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SOME APPLICATIONS OF NEARFIELDS

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Nearfields and near-rings are related to many other structures and needed for several representation theorems. Therefore it is important to gain knowledge about the structure of near-rings and nearfields and to find construction methods. The first examples of proper nearfields were constructed by L.E. Dickson 1905, they were finite. 30 years later H. Zassenhaus completely determined the finite nearfields his attention having been attracted to them by the study of certain permutation groups. By the axiomatization of Dickson's methods, H. Karzel succeeded in giving new examples of infinite nearfields. In the extensive generalization by J. Timm the so-called "Dicksonian processes" are the most important tool for constructions of nearfields and near-rings. The report (**44**) by H. Wähling is a summary of the results on nearfields obtained so far.

In this paper in the first section we will report on mutual connections of nearfields and neardomains with sharply 2- and 3-transitive permutation groups and with some geometric structures, the so-called rectangular 2-structures and hyperbola-structures.

We find further relations between nearfields and geometry in the theory of incidence groups due to E. Ellers and H. Karzel (**6**). This theory has recently been extended to non-linear geometries as Möbius-planes, Minkowski-planes, Burau-geometries by the authors, and spheric spaces by H. Hotje. For a treatment of all these structures there are needed special nearfield-extensions generalizing the concept of Dicksonian nearfields. Therefore we will give a short survey on "Dicksonian nearfield-extensions" and their properties in the second section. Apart from reporting known results, we will deal with recent advances made by G. Kist and H. Wähling. The research on this subject is still in progress.

1. Nearfields, neardomains, sharply 2-transitive groups and 2-structures

1.1. Neardomains and nearfields

The concept of a neardomain (Fastbereich) is a generalization of the concept of the nearfield. A set F provided with two binary operations "$+$" and "$\cdot$" is called a *neardomain* $(F, +, \cdot)$ if the following axioms are valid (**16**):

(Fb1) $(F, +)$ is a loop (that means: for all $a, b \in F$ each equation $a + x = b$ and $x + a = b$ has exactly one solution in F and there is a neutral element $0 \in F$ with $0 + a = a + 0 = a$).

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(Fb2) $\forall a, b \in F: a+b=0 \Rightarrow b+a=0$

(Fb3) $(F^*, \cdot)$ with $F^*:=F\backslash\{0\}$ is a group (with neutral element 1).

(Fb4) $\forall a \in F: 0 \cdot a = 0$

(Fb5) $\forall a, b, c \in F: a(b+c) = ab + ac$

(Fb6) $\forall a, b \in F \exists_1 d_{a,b} \in F^*$ such that: $\forall x \in F: a+(b+x)=(a+b)+d_{a,b}\cdot x$

A neardomain $(F, +, \cdot)$ is a *nearfield if* $(F, +)$ is a group, that means $d_{a,b}=1$ for all $a, b \in F$. A neardomain $(F, +, \cdot)$ is called *planar* or *projective if*

(Fb7) $\forall a, b, m \in F, \quad m \neq 1 \quad \exists x \in F: a+x=b+mx$

is valid. There are the following theorems:

Theorem 1.1. *A neardomain $(F, +, \cdot)$ is a nearfield if one of the following conditions is fulfilled*:
(a) $(F, +, \cdot)$ *is planar* (**16**, (5.6))
(b) $(F, +, \cdot)$ *is finite* (**16**, (5.7)) *or more generally*: $[F^*:Z] \in \mathbb{N}$ *for* $Z = \{z \in F^* \mid \forall a \in F: az = za\}$ (**29**, Corollary 1.)
(c) $(F, +)$ *is a commutative loop* (**27**, (2.11))
(d) $1+1+1=0$ (**27**, 3.7.)
Up to now it has been an open problem if there exist neardomains which are not nearfields. The results obtained by W. Kerby and H. Welfelscheid show, that the structure of a proper neardomain must be rather complicated (**24** §8, **30**).

1.2. Sharply n-transitive sets and groups

Let M be a set, S_M the symmetric group of M consisting of all permutations of M, $\Gamma \subset S_M$ and $n \in \mathbb{N}$. Then the pair (M, Γ) is called permutation set and permutation group if $\Gamma \leqq S_M$. A permutation set (M, Γ) is called *sharply n-transitive* if for any two n-tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in M^n$ with $|\{x_1, \ldots, x_n\}| = |\{y_1, \ldots, y_n\}| = n$ there is exactly one permutation $\gamma \in \Gamma$ with $\gamma(x_i) = y_i$ for all $i \in \{1, 2, \ldots, n\}$ and *symmetric*, if for any two permutations $\alpha, \beta \in \Gamma$ the existence of an $x \in M$ with $\alpha^{-1}\beta(x) = \beta^{-1}\alpha(x) \neq x$ implies $\alpha^{-1}\beta = \beta^{-1}\alpha$.
Here we know the following theorems:

Theorem 1.2. *Let (M, Γ) be a permutation set and $\sigma \in S_M$.*

(a) *If (M, Γ) is sharply n-transitive resp. symmetric, then also $(M, \sigma\Gamma)$, $(M, \Gamma\sigma)$ and (M, Γ^{-1})* (**18**, (1.1)).

