J. Appl. Prob. **42,** 121–137 (2005)
Printed in Israel

ON OPTIMALITY OF BOLD PLAY FOR PRIMITIVE CASINOS IN THE PRESENCE OF INFLATION

ROBERT W. CHEN,* *University of Miami*
LARRY A. SHEPP,** *Rutgers University*
YI-CHING YAO,*** *Academia Sinica*
CUN-HUI ZHANG,** *Rutgers University*

Abstract

Mr. G owes \$100 000 to a loan shark, and will be killed at dawn if the loan is not repaid in full. Mr. G has \$20 000, but partial payments are not accepted, and he has no other source of income or credit. The loan shark owns a primitive casino where one can stake any amount in one's possession, gaining r times the stake with probability w and losing the stake with probability $1 - w$ ($r > 0, 0 < w < 1$). Mr. G is permitted to gamble at the casino, but each time he places a bet, the amount of his debt is increased by a factor of $1 + \alpha$ ($\alpha \geq 0$). How should Mr. G gamble to maximize his chance of reaching his (moving) target and thereby surviving? Dubins and Savage showed that an optimal strategy is to stake boldly if the primitive casino is subfair or fair (i.e. $w(1 + r) \leq 1$) and the inflation rate α is 0. Intuitively, a positive inflation rate would motivate Mr. G to try to reach his goal as quickly as possible, so it seems plausible that the bold strategy is optimal. However, Chen, Shepp, and Zame found that, surprisingly, the bold strategy is no longer optimal for subfair primitive casinos with inflation if both $r > 1$ and α satisfies $1/r \leq \alpha < r$. They also conjectured that the bold strategy is optimal for subfair primitive casinos with inflation if $r < 1$. It is shown in the present paper that this conjecture is true provided that $w \leq \frac{1}{2}$. Furthermore, by introducing an interesting notion of sharp strategy, additional results are obtained on optimality of the bold strategy.

Keywords: Gambling problem; primitive casino; optimal strategy; bold strategy; sharp strategy

2000 Mathematics Subject Classification: Primary 60G40
Secondary 91A60

1. Formulation of the problem

Mr. G owes \$100 000 to a loan shark, and will be killed at dawn if the loan is not repaid in full. Mr. G has \$20 000, but partial payments are not accepted, and he has no other source of income or credit. The loan shark owns a primitive casino where one can stake any amount in one's possession, gaining r times the stake with probability w and losing the stake with probability $\bar{w} = 1 - w$ ($r > 0, 0 < w < 1$). Mr. G is permitted to gamble at the primitive casino, but each time he places a bet, the amount of his debt is increased by a factor of $1 + \alpha$

Received 31 March 2004; revision received 23 September 2004.
* Postal address: Department of Mathematics, University of Miami, Coral Gables, FL 33124-4250, USA.
** Postal address: Department of Statistics, Rutgers University, Piscataway, NJ 08855, USA.
*** Postal address: Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, R.O.C.
Email address: yao@stat.sinica.edu.tw

($\alpha \geq 0$). How should Mr. G (to be referred to as the gambler) gamble to maximize his chance of reaching the (moving) target and thereby surviving (i.e. achieve the goal)?

In the absence of inflation (i.e. $\alpha = 0$), the primitive casino is subfair if $w < 1/(1+r)$ and is fair if $w = 1/(1+r)$. Dubins and Savage [3] showed that in a subfair or fair primitive casino with zero inflation rate, the gambler should stake boldly since there is no other strategy that provides him with a higher probability of reaching the goal. Intuitively, a positive inflation rate would motivate him to try to reach the goal as quickly as possible. Therefore, we would naturally conjecture that he should again stake boldly. Indeed, Chen [1] proved that the bold strategy is optimal for subfair primitive casinos with inflation if $r = 1$ (the so-called red-and-black casino). However, Chen *et al.* [2] found that, surprisingly, the bold strategy is not optimal for subfair primitive casinos with inflation if both $r > 1$ and α satisfies $1/r \leq \alpha < r$. They also conjectured that the bold strategy is optimal for subfair primitive casinos if $r < 1$. We show, in Section 2, that this conjecture is true provided that $w \leq \frac{1}{2}$. In Section 3, we introduce an interesting notion of sharp strategy that facilitates the construction of a subset of the interval $(0, 1)$ with the property that the bold strategy is optimal if the initial fortune f belongs to this subset. In Section 4, we present upper and lower approximations for the value function with approximation errors decaying at a geometric rate. Section 5 contains some concluding remarks. Related work can be found in [4] and [5], which consider the case in which the future is discounted.

As in [1], [2], and [3], we formally formulate the above problem as a Dubins–Savage gambling problem in which the set of fortunes, the utility function, and the set of available gambles are, respectively, as follows:

$$F = [0, \infty),$$

$$u(f) = \begin{cases} 0 & \text{if } 0 \leq f < 1, \\ 1 & \text{if } f \geq 1, \end{cases}$$

$$\Gamma(f) = \begin{cases} \left\{ w\delta\left(\dfrac{f + rs}{1+\alpha}\right) + \bar{w}\delta\left(\dfrac{f-s}{1+\alpha}\right) : 0 \leq s \leq f \right\} & \text{if } 0 \leq f < 1, \\ \{\delta(f)\} & \text{if } 1 \leq f < \infty. \end{cases}$$

Here, for $0 \leq x < \infty$, $\delta(x)$ denotes the probability measure that assigns probability 1 to $\{x\}$. The reason that $\Gamma(f)$ consists only of $\delta(f)$ for $f \geq 1$ is that, when the gambler has a fortune $f \geq 1$, he has reached the goal already and need not gamble any more. Clearly, if $\alpha \geq r$, the gambler with an initial fortune $f < 1$ can never reach his goal, so we assume that $0 < \alpha < r$ throughout the rest of the paper.

2. Optimality of the bold strategy for $r \leq 1$ and $w \leq \frac{1}{2}$

For each integer $n \geq 1$, let f_{n-1} be the gambler's fortune before the nth play (with f_0 denoting the initial fortune). A strategy $\sigma = \{y_1, y_2, \dots\}$ is a sequence of stakes, where $0 \leq y_n \leq f_{n-1}$ is the gambler's stake on the nth play. Given the gambler's fortune $f_{n-1} < 1$ before the nth play and the stake y_n on the nth play, his fortune f_n (after the nth play and before the $(n+1)$th play) will be $(f_{n-1} + ry_n)/(1+\alpha)$ with probability w and $(f_{n-1} - y_n)/(1+\alpha)$ with probability $\bar{w}$. The value of the strategy σ is $V_\sigma(f) = \mathrm{P}\{f_n \geq 1 \text{ for some } n \geq 0 \mid f_0 = f\}$ for $f \geq 0$. The value of the game is defined as $V^*(f) = \sup\{V_\sigma(f)\}$, where the supremum is taken over all possible strategies σ.

https://doi.org/10.1239/jap/1110381375 Published online by Cambridge University Press

The bold stake at the fortune f is defined by $b(f) = \min\{f, (1+\alpha-f)/r\}$ if $0 \le f < 1$ and $b(f) = 0$ if $f \ge 1$. The gambler is said to use the bold strategy if he stakes the bold stake $b(f)$ whenever he has a fortune f (and stops playing as soon as he is either 'broke', i.e. his fortune equals 0, or reaches his goal). Let 'B' denote the bold strategy and $V_{\mathrm{B}}(f)$ the value of the bold strategy. It is obvious that $V_{\mathrm{B}}(0) = 0$ and $V_{\mathrm{B}}(f) = 1$ for $f \ge 1$. For $0 < f < 1$, we have, by considering one play,

$$V_{\mathrm{B}}(f) = \bar{w} V_{\mathrm{B}}\left(\frac{f - b(f)}{1+\alpha}\right) + w V_{\mathrm{B}}\left(\frac{f + rb(f)}{1+\alpha}\right).$$

Therefore,

$$V_{\mathrm{B}}(f) = \begin{cases} w V_{\mathrm{B}}\left(\dfrac{f}{\beta}\right) & \text{for } 0 \le f \le \beta, \\ w + \bar{w} V_{\mathrm{B}}\left(\dfrac{f-\beta}{r\beta}\right) & \text{for } \beta \le f < 1, \\ 1 & \text{for } f \ge 1, \end{cases} \tag{1}$$

where $\beta = (1+\alpha)/(1+r) < 1$ since $\alpha < r$, as noted earlier.

