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THE ALGEBRA OF DIFFERENTIALS OF INFINITE RANK

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Introduction. Let k and A denote commutative rings with identity and assume that A is a k-algebra. A qth order k-derivation δ of A into an A-module V is an element of $\mathrm{Hom}_k(A, V)$ such that for any $q + 1$ elements $a_0, \ldots, a_q$ of A, the following identity holds:

$$\delta(a_0a_1 \ldots a_q) = \sum_{i=1}^{q} (-1)^{i-1} \sum_{j_1<\ldots<j_i} a_{j_1} \ldots a_{j_i}\delta(a_0 \ldots \hat{a}_{j_1} \ldots \hat{a}_{j_i} \ldots a_q)$$

Thus, a 1st-order derivation is just an ordinary derivation of A into V.

In [**5**; **6**], Y. Nakai has summarized and refined the work of several authors (i.e. E. Kunz [**4**] and H. Osborn [**7**]) concerning the construction of a universal object $\Omega_k{}^q(A)$ for qth order derivations of A over k. Following the notation and terminology of Nakai, $\Omega_k{}^q(A)$ is defined to be an A-module having the following properties:

(α): There exists a canonical qth order k-derivation $\delta_k{}^q$ of A into $\Omega_k{}^q(A)$.

(β): $\Omega_k{}^q(A)$ is generated as an A-module by

$$\{\delta_k{}^q(a) | a \in A\}.$$

(γ): Given any A-module V together with a qth order k-derivation λ of A into V, there exists a unique A-module homomorphism Ψ mapping $\Omega_k{}^q(A)$ into V such that $\lambda = \Psi \circ \delta_k{}^q$.

The construction of $\Omega_k{}^q(A)$ together with a study of its functorial properties are carried out in [**6**].

Suppose now that V is a commutative A-algebra. Then a k-higher derivation of infinite rank $\delta = \{\delta_q\}$ from A into V is an infinite sequence of maps $\delta_1, \delta_2, \delta_3 \ldots$ such that

(1) each δ_i is an element of $\mathrm{Hom}_k(A, V)$, and

(2) for all $q \geqq 1$ and a, b in A, we have

$$\delta_q(ab) = a\delta_q(b) + \delta_1(a)\delta_{q-1}(b) + \ldots + \delta_q(a)b.$$

Henceforth, we shall abbreviate this last equation by writing $\delta_q(ab) = \sum_{i+j=q} \delta_i(a)\delta_j(b)$.

Recently the author and W. E. Kuan in [**1**] have used this notion of k-higher derivations of infinite rank to obtain some new results on analytic products of a variety $\mathscr{V}$ along a subvariety $\mathscr{W}$.

Thus, the following question naturally arises from Nakai's work. Does there exist a universal object for k-higher derivations of infinite rank? The purpose

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of this paper is to show that such an object does exist and satisfies properties analogous to α, β and γ for $\Omega_k{}^q(A)$. We shall also study the functorial properties of the universal object for k-higher derivations of infinite rank and finish with an example for polynomial rings.

Preliminaries. Through this paper, all rings will be assumed to be associative and commutative but not necessarily containing an identity. k will always denote a commutative ring with identity. We shall denote by A and B two rings with identities which are k-algebras via two ring homomorphisms $\theta_1 : k \to A$ and $\theta_2 : k \to B$ which take the identity of k to the identities in A and B respectively. Henceforth we shall not explicitly write the map θ_i. Thus, if a is an element of A and x an element of k, we shall write xa instead of $\theta_1(x)a$. In particular, if Ψ is a k-algebra homomorphism of A into B, then Ψ is said to be zero on k if $\Psi \circ \theta_1 = 0$. We shall write $\Psi(k) = 0$ if Ψ is zero on k.

Let V be a ring (commutative but not necessarily containing an identity). Then we shall call V an A-algebra if V is a unitary A-module such that $\nu_1 a = a\nu_1$ and

$$a(\nu_1\nu_2) = (a\nu_1)\nu_2 = \nu_1(a\nu_2)$$

for all ν_1, ν_2 in V and a in A. An A-algebra homomorphism from an A-algebra V_1 to an A-algebra V_2 is a ring homomorphism which is also an A-module homomorphism.

Let V be an A-algebra. By a k-higher derivation $\delta = \{\delta_q\}$ of A into V, we shall mean an infinite sequence $\delta_1, \delta_2, \ldots$ of maps $\delta_q : A \to V$ such that

(1) each δ_q is an element of $\mathrm{Hom}_k(A, V)$, and

(2) for all a and b in A and $q \geqq 1$, we have

$$\delta_q(ab) = a\delta_q(b) + \delta_1(a)\delta_{q-1}(b) + \ldots + \delta_{q-1}(a)\delta_1(b) + \delta_q(a)b.$$

We shall abbreviate this last equation (Leibniz's rule) by writing

$$\delta_q(ab) = \sum_{i+j=q} \delta_i(a)\delta_j(b).$$

Thus, a k-higher derivation of A into an A-algebra V is an infinite sequence of linear maps of A into V which are all zero on k and satisfy Leibniz's rule. We shall denote the collection of all such k-higher derivations of A into V by $\mathscr{H}_k(A, V)$.