(b) *If (M, Γ) is symmetric and sharply n-transitive, then $n \leqq 3$ or $|M| = 4$; for $|M| = 4$ and $n \geqq 3$ we have $\Gamma = S_4$* (**18**, (1.3)).

(c) *If (M, Γ) is symmetric, sharply 3-transitive, $|M| \geqq 3$ and Γ contains the identity id, then Γ is a group and there is a commutative field K such that Γ and $PGL(2, K)$ are isomorphic as permutation groups* (**18**, Theorem; **1**).

(d) *If M is finite and $|M|$ odd then (M,Γ) is symmetric* (**9**, Satz 13 and Satz 15) *and hence Γ isomorphic to $PGL(2,2^q)$, where $|M|=2^q+1$* (**8**).

(e) *If (M,Γ) is sharply 2-transitive, $\Gamma^{-1}=\Gamma, \alpha\Gamma\alpha\subset\Gamma$ for all $\alpha\in\Gamma$ and $id\in\Gamma$, then (M,Γ) is symmetric* (**18**).

(f) *If (M,Γ) is a sharply 2-transitive group, then (M,Γ) is symmetric* (follows from (e)).

1.3. 2-structures and hyperbola-structures

Let P be a set and $\mathfrak{G}_1,\mathfrak{G}_2$ two subsets of the powerset $\mathfrak{P}(P)$. The elements of P respectively $\mathfrak{G}_1\cup\mathfrak{G}_2$ are called points respectively generators. The triple $(P,\mathfrak{G}_1,\mathfrak{G}_2)$ is called a *Minkowski-lattice* if the following two axioms are satisfied:

(H.1) $\forall a\in P, \forall i\in\{1,2\}\exists_1[a]_i\in\mathfrak{G}_i: a\in[a]_i$

(H.2) $\forall A\in\mathfrak{G}_1, \forall B\in\mathfrak{G}_2: |\mathrm{A}\cap\mathrm{B}|=1$

Two points $a,b\in P$ are called joinable if $b\not\in[a]_1\cup[a]_2$. A quadruple $(P,\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{R})$ with $\mathfrak{R}\subset\mathfrak{P}(P)$ is called a *chain-structure* if $(P,\mathfrak{G}_1,\mathfrak{G}_2)$ is a Minkowski-lattice and if for each chain $K\in\mathfrak{R}$ and each generator $A\in\mathfrak{G}_1\cup\mathfrak{G}_2$ we have $|K\cap A|=1$. (**10**).

In a chain-structure $(P,\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{R})$ we can define the following permutations of P: For any $K, L\in\mathfrak{R}$ let be:

$$\hat{K}:\begin{cases}P\to P\\ x\to[[x]_1\cap K]_2\cap[[x]_2\cap K]_1\end{cases}$$

$$\langle LK\rangle_1:\begin{cases}P\to P\\ x\to[x]_1\cap[[[x]_2\cap K]_1\cap L]_2\end{cases}\quad\text{and}\quad\langle LK\rangle_2:\begin{cases}P\to P\\ x\to[x]_2\cap[[[x]_1\cap K]_2\cap L]_1\end{cases}$$

$\hat{K}$ is an involutorial permutation fixing exactly the points on K and interchanging the generators of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. $\langle LK\rangle_1$ and $\langle LK\rangle_2$ are fixing exactly the points on $L\cap K$ and mapping each generator of $\mathfrak{G}_i$ on a generator of $\mathfrak{G}_i$.

Therefore $\langle LK\rangle_1':\begin{cases}\mathfrak{G}_1\to\mathfrak{G}_1\\ A\to[[A\cap K]_2\cap L]_1\end{cases}$ is a faithful representation of $\langle LK\rangle_1$ on $\mathfrak{G}_1$.