Theorem 1. *Assume that* $0 \le \alpha < r \le 1$ *and* $0 < w \le \frac{1}{2}$. *Then,* V_{B} *is excessive; that is, for* $0 \le y \le f < 1$,

$$V_{\mathrm{B}}(f) \ge \overline{w} V_{\mathrm{B}}\left(\frac{f-y}{1+\alpha}\right) + w V_{\mathrm{B}}\left(\frac{f+ry}{1+\alpha}\right).$$

Remark 1. By Theorem 2.12.1 of [3], Theorem 1 implies that the bold strategy is optimal if $0 \le \alpha < r \le 1$ and $0 < w \le \frac{1}{2}$. In particular, Theorem 1 includes, as special cases, Theorem 2 of [1] and the case of the taxed coin with inflation (cf. Section 5.6 of [3]).

We first state two simple lemmas (without proof) that are needed in proving the theorem. Also, the simple fact that V_{B} is a nondecreasing function will be used (implicitly) several times in the proof of the theorem.

Lemma 1. *Suppose that* f *and* y *satisfy* $0 < y < f < 1$ *and*

$$\frac{f-y}{\beta(1+\alpha)} \ge \frac{(f+ry)/(1+\alpha) - \beta}{r\beta} > 0.$$

Define $f' = f'(f, y)$ *and* $y' = y'(f, y)$, *respectively, by*

$$f' = \frac{2f - (1-r)y}{1+\alpha} - \beta \quad \textit{and} \quad y' = \frac{\beta(1+\alpha) - (1-r)f - 2ry}{r(1+\alpha)}.$$

Then,

$$\frac{f' + ry'}{1+\alpha} = \frac{f-y}{\beta(1+\alpha)} \ge \frac{f'-y'}{1+\alpha} = \frac{(f+ry)/(1+\alpha) - \beta}{r\beta} > 0$$

and $0 \le y' < f'$.

Lemma 2. *Suppose that* f *and* y *satisfy* $0 < y < f < 1$ *and*

$$\frac{(f+ry)/(1+\alpha) - \beta}{r\beta} \ge \frac{f-y}{\beta(1+\alpha)} > 0.$$

Define $f' = f'(f, y)$ and $y' = y'(f, y)$, respectively, by

$$f' = \frac{(r^2+1)f + r(1-r)y - \beta(1+\alpha)}{r(1+\alpha)} \quad \textit{and} \quad y' = \frac{(1-r)f + 2ry - \beta(1+\alpha)}{r(1+\alpha)}.$$

Then,

$$\frac{f' + ry'}{1+\alpha} = \frac{(f+ry)/(1+\alpha) - \beta}{r\beta} \geq \frac{f' - y'}{1+\alpha} = \frac{f-y}{\beta(1+\alpha)} > 0$$

and $0 \leq y' < f'$.

Proof of Theorem 1. For notational simplicity, we write V for V_{B} throughout the proof. Define, for $0 \leq y \leq f < 1$,

$$H(f, y) = \overline{w}V\left(\frac{f-y}{1+\alpha}\right) + wV\left(\frac{f+ry}{1+\alpha}\right) - V(f) \tag{2}$$

and

$$S = \{(f, y) \colon 0 \leq y \leq f < 1, H(f, y) > 0\}.$$

We need to prove that $S = \varnothing$. Note that $H(f, y) \leq 0$ for $0 \leq y \leq (\alpha/r)f$, since

$$0 \leq \frac{f-y}{1+\alpha} \leq \frac{f+ry}{1+\alpha} \leq f,$$

and that

$$H(f, y) \leq H(f, b(f)) = 0 \quad \text{for} \quad b(f) \leq y \leq f < 1.$$

So, for $(f, y) \in S$, we have

$$0 < \frac{\alpha}{r} f < y < b(f), \tag{3}$$

which implies that

$$0 < \frac{f-y}{1+\alpha} < f < \frac{f+ry}{1+\alpha} < 1. \tag{4}$$

We now show that if $(f, y) \in S$, then there exists some $(f', y') \in S$ such that $H(f', y') \geq H(f, y)/\overline{w}$ (which implies that $S = \varnothing$, since H is bounded). In what follows, assume that $(f, y) \in S$. In view of (4), we treat the following four cases separately.

Case (i): $(f-y)/(1+\alpha) < f < (f+ry)/(1+\alpha) \leq \beta$. By (1) and (2),

$$\begin{aligned} V(f) &= wV\left(\frac{f}{\beta}\right) = w\left\{\overline{w}V\left(\frac{f/\beta - y/\beta}{1+\alpha}\right) + wV\left(\frac{f/\beta + ry/\beta}{1+\alpha}\right) - H\left(\frac{f}{\beta}, \frac{y}{\beta}\right)\right\} \\ &= \overline{w}V\left(\frac{f-y}{1+\alpha}\right) + wV\left(\frac{f+ry}{1+\alpha}\right) - wH\left(\frac{f}{\beta}, \frac{y}{\beta}\right), \end{aligned}$$

so that

$$H\left(\frac{f}{\beta}, \frac{y}{\beta}\right) = \frac{H(f, y)}{w} \geq \frac{H(f, y)}{\overline{w}}.$$

Case (ii): $\beta \le (f-y)/(1+\alpha) < f < (f+ry)/(1+\alpha)$. Let $f' = (f-\beta)/r\beta$ and $y' = (y+\alpha\beta)/r\beta$, which satisfy $0 < y' \le f' < 1$. By (1) and (2),

$$
\begin{aligned}
V(f) &= w + \overline{w}V(f') \\
&= w + \overline{w}\left\{\overline{w}V\left(\frac{f'-y'}{1+\alpha}\right) + wV\left(\frac{f'+ry'}{1+\alpha}\right) - H(f',y')\right\} \\
&\ge \overline{w}\left\{w + \overline{w}V\left(\frac{f'-y'}{1+\alpha}\right)\right\} + w\left\{w + \overline{w}V\left(\frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\right)\right\} - \overline{w}H(f',y') \\
&= \overline{w}V\left(\frac{f-y}{1+\alpha}\right) + wV\left(\frac{f+ry}{1+\alpha}\right) - \overline{w}H(f',y') \\
&= V(f) + H(f,y) - \overline{w}H(f',y'),
\end{aligned}
$$

where the inequality follows from the fact that

$$\frac{f'+ry'}{1+\alpha} \ge \frac{(f+ry)/(1+\alpha)-\beta}{r\beta}.$$

Hence, $H(f',y') \ge H(f,y)/\overline{w}$.

