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ON COMPLETELY PRINCIPALLY INJECTIVE RINGS

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A ring R is called right principally injective (right P-injective) if every R-linear map from a principal right ideal of R can be extended to R. If every ring homomorphic image of R is right P-injective, R is called completely right P-injective (right CP-injective). In this paper we characterise completely quasi-Frobenius rings in terms of CP-injectivity.

A ring R is called *right principally injective (right P-injective)* if every R-linear map $aR \to R$, $a \in R$, is given by left multiplication by an element of R, equivalently if $\ell[r(a)] = Ra$ for all $a \in R$ where $\ell(x)$ and $r(x)$ denote the left and right annihilator of a set x, respectively. We studied these rings in [**8**]; and commutative p-injective rings are discussed by Camillo in [**2**].

A ring R is called *completely right P-injective (right CP-injective)* if every ring image of R is right P-injective. Left P-injective and left CP-injective rings are defined analogously. In general, the prefix "completely" signifies that the property in question holds in every ring image of R. The class of completely quasi-Frobenius rings has been studied in detail, see Faith [**3**]. In this paper we characterise these rings in terms of P-injectivity.

A module M is *uniserial* if its submodules are linearly ordered by inclusion, and M is *serial* if it is a finite direct sum of uniserial submodules. A ring R is *right (uniserial) serial* if R_R is a right *(uniserial)* serial module, with a similar definition on the left, and a *serial ring* is one that is both right and left serial. A commutative uniserial ring is called a *valuation ring*. A ring R is called *right GPF-ring* (Generalised Pseudo-Frobenius) if R is a semiperfect right P-injective ring with essential right socle. GPF rings were studied in great detail in [**8**].

THEOREM 1. *Suppose R is a left perfect, right CP-injective ring. Then R is left Artinian and left serial.*

PROOF: Let A be a two-sided ideal of R. Then $\overline{R} = R/A$ is a right GPF-ring. By [**8**, Theorem 2.3], Soc $\left({}_{\overline{R}}\overline{R}\right)$ is finitely generated and essential as a left ideal of $\overline{R}$.

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It follows from [1, Proposition 5] that R is left Artinian. To show that R is left serial, we prove the following statement by induction on the index of nilpotency of $J = J(R)$:

(*) If $1 = e_1 + \cdots + c_n$ in R where the e_i are orthogonal local idempotents, then Re_i is uniserial for each i.

If $J = 0$ this is clear because R is semisimple. In general, let $1 = e_1 + \cdots + e_n$ as in (*). Writing $S = \operatorname{Soc}({}_R R)$, we have $Se_i = \operatorname{Soc} Re_i$ and this is simple and essential in Re_i for each i by [8, Theorem 2.3]. Moreover $\overline{R} = R/S$ inherits our hypotheses and, writing $\overline{r} = r + S$ for $r \in S$, we have $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_m$ for some $m \leqslant n$, and $\overline{R}\overline{e}_1 \cong Re_i/Se_i$ for each i. It follows by induction that Re_i/Se_i is uniserial for each i, so Re_i is uniserial, as required. ∎

The converse to Theorem 1 is not true. Indeed, if F is a field the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ is Artinian (meaning right and left Artinian) and serial, but is neither right nor left P-injective. However we get a converse for local rings.

The following result will be needed and extends the result (see Faith [3], p.133) that a commutative valuation ring is P-injective if and only if every element is a zero divisor or a unit. Let $Z(R_R)$ denote the right singular ideal of a ring R.

Lemma 1. *The following are equivalent for a left uniserial ring R.*

(1) *R is right P-injective.*

(2) $J(R) = Z(R_R)$

(3) *If $a \in R$, then either $Ra = R$ or $r(a)$ is essential in R_R.*

Proof: (1) $\Rightarrow$ (2) This follows by [8, Theorem 2.1].

(2) $\Rightarrow$ (3) By hypotheses $J = J(R)$ is the unique maximal left ideal of R. Hence $Ra \neq R$ implies $a \in Z(R_R)$ by (2).

(3) $\Rightarrow$ (1) If $b \in \ell(r(a)) - Ra$ then $Ra \subseteq Rb$ by hypothesis, say $a = cb$. Also, $r(b) \supseteq r[\ell(r(a))] = r(a)$, whence $bR \cap r(c) = 0$. Thus $r(c)$ is not essential so $Rc = R$ by (3). This implies that $b \in Ra$, a contradiction. ∎

Theorem 2. *Let R be a left uniserial, right perfect ring. Then:*

(1) *R is left Artinian and right CP-injective.*

(2) *R is left self-injective if and only if $bR = Rb$ for all $b \in R$.*

Proof: Write $J = J(R)$ so that $R = J^0 \supset J \supset J^2 \supset \cdots \supset J^{n-1} \supset J^n \supset \cdots$ is a composition series of R. Let $x_i \in J^i - J^{i+1}$, then $J^i = Rx_i$. Now $Rx_{i+1} = J^{i+1} = JJ^i = JRx_i = Jx_i$. Thus $x_{i+1} = t_i x_i$, for some $t_i \in J$. By left T-nilpotency of J, $J^n = 0$ for some n and so R is semiprimary.