A chain-structure $(P,\mathfrak{G}_1,\mathfrak{G}_2,\mathfrak{R})$ is called *symmetric*[1)] if:

(S) $\forall K\in\mathfrak{R}, \forall x\in P\backslash K, \forall L\in\mathfrak{R}$ with $x, \hat{K}(x)\in L: \hat{K}(L)=L$

and *rectangular*[1)] if:

(R) $\forall K, L, X\in\mathfrak{R}: \langle LK\rangle_i(X)\in\mathfrak{R}$

is valid. If in a chain-structure $(P,\mathfrak{E}_1,\mathfrak{E}_2,\mathfrak{R})$ for an $n\in\mathbf{N}$ the axioms

(H.3) For any n pairwise joinable points $a_1,a_2,\ldots,a_n\in P$ there is exactly one $K\in\mathfrak{R}$ with $a_1,a_2,\ldots,a_n\in K$.

[1)]For special chain-structures we find equivalent conditions for (S) in (**2**) and (**9**) and for (R) in (**2**) and (**16**).

(H.4) There are n pairwise joinable points

are valid, then $(P, \mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{R})$ is called *2-structure* for $n = 2$ and *hyperbola-structure* for $n = 3$.

For a 2-structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ the pair $(P, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{R})$ is an affine plane, if and only if to every $K \in \mathfrak{R}$ and every $a \in P \backslash K$ there is exactly one $L \in \mathfrak{R}$ such that $a \in L$ and $L \cap K = \emptyset$. A hyperbola-structure is called a *Minkowski-plane* if we have:

(H.5) $\forall K \in \mathfrak{R}$, $\forall q \in K$ and $\forall p \in P \backslash (K \cup [q]_1 \cup [q]_2) \exists_1 L \in \mathfrak{R} : p \in L$ and $K \cap L = q$.

Theorem 1.3. *Every symmetric hyperbola-structure is a Minkowski-plane* (**9**, Satz 11).

1.4. Derivations

There are close relations among neardomains, sharply 2-transitive groups and rectangular 2-structures:

Theorem 1.4. *Let* $(F, +, \cdot)$ *be a neardomain and* Γ *the set of all mappings* $\langle m, a \rangle : F \to F ; x \to a + m \cdot x$ *with* $a \in F, m \in F^*$. *Then the pair* (F, Γ) *is a sharply 2-transitive group.*

Remark. If we replace $(F, +, \cdot)$ by a quasifield, then (F, Γ) is only a sharply 2-transitive set; if $(F, +, \cdot)$ is a proper alternative field (= field of octaves), then (F, Γ) is a symmetric 2-transitive set but not a group (**34**).

Theorem 1.5. *Each sharply 2-transitive group* (M, Γ) *can be represented by a suitable uniquely determined neardomain according to* Theorem 1.4, (**14**, §11 or **24**, §6).

Theorem 1.6. (a) *Let* (M, Γ) *be a permutation set,* $P := M \times M$, $\mathfrak{G}_1 := \{\{a\} \times M : a \in M\}$, $\mathfrak{G}_2 := \{M \times \{a\} : a \in M\}$ *and* $\mathfrak{R} := \{\{(x, \gamma(x)) : x \in M\} : \gamma \in \Gamma\}$, *then* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *is a chain-structure.*
(b) *Let* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *be a chain-structure,* $E \in \mathfrak{R}$ *and* $\Gamma := \{\langle KE \rangle_1' : K \in \mathfrak{R}\}$, *then* $(\mathfrak{G}_1, \Gamma)$ *is a permutation set and the chain-structure derived from* $(\mathfrak{G}_1, \Gamma)$ *according to* (a) *is isomorphic to* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$.

Theorem 1.7. *Let* (M, Γ) *be a permutation set and* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *the associated chain-structure. Then the following statements are valid*:

(a) (M, Γ) *is a group* $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *is rectangular*[2)]

(b) (M, Γ) *is symmetric* $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *is symmetric*[2)]

(c) (M, Γ) *is sharply 2- respectively 3-transitive* $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *is a 2-structure respectively hyperbola-structure* (**16, 9, 21**)

[2)]The proof of (a) respectively of (b) is similar to (**16**, (4.1)) respectively to (**9**, p. 16).

From these theorems we see that there are one to one correspondences among neardomains, sharply 2-transitive groups and rectangular 2-structures.

Theorem 1.8. *Let* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *be a rectangular* 2-*structure and* $(F, +, \cdot)$ *the associated neardomain. Then* $(P, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{R})$ *is an affine plane if and only if* $(F, +, \cdot)$ *is planar and therefore a planar nearfield* (**16**, (5.5)).