Case (iii): $(f-y)/(1+\alpha) < f < \beta < (f+ry)/(1+\alpha)$. Since $y < b(f) = f$ (by (3)), we have

$$\beta < (f+ry)/(1+\alpha) < \{(1+r)f\}/(1+\alpha) = f/\beta < 1,$$

so that

$$
\begin{aligned}
V(f) &= wV\left(\frac{f}{\beta}\right) = w\left\{w + \overline{w}V\left(\frac{f/\beta-\beta}{r\beta}\right)\right\} \\
&= w^2 + \overline{w}wV\left(\frac{f/\beta-\beta}{r\beta}\right) = w^2 + \overline{w}V\left(\frac{f/\beta-\beta}{r}\right), \\
V\left(\frac{f-y}{1+\alpha}\right) &= wV\left(\frac{f-y}{\beta(1+\alpha)}\right), \\
V\left(\frac{f+ry}{1+\alpha}\right) &= w + \overline{w}V\left(\frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\right).
\end{aligned}
$$

(Note that $(f/\beta-\beta)/r < \beta$.) It follows that

$$\frac{H(f,y)}{\overline{w}} = wV\left(\frac{f-y}{\beta(1+\alpha)}\right) + wV\left(\frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\right) - V\left(\frac{f/\beta-\beta}{r}\right). \tag{5}$$

If

$$\frac{f/\beta-\beta}{r} \ge \max\left\{\frac{f-y}{\beta(1+\alpha)}, \frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\right\},$$

then the right-hand side of (5) is less than or equal to 0 (since $w \le \frac{1}{2}$), contradicting the assumption that $(f,y) \in S$. It now suffices to consider the following two subcases.

(iii.1) In this case,

$$\frac{f/\beta-\beta}{r} < \frac{f-y}{\beta(1+\alpha)} = \max\left\{\frac{f-y}{\beta(1+\alpha)}, \frac{1}{r\beta}\left(\frac{f+ry}{1+\alpha}-\beta\right)\right\}.$$

Let $f' = f'(f, y)$ and $y' = y'(f, y)$ be as defined in Lemma 1, so that $0 \le y' < f'$. Then,

$$\frac{f/\beta - \beta}{r} - f' = \frac{(1+r)f/(1+\alpha) - \beta}{r} - \frac{2f - (1-r)y}{1+\alpha} + \beta$$
$$= \frac{(1-r)(f + ry - \beta(1+\alpha))}{r(1+\alpha)} \ge 0,$$

since $r \le 1$ and $\beta < (f + ry)/(1+\alpha)$. Thus, we have $0 \le y' \le f' < 1$. Since $w \le \frac{1}{2}$,

$$V\left(\frac{f/\beta - \beta}{r}\right) \ge V(f')$$
$$= \overline{w}V\left(\frac{f' - y'}{1+\alpha}\right) + wV\left(\frac{f' + ry'}{1+\alpha}\right) - H(f', y')$$
$$\ge wV\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right) + wV\left(\frac{f - y}{\beta(1+\alpha)}\right) - H(f', y').$$

By (5), $H(f', y') \ge H(f, y)/\overline{w}$.

(iii.2) In this case,

$$\frac{f/\beta - \beta}{r} < \frac{1}{r\beta}\left(\frac{f + ry}{1+\alpha} - \beta\right) = \max\left\{\frac{f - y}{\beta(1+\alpha)}, \frac{1}{r\beta}\left(\frac{f + ry}{1+\alpha} - \beta\right)\right\}.$$

Let $f' = f'(f, y)$ and $y' = y'(f, y)$ be as defined in Lemma 2, so that $0 \le y' < f'$. Then,

$$\frac{f/\beta - \beta}{r} - f' = \frac{r(1-r)(f - y)}{r(1+\alpha)} \ge 0.$$

Thus, we have $0 \le y' \le f' < 1$. Since $w \le \frac{1}{2}$,

$$V\left(\frac{f/\beta - \beta}{r}\right) \ge V(f')$$
$$= \overline{w}V\left(\frac{f' - y'}{1+\alpha}\right) + wV\left(\frac{f' + ry'}{1+\alpha}\right) - H(f', y')$$
$$\ge wV\left(\frac{f - y}{\beta(1+\alpha)}\right) + wV\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right) - H(f', y').$$

By (5), $H(f', y') \ge H(f, y)/\overline{w}$.

Case (iv): $(f - y)/(1+\alpha) < \beta \le f < (f + ry)/(1+\alpha)$. Since $y < b(f) = (1 + \alpha - f)/r$ (by (3)), we have

$$\beta > \frac{f - y}{1+\alpha} > \frac{f - (1 + \alpha - f)/r}{1+\alpha} = \frac{f - \beta}{r\beta},$$

so that

$$V(f) = w + \overline{w}V\left(\frac{f - \beta}{r\beta}\right) = w + \overline{w}wV\left(\frac{f - \beta}{r\beta^2}\right),$$
$$V\left(\frac{f - y}{1+\alpha}\right) = wV\left(\frac{f - y}{\beta(1+\alpha)}\right),$$
$$V\left(\frac{f + ry}{1+\alpha}\right) = w + \overline{w}V\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right).$$

It follows that

$$\frac{H(f,y)}{w} = \overline{w}\biggl\{V\biggl(\frac{f-y}{\beta(1+\alpha)}\biggr) + V\biggl(\frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\biggr) - V\biggl(\frac{f-\beta}{r\beta^2}\biggr) - 1\biggr\}. \quad (6)$$

If

$$\frac{f-\beta}{r\beta^2} \geq \frac{f-y}{\beta(1+\alpha)} \quad \text{or} \quad \frac{f-\beta}{r\beta^2} \geq \frac{(f+ry)/(1+\alpha)-\beta}{r\beta}, \quad \text{i.e.}$$

$$y \geq \frac{1+\alpha-f}{r} \quad \text{or} \quad y \leq f-\beta\biggl(1-\frac{\alpha}{r}\biggr),$$

then the right-hand side of (6) is less than or equal to 0, contradicting the assumption that $(f,y) \in S$. Hence, we must have

$$\frac{1+\alpha-f}{r} > f-\beta\biggl(1-\frac{\alpha}{r}\biggr) \quad (\text{i.e. } f < 2\beta-\beta^2)$$

and

$$\frac{\alpha}{r}f \leq f-\beta\biggl(1-\frac{\alpha}{r}\biggr) < y < \frac{1+\alpha-f}{r} = b(f).$$

(Note that $f-\beta(1-\alpha/r) \geq f-f(1-\alpha/r) = (\alpha/r)f$.) Since $\beta \leq f/\beta - 1 + \beta < 1$,

$$V\biggl(\frac{f}{\beta}-1+\beta\biggr) = w + \overline{w}V\biggl(\frac{f/\beta-1+\beta-\beta}{r\beta}\biggr) = w + \overline{w}V\biggl(\frac{f-\beta}{r\beta^2}\biggr),$$

so that, by (6),

$$\frac{H(f,y)}{w} = \overline{w}V\biggl(\frac{f-y}{\beta(1+\alpha)}\biggr) + \overline{w}V\biggl(\frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\biggr) - V\biggl(\frac{f}{\beta}-1+\beta\biggr) + w - \overline{w}. \quad (7)$$

If

$$\frac{f}{\beta} - 1 + \beta \geq \max\biggl\{\frac{f-y}{\beta(1+\alpha)}, \frac{(f+ry)/(1+\alpha)-\beta}{r\beta}\biggr\},$$

then the right-hand side of (7) is less than or equal to 0, contradicting the assumption that $(f,y) \in S$. It suffices to consider the following two subcases. (Note that $f/\beta - 1 + \beta < 1$.)