Finally, we assume that the reader is familiar with the results which appear in [**5**; **6**].

1. Construction of the universal object for k-higher derivations. Let A and k be as in the preliminaries. We wish to construct an A-algebra $\Omega_k(A)$ having the following properties:

(a) There exists a k-higher derivation $\delta_k{}^A = \{\delta_{kq}{}^A\}$ from A into $\Omega_k(A)$.

(b) $\Omega_k(A)$ is generated as an A-algebra by the set $\{\delta_{kq}{}^A(a) | a \in A, q \geqq 1\}$.

(c) For any A-algebra V and any k-higher derivation $\lambda = \{\lambda_q\} \in \mathscr{H}_k(A, V)$, there exists a unique A-algebra homomorphism $\Psi : \Omega_k(A) \to V$ such that for all $q \geqq 1$, $\Psi \circ \delta_{kq}{}^A = \lambda_q$.

We note that (b) means that $\Omega_k(A)$ is generated as an A-module by all elements of the form $\delta_{ki_1}{}^A(a_{i_1})\delta_{ki_2}{}^A(a_{i_2}) \ldots \delta_{ki_n}{}^A(a_{i_n})$. Thus, $\Omega_k(A)$ does not contain an identity element.

We now proceed to the construction of $\Omega_k(A)$. Following Y. Nakai [**6**], we may construct for each $q \geqq 1$ a universal object $\Omega_k{}^q(A)$ for qth order derivations on A. Consider $A \otimes_k A$, and let $\varphi : A \otimes_k A \to A$ be given by $\varphi(\sum x_i \otimes y_i) = \sum x_i y_i$. Let I be the kernel of φ. We may view $A \otimes_k A$ as an A-module via $a(x \otimes y) = ax \otimes y$. Then I is an ideal in $A \otimes_k A$ and

$$0 \to I \to A \otimes_k A \xrightarrow{\varphi} A \to 0$$

is an exact sequence of A-modules. We note that I is generated as an A-module by all elements of the form $1 \otimes a - a \otimes 1$, a in A. Here 1 denotes the identity of A.

For all $q \geqq 1$, set $\Omega_k{}^q(A) = I/I^{q+1}$ and define

$$\delta_k{}^q : A \to \Omega_k{}^q(A) \quad \text{by} \quad \delta_k{}^q(a) = (1 \otimes a - a \otimes 1) + I^{q+1}.$$

Then one can readily verify that properties α, β and γ mentioned in the introduction hold.

Now let $S = \bigoplus_{q=1}^{\infty} I/I^{q+1}$, the direct sum of the A-modules $\Omega_k{}^q(A)$. Since each $\Omega_k{}^q(A)$ is generated as an A-module by $\{\delta_k{}^q(a) | a \in A\}$, we see that S is generated as an A-module by $\{\delta_k{}^q(a) | a \in A, q \geqq 1\}$. Set

$$S^n = S \otimes_A S \otimes_A \ldots \otimes_A S \text{ (the tensor product taken } n \text{ times).}$$

Let B_n be the A-submodule of S^n generated by all elements of the form

$$s_1 \otimes_A \ldots \otimes_A s_n - s_{\sigma(1)} \otimes_A \ldots \otimes_A s_{\sigma(n)}$$

where the s_i are in S, and σ is any permutation of $\{1, \ldots, n\}$. Set $\mathscr{S}(S) = \bigoplus_{n=1}^{\infty} S^n/B_n$. Then $\mathscr{S}(S)$ is just the usual symmetric algebra generated by S over A, but without the zero degree terms. Thus, $\mathscr{S}(S)$ is a commutative A-algebra without an identity element. Since S is generated as an A-module by the set $\{\delta_k{}^q(a) | a \in A, q \geqq 1\}$, $\mathscr{S}(S)$ is generated as an A-algebra by the same set. Thus, $\mathscr{S}(S)$ is generated as an A-module by all elements of the form $\{\delta_k{}^{i_1}(a_{i_1}) \ldots \delta_k{}^{i_n}(a_{i_n}) | a_{i_j} \in A, i_j \geqq 1\}$.

