following

$$(4.1)\quad \frac{1}{n}\|A\|^4 \leqslant \langle A_\sim \cdot A, A\rangle \leqslant \|A\|^4.$$

From this, we obtain the following

PROPOSITION 1. *Let $\bar{M}$ be an $(n + p)$-dimensional Kaehler manifold, and let M be an n-dimensional complex submanifold of $\bar{M}$. If $A^iA^j = A^jA^i$ for all i, j, then M is totally geodesic.*

Remark 2. If $\bar{M}$ is a Kaehler manifold and M its complex submanifold, then $A^iA^j = A^jA^i$ for all i, j if and only if $(\bar{R}_{X,Y}N)^\perp = R^\perp_{X,Y}N$ for any $X, Y \in \mathscr{X}(M)$, $N \in \mathscr{X}(M)^\perp$, where $R^\perp$ is the normal connection of M, because we can see, by the direct calculation,

$$(\bar{R}_{X,Y}N)^\perp = R_{X,Y}N - B(A^N(Y), X) + B(A^N(X), Y).$$

Let $\bar{M}$ be a real space form and M be a submanifold of $\bar{M}$. Then $A^iA^j = A^jA^i$ if and only if the normal connection of M is trivial ($R^\perp = 0$).

For the operator $A^{\sim}$, $JA^{\sim} = A^{\sim}J$ (see Simons [**7**, p. 76]). By the similar way as in the case of A^*, $A^{\sim}$ is represented by the matrix form, for a suitable basis,

$$A^{\sim} \equiv \begin{bmatrix} \mu_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \mu_{2p} \end{bmatrix}, \qquad \mu_{p+t} = \mu_t, \quad \mu_t \geqq 0.$$

Then we have

$$\langle A \cdot A^{\sim}, A \rangle = \sum_{i=1}^{2p} \langle A \cdot A^{\sim}(v_i), A^i \rangle = \sum_{i=1}^{2p} \mu_i \langle A^{\sim}(v_i), v_i \rangle = \sum_{i=1}^{2p} \mu_i^2.$$

From this we obtain the following:

$$(4.2) \quad \frac{1}{2p} ||A||^4 \leqslant \langle A \cdot A^{\sim}, A \rangle \leqslant \tfrac{1}{2} ||A||^4.$$

LEMMA 1. *Let M be a complex submanifold of complex dimension n $(n > 1)$ in $\bar{M}^{n+p}(c)$. Then M is Einstein if and only if $\langle A_{\sim} \cdot A, A \rangle = (1/n)||A||^4$.*

Proof. We have already that

$$\langle A_{\sim} \cdot A, A \rangle = 2 \sum_{t=1}^{2n} \langle (A^*)^2(e_t), e_t \rangle \quad \text{and} \quad Q = \tfrac{1}{2}(n+1)cI - A^*.$$

Therefore we obtain

$$\langle A_{\sim} \cdot A, A \rangle = \frac{1}{n} ||A||^4 - \frac{1}{n} K^2 + 2||Q||^2,$$

where $||Q||$ denotes the length of the Ricci operator Q. Generally, $K^2 \leqq 2n||Q||^2$ and equality holding if and only if M is Einstein. Hence M is Einstein if and only if $\langle A_{\sim} \cdot A, A \rangle = (1/n)||A||^4$.

If $\bar{M}$ is of constant holomorphic sectional curvature c, then the Simons' type formula is given by (see [**4**] and [**7**, p. 81])

$$(4.3) \quad \nabla^2 A = \tfrac{1}{2}(n+2)cA - A \cdot A^{\sim} - A_{\sim} \cdot A.$$

Here we notice that if the length of the second fundamental form A is constant, then $\langle \nabla A, \nabla A \rangle = -\langle \nabla^2 A, A \rangle$.

PROPOSITION 2. *Let M be an Einstein complex hypersurface of $\bar{M}^{n+1}(c)$. Then the second fundamental form A of M is parallel $(\nabla A = 0)$.*

Proof. By (4.2), (4.3) and Lemma 1, we get

$$\langle \nabla A, \nabla A \rangle = \frac{(n+2)}{2n} (||A||^2 - nc)||A||^2.$$

By Corollary 1 and (2.4), we get $||A||^2 = nc$, hence $\nabla A = 0$.

Remark 3. Let M be a complex hypersurface of $\bar{M}(c)$. If A is parallel, then Q is parallel. Hence by Corollary 1, M is Einstein. Consequently, M is Einstein if and only if $\nabla A = 0$. (Compare also Nomizu-Smyth [**3**, p. 507, Lemma 5].)

THEOREM 2. *Let M be an n-dimensional complex submanifold of $\bar{M}^{n+p}(c)$ with parallel second fundamental form* ($\nabla A = 0$). *If $c \leqq 0$, then M is totally geodesic. If $c > 0$, then the scalar curvature K of M satisfies*

$$K \geqq \frac{n^2c(n+p+1)}{(n+2p)}$$

and if equality holds, then M is an Einstein manifold.

Proof. From (4.1), (4.2) and (4.3), we get the following inequality:

$$0 = \langle \nabla A, \nabla A \rangle \geqslant \left(\frac{n+2p}{2pn} ||A||^2 - \frac{(n+2)c}{2} \right) ||A||^2.$$

Hence if $c \leqq 0$, then M is totally geodesic in $\bar{M}$.

Let $c > 0$. Then we obtain $||A||^2 \leqq pn(n+2)c/(n+2p)$. From this and (2.4), we can see that $K \geqq n^2c(n+p+1)/(n+2p)$. If equality holds, then $||A||^2 = pn(n+2)c/(n+2p)$ and hence, by (4.1), (4.2) and (4.3),

$$\left\{ \frac{(n+2)c}{2} - \frac{1}{2p} ||A||^2 \right\} ||A||^2 \geqslant \langle A_{\sim} \cdot A, A \rangle \geqslant \frac{1}{n} ||A||^4,$$

$$\frac{(n+2)c}{2} - \frac{1}{2p} ||A||^2 = \frac{1}{n} ||A||^2,$$

which imply $\langle A_{\sim} \cdot A, A \rangle = (1/n)||A||^4$. Therefore, by Lemma 1, M is an Einstein manifold.

Remark 4. For an Einstein complex hypersurface, Chern [**2**] proved that if $c \leqq 0$, then M is totally geodesic. By Proposition 2, our theorem is the extension of this. (See also Nomizu-Smyth [**3**] and Takahashi [**9**].)

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