(iv.1) In this case,

$$\frac{f}{\beta} - 1 + \beta < \frac{f-y}{\beta(1+\alpha)} = \max\biggl\{\frac{f-y}{\beta(1+\alpha)}, \frac{1}{r\beta}\biggl(\frac{f+ry}{1+\alpha} - \beta\biggr)\biggr\}.$$

Let $f' = f'(f,y)$ and $y' = y'(f,y)$ be as defined in Lemma 1, so that $0 \leq y' < f'$. Then,

$$\begin{aligned}\biggl(\frac{f}{\beta}-1+\beta\biggr) - f' &= \frac{(r-1)f+(1-r)y}{1+\alpha} - 1 + 2\beta \\ &= -(1-r)\biggl(\frac{f-y}{1+\alpha}\biggr) - 1 + 2\beta \\ &\geq -(1-r)\beta - 1 + 2\beta \\ &= \alpha \geq 0.\end{aligned}$$

https://doi.org/10.1239/jap/1110381375 Published online by Cambridge University Press

Thus, we have $0 \le y' \le f' < 1$. Therefore,

$$V\left(\frac{f}{\beta} - 1 + \beta\right) \ge V(f') = \overline{w}V\left(\frac{f' - y'}{1+\alpha}\right) + wV\left(\frac{f' + ry'}{1+\alpha}\right) - H(f', y')$$
$$= \overline{w}V\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right) + wV\left(\frac{f - y}{\beta(1+\alpha)}\right) - H(f', y')$$

and, by (7),

$$\frac{H(f, y)}{\overline{w}} \le \frac{H(f, y)}{w}$$
$$\le H(f', y') + (\overline{w} - w)\left(V\left(\frac{f - y}{\beta(1+\alpha)}\right) - 1\right) \le H(f', y').$$

(iv.2) In this case,

$$\frac{f}{\beta} - 1 + \beta < \frac{1}{r\beta}\left(\frac{f + ry}{1+\alpha} - \beta\right) = \max\left\{\frac{f - y}{\beta(1+\alpha)}, \frac{1}{r\beta}\left(\frac{f + ry}{1+\alpha} - \beta\right)\right\}.$$

Let $f' = f'(f, y)$ and $y' = y'(f, y)$ be as defined in Lemma 2, so that $0 \le y' < f'$. Then,

$$\left(\frac{f}{\beta} - 1 + \beta\right) - f' = \frac{(r-1)f - r(1-r)y}{r(1+\alpha)} - 1 + \beta\left(1 + \frac{1}{r}\right)$$
$$= -\left(\frac{1-r}{r}\right)\left(\frac{f + ry}{1+\alpha}\right) - 1 + \frac{1+\alpha}{r}$$
$$\ge -\left(\frac{1-r}{r}\right) - 1 + \frac{1+\alpha}{r} = \frac{\alpha}{r} \ge 0,$$

since $y < b(f) = (1 + \alpha - f)/r$. Thus, we have $0 \le y' \le f' < 1$. Therefore,

$$V\left(\frac{f}{\beta} - 1 + \beta\right) \ge V(f') = \overline{w}V\left(\frac{f' - y'}{1+\alpha}\right) + wV\left(\frac{f' + ry'}{1+\alpha}\right) - H(f', y')$$
$$= \overline{w}V\left(\frac{f - y}{\beta(1+\alpha)}\right) + wV\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right) - H(f', y')$$

and, by (7),

$$\frac{H(f, y)}{\overline{w}} \le \frac{H(f, y)}{w}$$
$$\le H(f', y') + (\overline{w} - w)\left(V\left(\frac{(f + ry)/(1+\alpha) - \beta}{r\beta}\right) - 1\right) \le H(f', y').$$

This completes the proof of Theorem 1.

3. Sharp strategy and further optimality results on the bold strategy

Let $\mathcal{B} = \{f : 0 < f < 1, V_{\mathrm{B}}(f) = V^*(f)\}$, the subset of the interval (0, 1) with the property that the bold strategy is optimal if and only if the initial fortune is in this subset. For $0 < f < \beta$, observe that

$$V^*(f) \ge wV^*\left(\frac{f}{\beta}\right) \ge wV_{\mathrm{B}}\left(\frac{f}{\beta}\right) = V_{\mathrm{B}}(f).$$

So, for $0 < f < \beta$, $f \in \mathcal{B}$ implies that $f/\beta \in \mathcal{B}$, since $V^*(f) = V_{\mathrm{B}}(f)$ implies that $V^*(f/\beta) = V_{\mathrm{B}}(f/\beta)$. For $\beta < f < 1$, we have

$$V^*(f) \geq \bar{w} V^*\left(\frac{f - b(f)}{1+\alpha}\right) + w \geq \bar{w} V_{\mathrm{B}}\left(\frac{f - b(f)}{1+\alpha}\right) + w = V_{\mathrm{B}}(f),$$

from which it follows that, for $\beta < f < 1$,

$$f \in \mathcal{B} \Rightarrow \frac{f - b(f)}{1+\alpha} = \frac{f - \beta}{r\beta} \in \mathcal{B}.$$

The following lemma shows that $\beta^n \in \mathcal{B},\ n = 1, 2, \ldots$, if $w < \beta$.

Lemma 3. *Let $p = (\log w)/(\log \beta)$, i.e. $w = \beta^p$. Suppose that $w < \beta$, i.e. $p > 1$. Then, $V^*(f) \leq f^p$ for $0 \leq f \leq 1$ and $V^*(\beta^n) = V_{\mathrm{B}}(\beta^n) = w^n,\ n = 1, 2, \ldots$.*

Proof. Let $\mathcal{C}(\alpha, r, w)$ denote the primitive casino under consideration, in which the gambler's fortune becomes $(f + ry)/(1+\alpha)$ with probability w and $(f - y)/(1+\alpha)$ with probability $\bar{w}$ if he stakes an amount y of the initial fortune f. Since

$$\frac{f + ry}{1+\alpha} \leq f + \left(\frac{1-\beta}{\beta}\right) y \quad \text{and} \quad \frac{f - y}{1+\alpha} \leq f - y,$$

the casino $\mathcal{C}(\alpha, r, w)$ is less favorable to the gambler than $\mathcal{C}(0, (1-\beta)/\beta, w)$. For the latter casino, by Theorem 6.8.1 of [3], the value function at f is bounded by f^p. (Note that 'r' in [3] (which is always between 0 and 1) has the following meaning: if the gambler stakes an amount y and wins, then he receives $y/r = y + (1/r - 1)y$, where $(1/r - 1)y$ is the gain. Thus the 'r' in [3] corresponds to '$1/(1+r)$' in our notation.) It follows that $V^*(f) \leq f^p$. Next, we have, by (1),

$$V_{\mathrm{B}}(\beta^n) = w^n = (\beta^n)^p \geq V^*(\beta^n),$$

implying that $V_{\mathrm{B}}(\beta^n) = V^*(\beta^n) = w^n$. This completes the proof.