Claim 1. If L is a left ideal of R, then $L = J^m$ for some $m = 0, 1, \cdots, n$.

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PROOF: If $L \neq 0$, let $L \subseteq J^m$, $L \not\subseteq J^{m+1}$. Then $J^{m+1} \subset L \subseteq J^m$ because ${}_RR$ is uniserial so $L = J^m$ because J^m/J^{m+1} is simple.

CLAIM 2. $r(J^m) = J^{n-m} = \ell(J^m)$ for all $m = 0, 1, 2, \ldots, n$.

PROOF: $J^{n-m} \subseteq r(J^m)$ so $r(J^m) = J^t$ where $t \leqslant n - m$. But then $0 = J^{m+t}$ so $m + t \geqslant n$. Hence $t = n - m$. Similarly $\ell(J^m) = J^{n-m}$.

(1) R is left Artinian by Claim 1. Since our hypotheses are inherited by images, it remains to show that R is right P-injective. By Lemma 1, it suffices to show that $Z(R_R) = J$. But if $a \in J$ we have $J^{n-1} \subseteq r(a)$, so it suffices to show that $J^{n-1} = \operatorname{Soc}(R_R)$ ($\operatorname{Soc}(R_R)$ is right essential because R is semiprimary). Let $\operatorname{Soc}(R_R) = J^m$. Then $J^{m+1} = \operatorname{Soc}(R_R)J = 0$, so $m \geqslant n - 1$; as $\operatorname{Soc}(R_R) \neq 0$, we have $m = n - 1$, as required.

(2) Since R is a left principal ideal ring (Claim 1), we show that R is left P-injective; equivalently that $r(\ell(b)) = bR$ for all $b \in R$. Write $Rb = J^m$ and $\ell(b) = J^t$. Then $J^{t+m} = J^tRb = 0$ so $t + m \geqslant n$. On the other hand $J^{n-m}b \subseteq J^{n-m}J^m = 0$, so $J^{n-m} \subseteq J^t$, whence $n - m \geqslant t$. It follows that $t + m = n$, so $r[\ell(b)] = r(J^{n-m}) = J^{n-(n-m)} = J^m = Rb$ by Claim 2. Now (2) follows. ∎

If R is assumed to be both left and right P-injective in Theorem 1, we obtain a much stronger conclusion.

THEOREM 3. *The following are equivalent for a ring R:*

(1) *R is left perfect and both right and left CP-injective.*
(2) *R is completely quasi-Frobenius.*

PROOF: Since (2) $\Rightarrow$ (1) is clear, assume (1). The hypotheses hold in any image of R, so it suffices to show that R is quasi-Frobenius. Theorem 1 implies that R is Artinian and serial. Moreover $Z(R_R) = J(R) = Z({}_RR)$ by [**8**, Theorem 2.1]. Now $\ell(J) = \operatorname{Soc}(R_R)$ because R is semiprimary, whence $\ell(Z({}_RR)) = \operatorname{Soc}(R_R)$. But $\operatorname{Soc}({}_RR)$. $Z({}_RR) = 0$ always holds, and it follows that $\operatorname{Soc}({}_RR) \subseteq \operatorname{Soc}(R_R)$. The other inclusion is similar, so $\operatorname{Soc}({}_RR) = \operatorname{Soc}(R_R)$. Now let $1 = e_1 + \cdots + e_n$ where the e_i are orthogonal local idempotents. Since R is left serial, it follows that $\operatorname{Soc}(Re_i)$ is simple for each i. Similarly $\operatorname{Soc}(e_iR)$ is simple for each i. Since R is (two-sided) Artinian and $\operatorname{Soc}({}_RR) = \operatorname{Soc}(R_R)$, this implies that R is quasi-Frobenius by [**6**, p.342]. ∎

A ring R is called *right 2-injective* if R-maps $T \to R$ can be extended to R for all 2-generated right ideals T of R. Then [**8**, Corollary 2.5] implies that a left perfect right 2-injective ring is left P-injective. Hence Theorem 3 gives:

THEOREM 4. *The following are equivalent for a ring R:*

(1) *R is left perfect and completely right 2-injective.*
(2) *R is completely quasi-Frobenius.*

If R is commutative, the hypotheses in (1) of Theorem (3) can be relaxed. A commutative ring R is called *min-injective* if, for each minimal ideal K of R, each R-linear map $K \to R$ is multiplication by an element of R (equivalently $\operatorname{ann}^2 K = K$ where $\operatorname{ann}^2 K = \operatorname{ann}(\operatorname{ann} K)$). R is called a *min-CS ring* if each (minimal) ideal is essential in a direct summand of R. Note that Z the ring of integers is completely min-injective and completely min-cs, but it is not P-injective.

The following Lemma will be needed.