From Theorems 1.2(c), 1.3 and 1.7 follows:

Theorem 1.9. *Let* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ *be a symmetric hyperbola-structure and* $(\mathfrak{G}_1, \Gamma)$ *the associated permutation set. Then* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ is a rectangular Minkowski-plane and $(\mathfrak{G}_1, \Gamma)$ *a sharply* 3-*transitive symmetric permutation group.*

Remark. By 1.7, 1.2(f) and the remark after 1.4 we have the following results for 2-structures: rectangularity implies symmetry, but not conversely. For hyperbola-structures: symmetry implies rectangularity by 1.9, but not conversely; for every rectangular hyperbola-structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ the associated sharply 3-transitive permutation group (compare 1.7) can be represented in a unique way by a KT-field $(F, +, \cdot, \sigma)$ (that is a neardomain $(F, +, \cdot)$ with an involutorial automorphism σ of $(F^*, \cdot)$ such that for all $a \in F \backslash \{0, -1\}$ we have $1 - \sigma(1 + a) = \sigma(1 + \sigma(a))$ (**28** p. 227)), and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{R})$ is symmetric if and only if $(F, +, \cdot)$ is a commutative field (σ is then the mapping $x \to x^{-1}$). But there are KT-fields $(F, +, \cdot, \sigma)$ where $(F, +, \cdot)$ is a proper nearfield (**28** §2). The question has not yet been answered if there are KT-fields $(F, +, \cdot, \sigma)$ where $(F, +, \cdot)$ is not a nearfield. One can however suppose, that proper KT-fields do not exist, since W. Kerby could prove the strong result: Each KT-field F of characteristic $F \equiv 1 (\mathrm{mod}\ 3)$ is a nearfield (**25**).

2. Nearfield-extensions and incidence groups

2.1. Dicksonian nearfield-extensions

Very general nearfield structures from which we can derive geometric structures are the Dicksonian nearfield-extensions defined by G. Kist.[3] Let $(F, +, \circ)$ be a nearfield and K a subnearfield of F. Then the nearfield-extension (F, K) is called

normal if $\forall a \in F : a \circ K = K \circ a$

central if $\forall a \in F \forall \lambda \in K : a \circ \lambda = \lambda \circ a$

quadratic if $\forall a \in F : a \circ a \in K + a \circ K$

Dicksonian if there is a map $\cdot : F \times K \to F$ such that $(K, +, \cdot)$ is a field, $(F, K, \cdot)$ is a right vector space over K and satisfies:

(D) For $K \neq F : \forall a \in F : a \circ K = a \cdot K$

[3] In (**32**) they were called "Fastkörpererweiterungen" defined by equivalent conditions.

(D') For $K = F: \forall a \in F^* = K^*$ the mapping $\psi_a : x \to a^{-1} \cdot (a \circ x)$ (a^{-1} denotes the inverse of a with respect to "$\cdot$") is an automorphism of the field $(K, +, \cdot)$.

(The dimension of the right vector space (F, K) is called the *rank* of the Dicksonian nearfield-extension and will be denoted by $[F:K]$; for $[F:K] = 1$ that means $F = K$, the concept of the Dicksonian nearfield-extension coincides with the concept of *Dicksonian nearfields* (**13**).)

Theorem 2.1. *Any Dicksonian nearfield-extension (F, K) satisfies:*

(a) *$a_l : x \to a \circ x$ is, for any $a \in F^*$, a bijective semilinear mapping of $(F, K, \cdot)$;*

(b) *$(K, +, \circ)$ is a Dicksonian nearfield;*

(c) *The map $\cdot : F \times K \to F$ can be chosen such that the neutral elements of $(F, \circ)$ and $(K, \cdot)$ coincide. In the case $K \neq F$ the mapping "$\cdot$" is then uniquely determined.*

For every nearfield $(F, +, \circ)$ and its *kernel* $K_F := \{z \in F : \forall x, y \in F : (x + y) \circ z = x \circ z + y \circ z\}$ the pair (F, K_F) is a Dicksonian nearfield-extension.

Theorem 2.2. *Any normal nearfield-extension (F, K) with $K \neq F$ is Dicksonian with respect to $\cdot : F \times K \to F$; $(a, \lambda) \to a \cdot \lambda := \lambda \circ a$* (**43**, Satz 1).

Theorem 2.3. *Any nearfield-extension (F, K) with $K \neq F$ and F finite is Dicksonian* (**32**, (3.11)).

Theorem 2.4. *Any quadratic Dicksonian nearfield-extension (F, K) with $[F:K] \geqq 3$ is a division algebra and therefore either a field of quaternions or a field-extension of characteristic 2 with $x^2 \in K$ for all $x \in F$* (**33**, Satz 3; **35**; **17**).