We now introduce the notion of sharp strategy, which helps us to find additional points in $\mathcal{B}$. The sharp stake at the fortune f is defined by

$$s(f) = \begin{cases} \dfrac{\{(1+\alpha)\beta^n - f\}}{r} & \text{if } \beta^{n+1} \leq f < \beta^n, \\ 0 & \text{if } f \geq 1, \end{cases}$$

where n is a nonnegative integer. The gambler is said to use the sharp strategy if he makes the sharp stake $s(f)$ whenever he has a fortune f (and stops playing as soon as he is either broke or reaches his goal). Let 'S' denote the sharp strategy and $V_{\mathrm{S}}(f)$ the value of the sharp strategy. The next lemma shows that V_{B} and V_{S} are identical. We note in passing that the sharp stake is related to the notion of conserving stake introduced in [3].

Lemma 4. $V_{\mathrm{B}}(f) = V_{\mathrm{S}}(f)$ *for all* $f \geq 0$.

Proof. For each $k = 1, 2, \ldots$, let 'kSB' denote the strategy that the gambler makes the sharp stake for the first k plays and then makes the bold stake for the remaining plays (and stops playing as soon as he is either broke or reaches his goal). The value of the strategy kSB is denoted $V_{k\mathrm{SB}}$. We first show that $V_{\mathrm{B}}(f) = V_{1\mathrm{SB}}(f)$ for all $f \geq 0$. It suffices to consider $0 < f < 1$. Let $n \geq 0$ be the unique integer satisfying $\beta^{n+1} \leq f < \beta^n$. Then, by (1),

$$\begin{aligned} V_{1\mathrm{SB}}(f) &= w V_{\mathrm{B}}(\beta^n) + \bar{w} V_{\mathrm{B}}\left(\frac{f - s(f)}{1+\alpha}\right) \\ &= w^{n+1} + \bar{w} w^n V_{\mathrm{B}}\left(\frac{f - s(f)}{\beta^n(1+\alpha)}\right), \\ V_{\mathrm{B}}(f) &= w^n V_{\mathrm{B}}(\beta^{-n} f) \\ &= w^{n+1} + w^n \bar{w} V_{\mathrm{B}}\left(\frac{\beta^{-n} f - \beta}{r\beta}\right). \end{aligned}$$

It follows that $V_{\mathrm{B}}(f) = V_{1\mathrm{SB}}(f)$ since

$$\frac{f - s(f)}{\beta^n(1+\alpha)} = \frac{\beta^{-n} f - \beta}{r\beta},$$

by the definition of $s(f)$. We now show that

$$V_{k\mathrm{SB}}(f) = V_{(k+1)\mathrm{SB}}(f) \quad \text{for } 0 < f < 1 \text{ and } k = 1, 2, \ldots.$$

Noting that the two strategies kSB and $(k+1)$SB are identical for the first k plays, let X denote the gambler's fortune after k plays using either strategy. Here, we use the convenient convention that X is defined to be 0 or 1 if the gambler is broke or reaches his goal, respectively, before the kth play. Then,

$$V_{(k+1)\mathrm{SB}}(f) = \mathrm{E}\, V_{1\mathrm{SB}}(X) = \mathrm{E}\, V_{\mathrm{B}}(X) = V_{k\mathrm{SB}}(f).$$

We have shown that $V_{\mathrm{B}}(f) = V_{k\mathrm{SB}}(f)$ for all $f \geq 0$ and $k = 1, 2, \ldots$. Since, if he uses the sharp strategy, the gambler is either broke or reaches his goal in a finite number of plays with probability 1, it follows that $V_{k\mathrm{SB}}(f) \to V_{\mathrm{S}}(f)$ as $k \to \infty$, implying that $V_{\mathrm{B}}(f) = V_{\mathrm{S}}(f)$. This completes the proof of Lemma 4.

We now assume that $0 < \alpha < r$ and $w < \beta$, so that $\beta = (1+\alpha)/(1+r) < 1$ and $p = (\log w)/(\log \beta) > 1$. Define

$$H(x) = \beta^p + (1 - \beta^p)x^p - \left(\frac{1+rx}{1+r}\right)^p, \qquad 0 \leq x \leq 1.$$

Noting that $H(0) = \beta^p - (1/(1+r))^p > 0$ and $H(1) = 0$, let $x_0 = \inf\{x \leq 1 \colon H(x) = 0\}$ with $0 < x_0 \leq 1$. Also, let

$$k_0 = \inf\{k \in \mathbb{Z} \colon \beta^k < x_0,\ \beta + r\beta^{k+1} < 1\} \tag{8}$$

and

$$a_0 = \min\{1, \beta + r\beta x_0\}, \tag{9}$$

so that $\beta < a_0 \leq 1$. For $n = 1, 2, \ldots$, let $D_n = \{\beta^n + r\beta^{n+k} \colon k = k_0, k_0 + 1, \ldots\}$, which is contained in (β^n, β^{n-1}). We will show that $D_n \subset \mathcal{B}$.

Remark 2. It can be readily checked that $H'(x) = \mathrm{d}H(x)/\mathrm{d}x$ has exactly one zero, or none, in $(0, 1)$ if $w(1+r) < 1$ or $w(1+r) \ge 1$, respectively. As a result, we have $x_0 = 1$ for $w(1+r) \ge 1$. On the other hand, for $w(1+r) < 1$ and with x_1 denoting the unique zero of $H'(x)$ in $(0, 1)$, the facts that $H'(x) < 0$ for $x < x_1$, $H'(x) > 0$ for $x > x_1$, $H(0) > 0$, and $H(1) = 0$ show that x_0 is the only zero of $H(x)$ in $(0, 1)$, and that $0 < x_0 < x_1 < 1$.

Lemma 5. *Assume that $0 < \alpha < r$ and $w < \beta$, and let $V_0(f) = f^p$ for $f \ge 0$. Define the following functions recursively. For $n = 0, 1, \ldots$, let*

$$U_n(y; f) = \bar{w} V_n\left(\frac{f-y}{1+\alpha}\right) + w V_n\left(\frac{f+ry}{1+\alpha}\right) \quad \text{for } 0 \le y \le b(f) \text{ and } 0 \le f < \beta^n a_0, \tag{10}$$

$$V_{n+1}(f) = \begin{cases} \sup\{U_n(y; f) \colon 0 \le y \le b(f)\} & \text{for } \beta^{n+1} \le f < \beta^n a_0, \\ V_0(f) & \text{for } 0 \le f < \beta^{n+1}. \end{cases} \tag{11}$$

Then,

$$V^*(f) \le V_{n+1}(f) \le V_0(f) \quad \text{for } 0 \le f < \beta^n a_0 \text{ and } n = 0, 1, \ldots.$$

Proof. For $0 \le f < a_0$,

$$\begin{aligned} V^*(f) &= \sup\left\{\bar{w} V^*\left(\frac{f-y}{1+\alpha}\right) + w V^*\left(\frac{f+ry}{1+\alpha}\right) \colon 0 \le y \le f\right\} \\ &= \sup\left\{\bar{w} V^*\left(\frac{f-y}{1+\alpha}\right) + w V^*\left(\frac{f+ry}{1+\alpha}\right) \colon 0 \le y \le b(f)\right\} \\ &\le \sup\left\{\bar{w} V_0\left(\frac{f-y}{1+\alpha}\right) + w V_0\left(\frac{f+ry}{1+\alpha}\right) \colon 0 \le y \le b(f)\right\} \\ &= \sup\{U_0(y; f) \colon 0 \le y \le b(f)\}, \end{aligned} \tag{12}$$

where the first equality is due to the optimality of V^*; the second equality follows from V^* being nondecreasing and the fact that $V^*(x) = 1$ for $x \ge 1$; and the inequality follows from $V^* \le V_0$ (cf. Lemma 3). By (11) (with $n = 0$) and (12), we have $V^*(f) \le V_1(f)$ for $0 \le f < a_0$. A similar argument, along with induction on n, yields

$$V^*(f) \le V_{n+1}(f) \quad \text{for } 0 \le f < \beta^n a_0 \text{ and } n = 0, 1, \ldots.$$