LEMMA 2. *Every commutative, semiprime P-injective ring R is (von Neumann) regular.*

PROOF: Given $a \in R$ we have $\operatorname{ann}(a^2) \subseteq \operatorname{ann}(a)$ because R is semiprime. Hence $\sigma\colon a^2R \to aR$ is well-defined by $\sigma(a^2r) = ar$. Since R is P-injective, $\sigma = b\cdot$ is multiplication by $b \in R$. Hence $a = \sigma(a^2) = ba^2 = aba$. ∎

THEOREM 5. *The following are equivalent for a commutative ring R:*

(1) *R is completely quasi-Frobenius.*
(2) *R is perfect and completely min-injective.*
(3) *R is perfect and is a completely min-cs ring.*
(4) *R has Krull dimension and is completely P-injective.*
(5) *R is completely GPF-ring.*

PROOF: Clearly, (1) implies each of (2), (3), (4) and (5).

(2) $\Rightarrow$ (1) It is routine to verify that a finite product of commutative rings is min-injective if and only if each factor is min-injective. Hence we may assume that R is local. Moreover (2) is inherited by ring images, so it suffices to show that R is quasi-Frobenius. Now $S = \operatorname{Soc}(R)$ is essential in R (R is perfect) and S is homogeneous (two isomorphic simple ideals are equal because R is min-injective). It follows that S is simple, so R is uniform. Furthermore, each non-zero R-module has a maximal submodule (R is perfect) so R is Noetherian by a theorem of Shock [9]. As $J(R)$ is nil, this implies that R is semiprimary, hence Artinian. Now R is quasi-Frobenius by [**6**, p.342].

(3) $\Rightarrow$ (1). As (3) is inherited by ring images, we show that R is quasi-Frobenius. Write $S = \operatorname{Soc}(R)$ and $J = J(R)$. We have $S = \operatorname{ann} J$ (as R/J is semisimple) and it follows that $\operatorname{ann}^2 S = S$. This gives:

CLAIM 1. $\operatorname{ann}^2 K = K$ for all simple ideals K.

PROOF: First $K \subseteq S$ so $\operatorname{ann}^2 K \subseteq \operatorname{ann}^2 S = S$, whence $\operatorname{ann}^2 K$ is semisimple. Since $K \subseteq \operatorname{ann}^2 K$, it suffices to show that $K \subseteq \operatorname{ann}^2 K$ is an essential extension. But $K \subseteq Re$ is essential for some $e^2 = e \in R$, so $K \subseteq \operatorname{ann}^2 K \subseteq \operatorname{ann}^2 Re = Re$.

The claim shows that R is min-injective, so (2) $\Rightarrow$ (1) completes the proof.

(4) $\Rightarrow$ (1). As (4) is inherited by images, we show that R is quasi-Frobenius.

CLAIM 2. Every prime ideal of R is maximal.

PROOF: R/P is regular by Lemma 2, and so is semisimple (it has Krull-dimension and so is finite dimensional). Since P is prime, R/P is simple.

Writing $J = J(R)$, it follows from Claim 2 that J is nil and so is nilpotent (see [4]). Furthermore, R/J is regular (by Lemma 2) and finite dimensional (it has Krull dimension), and so is semisimple. Thus R is semiprimary and we are done by (2) $\Rightarrow$ (1).

(5) $\Rightarrow$ (2). By [**8**, Theorem 2.3] and [**1**, Proposition 5], R is Artinian. Since GPF-rings are P-injective we are done. ∎

REMARKS. (i) It is easy to see that every regular ring is a left and right CP-injective ring and by Theorem 3 the converse is not true. In fact Z_{q^2}, where q is a prime number, is a commutative CP-injective ring which is not regular.

(ii) In [**2**, Remark 2 on p.36] Camillo has an example of a commutative, semiprimary, local P-injective ring which is not injective.

(iii) See Faith [**3**, Proposition 25.4.6B, p.238] for a complete description of the class of completely QF-rings. They are precisely the Artinian principal ideal rings.

(iv) The ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field, is an Artinian completely CS-ring which is not right min-injective. However, every proper homomorphic image of R is injective.

(v) In general a module ${}_RM$ is called a (min) CS-module if every (simple) submodule of M is essential in a summand of M. (min) CS-modules are called (simple)-extending modules by Harada [**5**]. For a full account of CS-modules see Mohamed and Müller [**7**].

REFERENCES

[1] J.A. Beachy, 'On quasi-artinian rings', *J. London Math. Soc.* **3** (1971), 449–452.

[2] V. Camillo, 'Commutative rings whose principal ideals are annihilators', *Portugal. Math.* **46** (1989), 33–37.

[3] C. Faith, *Algebra II, Ring theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[4] R. Gordon and J.C. Robson, 'Krull dimension', *Mem. Amer. Math. Soc.* **133** (1973).

[5] M. Harada, 'On modules with extending properties', *Osaka J. Math.* **19** (1982), 203–215.

[6] F. Kasch, *Modules and rings*, London. Math. Soc. Monographs **17** (Academic Press, New York, 1982).

[7] S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Math. Soc. Lecture Notes Series **147** (Cambridge University Press, Oxford, 1990).

[8] W.K. Nicholson and M.R. Yousif, 'Principally injective rings', *J. Algebra* (to appear).

[9] R.C. Shock, 'Dual generalizations of the artinian and neotherian conditions', *Pacific J. Math.* **54** (1974), 227–235.

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