Since $\Omega_k{}^q(A) \subset S \subset \mathscr{S}(S)$, each $\delta_k{}^q$ may be viewed as a qth order k-derivation of A into $\mathscr{S}(S)$. Thus, for each $q \geqq 1$ we have a qth order derivation $\delta_k{}^q$ of A into $\mathscr{S}(S)$, and $\mathscr{S}(S)$ is generated as an A-algebra by

$$\{\delta_k{}^q(a) | a \in A, q \geqq 1\}.$$

Now let J denote the ideal in $\mathscr{S}(S)$ generated by all elements of the form

$$\delta_k{}^q(ab) - \sum_{i+j=q} \delta_k{}^i(a)\delta_k{}^j(b) \quad a, b \text{ in } A, q \geqq 1.$$

Let $\pi : \mathscr{S}(S) \to \mathscr{S}(S)/J$ be the natural projection of $\mathscr{S}(S)$ onto $\mathscr{S}(S)/J$. Set $\Omega_k(A) = \mathscr{S}(S)/J$, and for each $q \geqq 1$, $\delta_{kq}{}^A = \pi \circ \delta_k{}^q$. Then clearly $\delta_k{}^A = \{\delta_{kq}{}^A\} \in \mathscr{H}_k(A, \Omega_k(A))$. We also note the $\Omega_k(A)$ is generated as an A-algebra by $\{\delta_{kq}{}^A(a)|a \in A, q \geqq 1\}$ since $\mathscr{S}(S)$ is generated as an A-algebra by $\{\delta_k{}^q(a)|a \in A, q \geqq 1\}$. Hence, $(\Omega_k(A), \delta_k{}^A)$ will be the desired universal object if it has the universal mapping property (c)

THEOREM 1. *Let V be any A-algebra and $\lambda = \{\lambda_q\} \in \mathscr{H}_k(A, V)$. Then there exists a unique A-algebra homomorphism $\Psi : \Omega_k(A) \to V$ such that for all $q \geqq 1$,*

$$\Psi \circ \delta_{kq}{}^A = \lambda_q.$$

Proof. By [**6**, Proposition 5], λ_q is a qth order k-derivation of A into V. Hence by the universal mapping property of $(\Omega_k{}^q(A), \delta_k{}^q)$, there exists a unique A-module homomorphism $h_q : \Omega_k{}^q(A) \to V$ such that $h_q \circ \delta_k{}^q = \lambda_q$. By setting $\Psi_0 = \bigoplus_{q=1}^\infty h_q : S \to V$, we obtain a unique A-module homomorphism of S into V such that for all $q \geqq 1$, $\Psi_0 \circ \delta_k{}^q = \lambda_q$. Since $\mathscr{S}(S)$ is the symmetric algebra generated by S over A (except for terms of degree zero), and V is commutative, we may extend Ψ_0 to an A-algebra homomorphism $\Psi_1 : \mathscr{S}(S) \to V$ by setting

$$\Psi_1(s_1 \otimes_A \ldots \otimes_A s_n + B_n) = \Psi_0(s_1)\Psi_0(s_2) \ldots \Psi_0(s_n).$$

We note that if a is an element of A, then for all $q \geqq 1$ $\Psi_1(\delta_k{}^q(a)) = \Psi_0\delta_k{}^q(a) = \lambda_q(a)$.

Suppose

$$\delta_k{}^q(ab) - \sum_{i+j=q} \delta_k{}^i(a)\delta_k{}^j(b)$$

is a typical generator of the ideal J in $\mathscr{S}(S)$. Since Ψ_1 is an A-algebra homomorphism, we have

$$\Psi_1\{\delta_k{}^q(ab) - \sum_{i+j=q} \delta_k{}^i(a)\delta_k{}^j(b)\} =$$
$$\Psi_1 \circ \delta_k{}^q(ab) - \sum_{i+j=q} \Psi_1 \circ \delta_k{}^i(a)\Psi_1 \circ \delta_k{}^j(b) = \lambda_q(ab) - \sum_{i+j=q} \lambda_i(a)\lambda_j(b) = 0$$

since λ is a k-higher derivation on A. Hence, $\Psi_1(J) = 0$. Thus, Ψ_1 induces an A-algebra homomorphism Ψ of $\Omega_k(A)$ into V such that for all $q \geqq 1$, $\Psi \circ \delta_{kq}{}^A = \lambda_q$. Since $\Omega_k(A)$ is generated as an A-algebra by

$$\{\delta_{kq}{}^A(a)|a \in A, q \geqq 1\},$$

Ψ is obviously unique.

COROLLARY. *Let A be a k-algebra with identity. Then there exists an A-algebra $\Omega_k(A)$ and a k-higher derivation $\delta_k{}^A \in \mathscr{H}_k(A, \Omega_k(A))$ such that properties* (a), (b) *and* (c) *are satisfied.*

We shall call $(\Omega_k(A), \delta_k{}^A)$ the universal object associated with A, and property (c) will be referred to as the universal mapping property (U.M.P.) of $(\Omega_k(A), \delta_k{}^A)$. Clearly the A-algebra $\Omega_k(A)$ is unique up to isomorphism.