Next, we show that

$$V_{n+1}(f) \le V_0(f) \quad \text{for } 0 \le f < \beta^n a_0 \text{ and } n = 0, 1, \ldots. \tag{13}$$

For $\beta \le f < a_0$,

$$\begin{aligned} V_1(f) &= \sup\{U_0(y; f) \colon 0 \le y \le b(f)\} \\ &= \sup\left\{\bar{w} V_0\left(\frac{f-y}{1+\alpha}\right) + w V_0\left(\frac{f+ry}{1+\alpha}\right) \colon 0 \le y \le b(f)\right\} \\ &\le \sup\left\{\bar{w} V_0\left(\frac{f-y}{1+\alpha}\right) + w V_0\left(\frac{f+ry}{1+\alpha}\right) \colon 0 \le y \le f\right\} \\ &= \max\left\{V_0\left(\frac{f}{1+\alpha}\right), w V_0\left(\frac{f}{\beta}\right)\right\} \\ &= f^p = V_0(f), \end{aligned}$$

where we have used the fact that $\bar{w}V_0((f-y)/(1+\alpha)) + wV_0((f+ry)/(1+\alpha))$ is convex in $y \in [0, f]$, so that its maximum is attained at either $y = 0$ or $y = f$. This establishes (13) for $n = 0$. A similar argument, along with induction on n, yields (13) for all n. This completes the proof of Lemma 5.

Remark 3. It follows from (11) and (13) that $V_{n+1}(f) \le V_n(f)$ for $0 \le f < \beta^n a_0$.

Theorem 2. *Assume that* $0 < \alpha < r$ *and* $w < \beta$. *Then* $D_n \subset \mathcal{B}$; *that is,*

$$V_{\mathrm{B}}(\beta^n + r\beta^{n+k}) = V^*(\beta^n + r\beta^{n+k}) \quad \textit{for } k = k_0, k_0+1, \dots \textit{ and } n = 1, 2, \dots.$$

Proof. By Lemmas 3 and 5, we have, for $n = 0, 1, \dots,$

$$V_0(\beta^{n+1}) = w^{n+1} = V^*(\beta^{n+1}) \le V_{n+1}(\beta^{n+1}) \le V_0(\beta^{n+1}) \quad (\text{since } \beta^{n+1} < \beta^n a_0),$$

implying that

$$V_{n+1}(\beta^{n+1}) = w^{n+1}.$$

It follows that $V_{n+1}(f)$ is continuous at $f = \beta^{n+1}$. Standard arguments show that $V_{n+1}(f)$ is also continuous at all other points in $[0, \beta^n a_0)$. So, for a fixed $0 \le f < \beta^n a_0$ ($n = 0, 1, \dots$), $U_n(y; f)$ is continuous in $y \in [0, b(f)]$. Define, for $0 \le f < \beta^n a_0,\ n = 0, 1, \dots,$

$$y_n(f) = \arg\max\{U_n(y; f) \colon 0 \le y \le b(f)\},$$

so that $V_{n+1}(f) = U_n(y_n(f); f)$ for $\beta^{n+1} \le f < \beta^n a_0$. (Here, $y_n(f)$ is taken to be the smallest value of y at which the maximum is attained, if more than one exists.) We first establish the following facts: for $n = 0, 1, \dots,$

$$y_n(f) = s(f) = \frac{(1+\alpha)\beta^n - f}{r} \quad \text{for } \beta^{n+1} \le f < \beta^n a_0, \tag{14}$$

$$V_{n+1}(f) = w^n V_1\left(\frac{f}{\beta^n}\right) \quad \text{for } 0 \le f < \beta^n a_0. \tag{15}$$

We proceed by induction on n. First consider (14) with $n = 0$. Since $U_0(y; f)$ is strictly convex in y, $\sup\{U_0(y; f) \colon 0 \le f \le b(f)\}$ is attained at either $y = 0$ or $y = b(f)$. For $\beta \le f < a_0$,

$$\begin{aligned} U_0(b(f); f) - U_0(0; f) &= w + \bar{w}\left(\frac{f-\beta}{r\beta}\right)^p - \left(\frac{f}{1+\alpha}\right)^p \\ &= H\left(\frac{f-\beta}{r\beta}\right) > 0 \end{aligned}$$

since $(f-\beta)/r\beta < (a_0-\beta)/r\beta \le x_0$, by (9). It follows that $y_0(f) = b(f) = s(f)$ for $\beta \le f < a_0$. This proves (14) for $n = 0$. The case $n = 0$ in (15) is trivial. Incidentally, later we will need the fact that

$$V_1(f) = U_0(b(f); f) = w + \bar{w}\left(\frac{f-\beta}{r\beta}\right)^p, \qquad \beta \le f < a_0, \quad \text{is strictly convex.} \tag{16}$$

Suppose that (14) and (15) hold for $n = m - 1$ ($m \geq 1$). We now prove that (14) and (15) hold for $n = m$.

To prove (14) with $n = m$, fix an f such that $\beta^{m+1} \leq f < \beta^m a_0$. For $y < s(f) = \{(1+\alpha)\beta^m - f\}/r$,

$$\begin{aligned} U_m(y; f) &= \bar{w} V_m\left(\frac{f-y}{1+\alpha}\right) + w V_m\left(\frac{f+ry}{1+\alpha}\right) \\ &= \bar{w}\left(\frac{f-y}{1+\alpha}\right)^p + w\left(\frac{f+ry}{1+\alpha}\right)^p \end{aligned}$$

is strictly convex in y, where the second equality follows from (11) (with $n = m - 1$) since

$$\frac{f-y}{1+\alpha} \leq \frac{f+ry}{1+\alpha} < \frac{f+rs(f)}{1+\alpha} = \beta^m.$$

Therefore,

$$\begin{aligned} \sup\{U_m(y; f) \colon 0 \leq y < s(f)\} &= \max\{U_m(0; f), U_m(s(f)-; f)\} \\ &= \max\{U_m(0; f), U_m(s(f); f)\} \\ &= U_m(s(f); f) \end{aligned}$$

since

$$\begin{aligned} U_m(s(f); f) - U_m(0; f) &= w(\beta^m)^p + \bar{w}\left(\frac{f-\beta^{m+1}}{r\beta}\right)^p - \left(\frac{f}{1+\alpha}\right)^p \\ &= \beta^{mp}\left\{w + \bar{w}\left(\frac{f/\beta^m - \beta}{r\beta}\right)^p - \left(\frac{1 + r((f/\beta^m - \beta)/r\beta)}{1+r}\right)^p\right\} \\ &= \beta^{mp} H\left(\frac{f/\beta^m - \beta}{r\beta}\right) \\ &> 0, \end{aligned}$$

where $(f/\beta^m - \beta)/r\beta < (a_0 - \beta)/r\beta \leq x_0$.