Theorem 2.5. *Let (F, K) be a normal nearfield-extension with $K \neq F$. Then:*

(a) *For $[F:K] \geqq 3$ we have: F^*/K^* commutative $\Leftrightarrow$ F is a commutative field* (**15**).

(b) *For $[F:K] \geqq 3$ we have: $\forall a, b, c \in F : (a + b)c \in Kac + Kbc \Leftrightarrow F$ is a field* (**42**).

(c) *For $[F:K] = 2$ and F finite we have: F^*/K^* commutative $\Leftrightarrow F$ is commutative or F is one of the Dickson-nearfields $DF(9,3)$ or $DF(64,4)$ with 9 or 64 elements* (**12**, Satz 12).

Remark. Up to now it has been an open problem if we can drop the condition "finite" in 2.5(c).

2.2. Incidence groups as derivations of normal nearfield-extensions

By an incidence group $(G, \cdot, \gamma)$ we understand a set G which is provided with a group structure "$\cdot$" and a geometric structure "γ" such that for every $a \in G$ the mapping $a_l : G \to G; x \to ax$ is an automorphism of (G, γ).

An incidence group $(G, \cdot, \gamma)$ is called *projective* respectively *punctured affine* if $\gamma = \mathfrak{G} \subset \mathfrak{P}(G)$ is the set of "lines" such that $(G, \mathfrak{G})$ is a projective space respectively punctured affine space, that is a space gained from an affine space by omitting one point.[4)]

We have the following representation theorems (**12**, **32**):

Theorem 2.6. (a) *Let (F, K) be a normal nearfield-extension with $[F:K] \geqq 3$ and $\varphi: F^* \to F^*/K^*; x \to K^*x$. Then the factorial group $(F^*/K^*, \cdot)$ together with the set of lines $\mathfrak{G} := \{\varphi(Ka + Kb)^* : a, b \in F^* : Ka \neq Kb\}$ forms a desarguesian projective incidence group.*

(b) *Any desarguesian projective incidence group $(G, \cdot, \mathfrak{E})$ can be derived from a suitable normal nearfield-extension according to (a).*

Theorem 2.7. (a) *Let (F, K) be a Dicksonian nearfield-extension with $F \neq K \neq \mathbb{Z}_2$. $G := F^*$ and $\mathfrak{G} := \{(a + b \cdot K)^* : a \in F, b \in F^*\}$. Then $(F^*, \circ, \mathfrak{G})$ is a punctured affine incidence group.*

(b) *Any desarguesian punctured affine incidence group can be derived according to (a).*

Because of the Theorems 2.6 and 2.7 it is important in geometry to gain knowledge of construction methods for nearfields and to study their properties especially the lattice of subnearfields. Some work in this respect has been done by F. Pokropp (**36**), H. Wähling (**44**), G. Kist (**32**) and for the case of near-rings by J. Timm (**40**), (**41**).

An other class of incidence groups derivable by normal nearfield-extensions are the Möbius- and Burau-incidence-groups. A pair $(M, \mathfrak{K})$ is called *Möbius-plane* if M is a set of "points" and $\mathfrak{K} \subset \mathfrak{P}(M)$ (set of circles) such that the following axioms hold:

(M1) $|\mathfrak{K}| \geqq 2$ and $|C| \geqq 3$ for all $C \in \mathfrak{K}$;

(M2) For any three distinct points $a, b, c \in M$ there is exactly one $C \in \mathfrak{K}$ with $a, b, c \in C$;

(M3) $\forall K \in \mathfrak{K}$, $\forall q \in K$ and $p \in M \backslash K \exists_1 L \in \mathfrak{K} : p \in L$ and $K \cap L = q$.

Let $(P, \mathfrak{G})$ be a projective space, $\mathfrak{K} \subset \mathfrak{P}(P)$ and for $X \subset P$ let $\overline{X}$ be the projective closure of X in $(P, \mathfrak{G})$ and $\mathfrak{K}(X) := \{K \in \mathfrak{K} : K \subset X\}$. Then the triple $(P, \mathfrak{G}, \mathfrak{K})$ is called a *Burau-geometry* (**22**) if (B1), (B2), (B3) hold:

(B1) $\forall K \in \mathfrak{K} : \overline{K} \in \mathfrak{G}$

(B2) $\forall M \in \mathfrak{G} : (M, \mathfrak{K}(M))$ is a Möbius-plane

(B3) For any five distinct points $a, b, c, d, e \in P$ with $c = \overline{a, b} \cap \overline{d, e}$ holds: Any line $L \in \mathfrak{G}$ passing through $f := \overline{a, e} \cap \overline{b, d}$, which meets the circle through a, b, c meets also the circle through c, d, e.

An incidence-group $(G, \cdot, \gamma)$ is called *Möbius*- respectively *Burau-incidence-group* if (G, γ) is a Möbius-plane for $\gamma = \mathfrak{K}$ respectively Burau-geometry for $\gamma = (\mathfrak{G}, \mathfrak{K})$.

[4)]All geometric notions which are not defined in this paper can be found in (**23**).

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Theorem 2.8. (a) *Let* (F, L, K) *be a triple of distinct nearfields such that* $(F, L), (F, K)$ *and* (L, K) *are normal nearfield-extensions with* $[L:K]=2$ *and* $\varphi: F^* \to F^*/L^*; x \to L^*x$. *Then the following B-derivation* $B(F, L, K) = (G, \cdot, \mathfrak{G}, \mathfrak{K})$ *gives us a Burau-incidence-group for* $[F:L] \geqq 3$ *and a Miquelian (cf.* (3)*) Möbius-incidence-group for* $[F:L]=2$:
$(G, \cdot) := (F^*/L^*, \cdot)$, $\mathfrak{G} := \{\varphi(La + Lb)^*; a, b \in F^*: La \neq Lb\}$,
$\mathfrak{K} := \{\varphi(Ka + Kb)^*: a, b \in F^*: La \neq Lb\}$

(b) *Any Burau- and Miquelian Möbius-incidence-group can be represented according to* (a). **(19)**, **(20)**

2.3 Minkowski-incidence-groups

An incidence group $(G, \cdot, \gamma)$, where (G, γ) for $\gamma = (\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a Minkowski-plane (cf. 1.3), is called a *Minkowski-incidence-group*. Any symmetric Minkowski-plane $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ can be represented algebraically in the following manner (cf. 1.6, 1.2(c)): Let K be a commutative field, $E := K^{2*}/K^*$, $P := E \times E$, $\mathfrak{G}_1 := \{\{a\} \times E : a \in E\}$, $\mathfrak{G}_2 := \{E \times \{a\} : a \in E\}$ and $\mathfrak{K} := \{\{(x, \kappa(x) : x \in E\} : \kappa \in PGL(2, K)\}$. By the fundamental-theorem for symmetric Minkowski-planes the mapping $(x, y) \to (\gamma(x), \rho\gamma(y))$ and $(x, y) \to (\gamma(y), \rho\gamma(x))$ are automorphisms of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ if $\gamma \in P\Gamma L(2, K)$ and $\rho \in PGL(2, K)$ and every automorphism of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ can be obtained in this way. Therefore the following theorem gives us a construction method for symmetric Minkowski-incidence-groups:

Theorem 2.9. *Let* $(F, +, \circ, \cdot)$ *be a set with three binary operations* $+, \circ, \cdot$ *such that* $F_1 := (F, +, \circ)$ *and* $F_2 := (F, +, \cdot)$ *are nearfields and let* K *be a subnearfield of* F_1 *and* F_2 *such that* $\circ|F \times K = \cdot\,|F \times K$ *and* K *lies in the centre of* F_1 *and* F_2 *(that means* (F_1, K) *and* (F_2, K) *are central nearfield-extensions) and* $[F_1:K] = [F_2:K] = 2$. *Further let* $E_1 = F_1^*/K^*$, $E_2 = F_2^*/K^*$ *and* $\varphi: E_2 \to \mathrm{Aut}(F_1, K)$, $\psi: E_1 \to \mathrm{Aut}(F_2, K)$ *two mappings with*

$$\left.\begin{aligned} \varphi_{b \cdot \psi_a(y)} &= \varphi_b \varphi_y \\ \psi_{a \circ \varphi_b(x)} &= \psi_a \psi_x \end{aligned}\right\} \quad \textit{for all } a, b, x, y \in E := E_1 = E_2$$

For $(a, b) \,\square\, (x, y) := (a \circ \varphi_b(x), b \cdot \psi_a(y))$ *the quintuple* $(G = E \times E, \square, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is a Minkowski-incidence-group.*

Special cases:

(α) $\varphi_b = \mathrm{Id}_{F_1}$, $\psi_a = \mathrm{Id}_{F_2}\, \forall a \in E_1, b \in E_2 : (G, \square)$ is the direct product of $(E_1, \circ)$ and $(E_2, \cdot)$

(β) $\varphi_b = \mathrm{Id}_{F_1}\, \forall b \in E_2$ or $\psi_a = \mathrm{Id}_{F_2}\, \forall a \in E_1 : (G, \square)$ is the semidirect product of $(E_1, \circ)$ and $(E_2, \cdot)$