For $s(f) \leq y \leq b(f) = f$,

$$U_m(y; f) = \bar{w} V_m\left(\frac{f-y}{1+\alpha}\right) + w V_m\left(\frac{f+ry}{1+\alpha}\right).$$

Since

$$\frac{f-y}{1+\alpha} \leq \frac{f-s(f)}{1+\alpha} = \frac{f-\beta^{m+1}}{r\beta} < \frac{\beta^m a_0 - \beta^{m+1}}{r\beta} \leq \beta^m \quad \text{(by (9))},$$

we have, by (11) (with $n = m - 1$),

$$V_m\left(\frac{f-y}{1+\alpha}\right) = \left(\frac{f-y}{1+\alpha}\right)^p,$$

which is strictly convex in y. Moreover, since

$$\beta^{m-1} a_0 > \frac{f}{\beta} = \frac{f+rf}{1+\alpha} \geq \frac{f+ry}{1+\alpha} \geq \frac{f+rs(f)}{1+\alpha} = \beta^m,$$

we have, by the induction hypothesis,

$$V_m\left(\frac{f+ry}{1+\alpha}\right) = w^{m-1}V_1\left(\frac{f+ry}{(1+\alpha)\beta^{m-1}}\right),$$

which is strictly convex in y, by (16). Therefore, $U_m(y; f)$ is strictly convex in $y \in [s(f), f]$, implying that

$$\sup\{U_m(y; f)\colon s(f) \le y \le f\} = \max\{U_m(s(f); f), U_m(f; f)\}.$$

Verifying that $U_m(s(f); f) = U_m(f; f)$, we have

$$\sup\{U_m(y; f)\colon 0 \le y \le f\} = U_m(s(f); f)$$

and $y_m(f) = s(f)$, proving (14) with $n = m$.

We now prove (15) with $n = m$. For $0 \le f < \beta^{m+1}$, by (11),

$$w^m V_1\left(\frac{f}{\beta^m}\right) = w^m\left(\frac{f}{\beta^m}\right)^p = f^p = V_{m+1}(f)$$

and, for $\beta^{m+1} \le f < \beta^m a_0$,

$$\begin{aligned}
V_{m+1}(f) &= U_m(y_m(f); f) = U_m(s(f); f) \quad \text{(by (14) with } n = m)\\
&= \bar{w}V_m\left(\frac{f-\beta^{m+1}}{r\beta}\right) + wV_m(\beta^m)\\
&= \bar{w}w^{m-1}V_1\left(\frac{f-\beta^{m+1}}{r\beta^m}\right) + ww^{m-1}V_1(\beta)\\
&\qquad \text{(by the induction hypothesis (15) with } n = m-1)\\
&= \bar{w}w^{m-1}\left(\frac{f-\beta^{m+1}}{r\beta^m}\right)^p + w^m\beta^p \quad \text{(since } V_1(x) = x^p \text{ for } x \le \beta)\\
&= w^m\left\{\bar{w}\left(\frac{f-\beta^{m+1}}{r\beta^{m+1}}\right)^p + w\right\}\\
&= w^m U_0\left(s\left(\frac{f}{\beta^m}\right); \frac{f}{\beta^m}\right) = w^m U_0\left(y_0\left(\frac{f}{\beta^m}\right); \frac{f}{\beta^m}\right)\\
&= w^m V_1\left(\frac{f}{\beta^m}\right),
\end{aligned}$$

proving (15) with $n = m$.

Finally, we are ready to show that $V_{\mathrm{B}}(f) = V^*(f)$ for $f = \beta^n + r\beta^{n+k}$ ($k \ge k_0, n \ge 1$). Since, by (8) and (9),

$$\begin{aligned}
f = \beta^n + r\beta^{n+k} &\le \beta^{n-1}(\beta + r\beta^{k_0+1})\\
&< \beta^{n-1}\min\{1, \beta + r\beta x_0\}\\
&= \beta^{n-1}a_0,
\end{aligned}$$

it follows that

$$\begin{aligned}
V^*(f) &\le V_n(f) \quad \text{(by Lemma 5)}\\
&= w^{n-1}V_1\left(\frac{f}{\beta^{n-1}}\right) \quad \text{(by (15))}\\
&= w^{n-1}U_0\left(y_0\left(\frac{f}{\beta^{n-1}}\right);\ \frac{f}{\beta^{n-1}}\right) \quad \text{(by (11))}\\
&= w^{n-1}U_0\left(s\left(\frac{f}{\beta^{n-1}}\right);\ \frac{f}{\beta^{n-1}}\right) \quad \text{(by (14))}\\
&= w^{n-1}\{\bar{w}V_0(\beta^k) + wV_0(1)\} \quad \text{(by (10))}\\
&= w^{n-1}\{\bar{w}(\beta^k)^p + w\}\\
&= w^n + \bar{w}w^{n+k-1}.
\end{aligned}$$

However, by (1),

$$\begin{aligned}
V_{\mathrm{B}}(f) &= w^{n-1}V_{\mathrm{B}}(\beta + r\beta^{k+1})\\
&= w^{n-1}(w + \bar{w}V_{\mathrm{B}}(\beta^k))\\
&= w^{n-1}(w + \bar{w}w^k)\\
&= V_n(f),
\end{aligned}$$

implying that $V^*(f) = V_{\mathrm{B}}(f) = V_n(f)$. This completes the proof of Theorem 2.

4. Upper and lower approximations for the value function V^*

While it is difficult to compute the value function V^*, the following recursive procedure provides upper and lower approximations for V^*, with approximation errors decaying at a geometric rate. Again, we assume that $0 < \alpha < r$ and $w < \beta$. Let $p = (\log w)/(\log \beta) > 1$ and $R_0(f) = f^p,\ f \ge 0$. Define, for $n = 0, 1, \dots,$

$$R_{n+1}(f) = \begin{cases} \sup\left\{\bar{w}R_n\left(\dfrac{f-y}{1+\alpha}\right) + wR_n\left(\dfrac{f+ry}{1+\alpha}\right) : 0 \le y \le b(f)\right\} & \text{for } 0 \le f < 1,\\ 1 & \text{for } f = 1. \end{cases} \tag{17}$$

For $0 \le f < 1$, let $b_{n+1}(f)$ denote the (smallest) value of y that maximizes

$$\bar{w}R_n\left(\frac{f-y}{1+\alpha}\right) + wR_n\left(\frac{f+ry}{1+\alpha}\right), \qquad 0 \le y \le b(f).$$

Next, recursively define L_n, as in (17), with the initial condition R_0 replaced by $L_0(f) = \mathbf{1}_{\{1\}}(f),\ 0 \le f \le 1$, where $\mathbf{1}_A$ denotes the indicator function of the set A. (Note that the R_n are related to the V_n defined in Lemma 5, whereas the L_n are, in fact, the U_n introduced in a more general context in Section 2.15 of [3].)

Lemma 6. $L_0 \le L_1 \le \cdots \le V^* \le \cdots \le R_1 \le R_0$.