By 2.9, we do not get all Minkowski-incidence-groups, as the following example shows: Let (F, K) be a central nearfield-extension with $[F:K]=2$ such that there is a subgroup U of $(F^*, \cdot)$ of index 2 and $H := U/K^*$. Then $(G, \square, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a

Minkowski-incidence-group with respect to

$$(a, b) \square (x, y) := \begin{cases} (a \cdot x, b \cdot y) & \text{if} \quad a \cdot b \in H \\ (a \cdot y, b \cdot x) & \text{if} \quad a \cdot b \notin H. \end{cases}$$

REFERENCES

(1) R. Artzy, A Pascal theorem applied to Minkowski geometry, *J. Geometry* **3** (1973), 93–105.

(2) W. Benz, Permutations and plane sections of a ruled quadric, *Ist. Naz. di Alta Matematica* **5** (1970), 325–339.

(3) W. Benz, *Vorlesungen über Geometrie der Algebren*, (Berlin, Heidelberg, New York, 1973).

(4) L. E. Dickson, Definitions of a group and a field by independent postulates, *Trans. Amer. Math. Soc.* **6** (1905), 198–204.

(5) L. E. Dickson, On finite algebras, *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.* II (1905), 358–393.

(6) E. Ellers and H. Karzel, Kennzeichnung elliptischer Gruppenräume, *Abh. Math. Sem. Univ. Hamburg* **26** (1963), 55–77.

(7) E. Ellers and H. Karzel, Endliche Inzidenzgruppen, *Abh. Math. Sem. Univ. Hamburg* **27** (1964), 250–264.

(8) W. Heise, Minkowski-Ebenen gerader Ordnung, *J. Geometry* **5** (1974), 83.

(9) W. Heise, and H. Karzel, Symmetrische Minkowski-Ebenen, *J. Geometry* **3** (1973), 5–20.

(10) W. Heise, and H. Karzel, Laguerre- und Minkowski-m-Strukturen, *Rend. Ist. Mat. Univ. Trieste* **4** (1972), 1–9.

(11) H. Hotje, σ-Inzidenzgruppen, *J. Geometry* (to appear).

(12) H. Karzel, Bericht über projektive Inzidenzgruppen, *Jber. Deutsch. Math.-Verein.* **67** (1964), 58–92.

(13) H. Karzel, Unendliche Dicksonsche Fastkörper, *Arch. Math.* **16** (1965), 247–256.

(14) H. Karzel, Inzidenzgruppen, *Vorlesungsausarbeitung von* I. Pieper und K. Sörensen, (Univ. Hamburg, 1965).

(15) H. Karzel, Normale Fastkörper mit kommutativer Inzidenzgruppe, *Abh. Math. Sem. Univ. Hamburg* **28** (1965), 124–132.

(16) H. Karzel, Zusammenhänge zwischen Fastbereichen, scharf zweifach transitiven Permutationsgruppen und 2-Strukturen mit Rechtecksaxiom, *Abh. Math. Sem. Univ. Hamburg* **32** (1968), 191–206.

(17) H. Karzel, Kinematic spaces, *Symposia Mathematica. Ist. Naz. di Alta Matematica* **11** (1973), 413–439.

(18) H. Karzel, Symmetrische Permutationsmengen, *Aequationes Math.* **17** (1978), 83–90.

(19) H. Karzel, Einige neuere Beiträge zur Theorie der Inzidenzgruppen, (to appear).

(20) H. Karzel, and G. Kist, Burau-Inzidenzgruppen, (to appear).

(21) H. Karzel, and H.-J. Kroll, Gruppen von Projektivitäten in Zwei- und Hyperbelstrukturen, *FU Berlin, LENZ-Festband* (1976), 125–134.

(22) H. KARZEL, G. KIST and H.-J. KROLL, Burau-Geometrien, *Resultate der Mathematik* **2** (1979), 88–104.

(23) H. KARZEL, K. SÖRENSEN and D. WINDELBERG, *Einführung in die Geometrie* (Göttingen, 1973).

(24) W. KERBY, On infinite sharply multiply transitive groups, *Hamb. Math. Einzelschriften* 6 (Göttingen, 1971).

(25) W. KERBY, Sharply 3-transitive groups of characteristic $\equiv 1$ mod. 3, *J. Algebra* **32** (1974), 240–245.

(26) W. KERBY, Sharply 2- and 3-transitive groups with kernels of finite index, *Aequationes Math.* **14** (1976), 137–141.