Proof. Note that V^* satisfies, for $0 \le f < 1$,

$$V^*(f) = \sup\left\{\bar{w}V^*\left(\frac{f-y}{1+\alpha}\right) + wV^*\left(\frac{f+ry}{1+\alpha}\right) : 0 \le y \le b(f)\right\}.$$

Since $L_0(f) \leq V^*(f) \leq R_0(f)$ by Lemma 3, it follows by induction that $L_n(f) \leq V^*(f) \leq R_n(f)$ for all n. The monotonicity of the L_n follows, by induction along with the simple fact that $L_1(f) \geq L_0(f) = \mathbf{1}_{\{1\}}(f)$. The monotonicity of the R_n will also follow by induction if we can establish that $R_1(f) \leq R_0(f)$. By definition, for $0 \leq f < 1$,

$$\begin{aligned}
R_1(f) &= \sup\left\{\bar{w} R_0\left(\frac{f-y}{1+\alpha}\right) + w R_0\left(\frac{f+ry}{1+\alpha}\right) : 0 \leq y \leq b(f)\right\} \\
&\leq \sup\left\{\bar{w} R_0\left(\frac{f-y}{1+\alpha}\right) + w R_0\left(\frac{f+ry}{1+\alpha}\right) : 0 \leq y \leq f\right\} \\
&= \max\left\{R_0\left(\frac{f}{1+\alpha}\right), w R_0\left(\frac{f}{\beta}\right)\right\} \\
&= f^p = R_0(f),
\end{aligned}$$

where the second equality follows from the convexity of both $R_0((f-y)/(1+\alpha))$ and $R_0((f+ry)/(1+\alpha))$ in y. This completes the proof of Lemma 6.

Theorem 3. *Let* $\rho = \max\{1/(1+\alpha), w/\beta\} < 1$. *Then*

$$R_n(f) - L_n(f) \leq \rho^n f, \qquad 0 \leq f < 1. \tag{18}$$

Proof. The functions R_n and L_n may be interpreted as the value functions for the following gambling system. In each game, the gambler with fortune f $(0 \leq f < 1)$ can stake any amount $y \leq b(f)$, resulting in fortune $(f-y)/(1+\alpha)$ or $(f+ry)/(1+\alpha)$ with respective probabilities $\bar{w}$ and w. If $f = 1$, the gambler's fortune remains equal to 1 with probability 1 at the end of the game. The gambler is required to play exactly n (independent) games. Then R_n or L_n is the optimal value if the gambler's utility function is R_0 or L_0, respectively. To show (18), let $X_i, i = 0, \ldots, n-1$, denote the fortune at the end of the $(n-i)$th game, when the gambler (with initial fortune $X_n = f$) plays optimally with respect to the utility function R_0. (Here, the subscript i in X_i indicates the number of remaining games.) Recall that $b_{n+1}(f)$ denotes the smallest value of y that maximizes $\bar{w} R_n((f-y)/(1+\alpha)) + w R_n((f+ry)/(1+\alpha))$ over $0 \leq y \leq b(f)$. Then the X_i are (backwards) Markov, and

$$\begin{aligned}
\mathrm{P}(X_{i-1} = 1 \mid X_i) = 1 \quad &\text{if } X_i = 1, \\
\mathrm{P}\left(X_{i-1} = \frac{X_i - b_i(X_i)}{1+\alpha} \;\middle|\; X_i\right) = \bar{w} \quad &\text{if } X_i < 1, \\
\mathrm{P}\left(X_{i-1} = \frac{X_i + r b_i(X_i)}{1+\alpha} \;\middle|\; X_i\right) = w \quad &\text{if } X_i < 1.
\end{aligned}$$

Clearly,

$$\begin{aligned}
R_n(f) &= \mathrm{E}\, X_0^p, \\
L_n(f) &\geq \mathrm{E}\, \mathbf{1}_{\{X_0 = 1\}} = \mathrm{P}(X_0 = 1),
\end{aligned}$$

meaning that

$$R_n(f) - L_n(f) \leq \mathrm{E}\, X_0^p \,\mathbf{1}_{\{X_0 < 1\}} \leq \mathrm{E}\, X_0 \,\mathbf{1}_{\{X_0 < 1\}}.$$

We now prove that

$$\mathrm{E}(X_{i-1} \mathbf{1}_{\{X_{i-1} < 1\}} \mid X_i) \leq \rho X_i \mathbf{1}_{\{X_i < 1\}}, \tag{19}$$

from which (18) follows immediately. Given $X_i = x < 1$, the conditional expectation of X_{i-1} equals

$$\bar{w}\frac{x - b_i(x)}{1+\alpha} + w\frac{x + rb_i(x)}{1+\alpha} = \frac{x}{1+\alpha} + b_i(x)\frac{(1+r)w - 1}{1+\alpha},$$

which is bounded by $x/(1+\alpha)$ if $(1+r)w \le 1$, or by $x(1+r)w/(1+\alpha) = xw/\beta$ if $(1+r)w > 1$ (the latter bound being due to the fact that $b_i(x) \le x$). This proves (19) and, hence, (18).

Remark 4. Similar results on the geometric convergence of algorithms for finite gambling problems can be found in [6].

5. Concluding remarks

It was conjectured in [2] that the bold strategy is optimal for $0 < \alpha < r < 1$ and $w \le 1/(1+r)$. In view of Theorem 1, the conjecture remains unproven for $0 < \alpha < r < 1$ and $\frac{1}{2} < w \le 1/(1+r)$. We have made an attempt to find a counterexample but have not been successful. Thus, *we* conjecture that the bold strategy is optimal for $0 < \alpha < r < 1$ and $\frac{1}{2} < w \le 1/(1+r)$.

In the absence of inflation ($\alpha = 0$), the primitive casino is superfair if $w > 1/(1+r)$, in which case the bold strategy is not optimal since making small (infinitesimal) bets would enable the gambler to reach his goal with certainty. (See [7] for interesting related results.) In the presence of inflation ($\alpha > 0$), the primitive casino may be referred to as subfair, fair, or superfair if $w < \beta$, $w = \beta$, or $w > \beta$, respectively, since the (discounted) expected gain $w(1+r)y/(1+\alpha) - y$ for stake y (taking inflation into account) is negative, zero, or positive, respectively. However, if the gambler stakes only a part of the total fortune, then the remaining part is to be discounted due to inflation, so that his 'overall' discounted expected gain may be negative even when $w > \beta$. Thus, it is not wise to make small bets in the presence of inflation. Our Theorem 2 shows, for the case in which $w < \beta$, that the bold strategy is optimal if the initial fortune equals β^n or $\beta^n + r\beta^{n+k}$ for $n = 1, 2, \ldots$ and $k = k_0, k_0 + 1, \ldots$, where k_0 is given in (8). To the best of the authors' knowledge, the issue of optimality of the bold strategy has not been studied for the case $w \ge \beta$ in the literature.

Acknowledgements

We are grateful to two referees for their helpful comments. In particular, the motivating story was suggested by one of them.

References

[1] Chen, R. (1978). Subfair 'red-and-black' in the presence of inflation. *Z. Wahrscheinlichkeitsth.* **42,** 293–301.
[2] Chen, R. W., Shepp, L. A. and Zame, A. (2004). A bold strategy is not always optimal in the presence of inflation. *J. Appl. Prob.* **41,** 587–592.
[3] Dubins, L. E. and Savage, L. J. (1965). *How To Gamble If You Must: Inequalities for Stochastic Processes*. McGraw-Hill, New York.
[4] Dubins, L. E. and Teicher, H. (1967). Optimal stopping when the future is discounted. *Ann. Math. Statist.* **38,** 601–605.
[5] Klugman, S. (1977). Discounted and rapid subfair red-and-black. *Ann. Statist.* **5,** 734–745.
[6] Ramakrishnan, S. and Sudderth, W. (1998). Geometric convergence of algorithms in gambling theory. *Math. Operat. Res.* **23,** 568–575.
[7] Ruth, K. (1999). Favorable red and black on the integers with a minimum bet. *J. Appl. Prob.* **36,** 837–851.