(27) W. KERBY and H. WEFELSCHEID, Bemerkungen über Fastbereiche und scharf zweifach transitive Gruppen, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 20–29.

(28) W. KERBY and H. WEFELSCHEID, Über eine scharf 3-fach transitiven Gruppen zugeordnete algebraische Struktur, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 225–235.

(29) W. KERBY and H. WEFELSCHEID, Conditions of finiteness on sharply 2-transitive groups, *Aequationes Math.* **8** (1972), 287–290.

(30) W. KERBY and H. WEFELSCHEID, The maximal sub near-field of a near-domain, *J. Algebra* **28** (1974), 319–325.

(31) W. KERBY and H. WEFELSCHEID, Über eine Klasse von scharf 3-fach transitiven Gruppen, *J. Reine Angew. Math.* **268/269** (1974), 17–26.

(32) G. P. KIST, Punktiert-affine Inzidenzgruppen und Fastkörpererweiterungen, *Abh. Math. Sem. Univ. Hamburg* **44** (1975), 233–248.

(33) G. P. KIST, Kinematische punktiert-affine Inzidenzgruppen, *Beiträge zur geometrischen Algebra: proceedings d. Symposiums über Geometrische Algebra vom* 29.3.–3.4. 1976 *in Duisburg hrsg.* von H. J. ARNOLD... (Basel, Stuttgart, 1977), 179–183.

(34) H. KÜHLBRANDT, Scharf 2-fach transitive Permutationsmengen und Quasibereiche, (to appear).

(35) J. M. OSBORNE, Quadratic division algebras, *Trans. Amer. Math. Soc.* **105** (1962), 202–221.

(36) F. POKROPP, Dicksonsche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **30** (1967), 188–219.

(37) F. POKROPP, Isomorphe Gruppen und Fastkörperpaare, *Arch. Math.* **18** (1967), 235–240.

(38) E. M. SCHRÖDER, Zur Theorie subaffiner Inzidenzgruppen, *J. Geometry* **3** (1973), 31–69.

(39) J. TIMM, Zur Theorie der (nicht notwendig assoziativen) Fastringe, *Abh. Math. Sem. Univ. Hamburg* **35** (1970), 14–31.

(40) J. TIMM, Zur Konstruktion von Fastringen I, *Abh. Math. Sem. Univ. Hamburg* **35** (1970), 57–74.

(41) J. TIMM, Zur Theorie der Fastringkonstruktionen II, *Abh. Math. Sem. Univ. Hamburg* **36** (1971), 16–32.

(42) H. WÄHLING, Darstellung zweiseitiger Inzidenzgruppen durch Divisionsalgebren, *Abh. Math. Sem. Univ. Hamburg* **30** (1967), 220–240.

(43) H. WÄHLING, Invariante und vertauschbare Teilfastkörper, *Abh. Math. Sem. Univ. Hamburg* **33** (1969), 197–202.

(44) H. WÄHLING, Bericht über Fastkörper, *Jber. Deutsch. Math.-Verein.* **76** (1974), 41–103.

(45) H. WÄHLING, Automorphismen Dicksonscher Fastkörperpaare mit kleiner Dicksongruppe, *Abh. Math. Sem. Univ. Hamburg* **44** (1975), 122–138.

(46) H. WÄHLING, Normale Fastkörper mit kommutativer bzw. zweiseitiger Inzidenzgruppe, *Math. Z.* **147** (1976), 65–78.

(47) H. WÄHLING, Ein Zassenhauskriterium für unendliche Fastkörper, *Arch. Math.* **28** (1977), 255–260.

(48) H. WÄHLING, Projektive Inzidenzgruppoide und Fastalgebren, *J. Geometry* **9** (1977), 109–126.

(49) H. WÄHLING, Beispiel eines projektiven Fastkörpers mit nichtprojektiven Teilfastkörpern, *Arch. Math.* **28** (1977), 393–394.

(50) H. WEFELSCHEID, Die Automorphismengruppe der Hyperbelstrukturen, *Beiträge zur geometrischen Algebra: proceedings d. Symposiums über Geometrische Algebra vom* 29.3.–3.4. 1976 *in Duisburg hrsg. von* H. J. ARNOLD... (Basel, Stuttgart, 1977), 347–343.

(51) H. WEFELSCHEID, Zur Charakterisierung einer Klasse von Hyperbelstrukturen, *J. Geometry* **9** (1977), 127–133.

(52) H. ZASSENHAUS, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen, *Abh. Math. Sem. Univ. Hamburg* **11** (1936), 17–40.

(53) Über endliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **11** (1936), 187–220.

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