We remind the reader again that $\Omega_k(A)$ is an A-algebra which does not contain an identity.

2. Functorial properties of $\Omega_k(A)$**.** In this section we explore the functorial properties of $\Omega_k(A)$.

Let A and B be two k-algebras with identities, and let $h : A \to B$ be a k-algebra homomorphism taking the identity of A to that of B. Let $\delta_k^A = \{\delta_{kq}^A\}$ and $\delta_k^B = \{\delta_{kq}^B\}$ be the canonical k-higher derivations of A and B into $\Omega_k(A)$ and $\Omega_k(B)$ respectively.

Now we may regard $\Omega_k(B)$ as an A-algebra via h. Since $\delta_k^B \circ h$ is a k-higher derivation of A into $\Omega_k(B)$, the U.M.P. of $(\Omega_k(A), \delta_k^A)$ implies that there exists a unique A-algebra homomorphism $h^* : \Omega_k(A) \to \Omega_k(B)$ such that for all $q \geqq 1$, $h^*\delta_{kq}^A = \delta_{kq}^B h$. The map h need not be a monomorphism but by abuse of notation we shall identify A with $h(A)$ in B. Hence, if a is an element of A, then $h^*\delta_{kq}^A(a) = \delta_{kq}^B(a)$. Suppose C is a third k-algebra with identity, and $g : B \to C$ is a k-algebra homomorphism taking the identity of B to that of C. Let g^* be the induced mapping between $\Omega_k(B) \to \Omega_k(C)$. Then via gh, C may be viewed as an A-algebra. One can easily verify that $g^* \circ h^* = (gh)^*$: $\Omega_k(A) \to \Omega_k(C)$.

We may use the A-algebra map h^* to define a B-algebra homomorphism $\mu : B \otimes_A \Omega_k(A) \to \Omega_k(B)$ as follows:

$$\mu\left(\sum_{i=1}^{n} b_i \otimes_A x_i\right) = \sum_{i=1}^{n} b_i h^*(x_i).$$

Here $x_1, \ldots, x_n$ are elements of $\Omega_k(A)$, and $b_1, \ldots, b_n$ are elements of B. The image of μ in $\Omega_k(B)$ is a B-subalgebra, but not necessarily an ideal of $\Omega_k(B)$. Let $[\mathrm{Im}\, \mu]$ denote the ideal of $\Omega_k(B)$ generated by the image of μ.

Let us set $\Omega_k(B/A) = \Omega_k(B)/[\mathrm{Im}\, \mu]$, and let $j : \Omega_k(B) \to \Omega_k(B/A)$ be the natural projection. Then $\Omega_k(B/A)$ is a B-algebra, and j is a B-algebra homomorphism. For each $q \geqq 1$, set $\hat{\delta}_q = j \circ \delta_{kq}^B$. Then $\hat{\delta} = \{\hat{\delta}_q\} \in \mathscr{H}_k(B, \Omega_k(B/A))$. If a is an element of A, then for all $q \geqq 1$,

$$\hat{\delta}_q(a) = j \circ \delta_{kq}^B(a) = j \circ h^*\delta_{kq}^A(a) = 0$$

since $h^*\delta_{kq}^A(a)$ is in $\mathrm{Im}\, \mu$. Hence $\hat{\delta}$ is a k-higher derivation of B into $\Omega_k(B/A)$ which is zero an A. Now we can show that $(\Omega_k(B/A), \hat{\delta})$ has the universal mapping property with respect to all k-higher derivations on B which are zero on A.

LEMMA 1. *Suppose V is any B-algebra, and $\lambda = \{\lambda_q\} \in \mathscr{H}_k(B, V)$ such that $\lambda_q(A) = 0$ for all q. Then there exists a unique B-algebra homomorphism $\alpha : \Omega_k(B/A) \to V$ such that $\alpha \circ \hat{\delta} = \lambda$.*

Proof. By the U.M.P. of $(\Omega_k(B), \delta_k^B)$, there exists a unique B-algebra homomorphism $\alpha_0 : \Omega_k(B) \to V$ such that $\alpha_0 \circ \delta_k^B = \lambda$. Suppose a is an

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element of A. Then for all $q \geqq 1$, we have

$$(*): \qquad 0 = \lambda_q(a) = \alpha_0 \delta_{kq}{}^B(a) = \alpha_0(h^* \circ \delta_{kq}{}^A(a)).$$

Now let z be an arbitrary element of $\operatorname{Im} \mu$. Then $z = \mu(\sum_{i=1}^n b_i \otimes_A x_i)$ for some $b_1, \ldots, b_n$ in B and $x_1, \ldots, x_n$ in $\Omega_k(A)$. Since the tensor product is over A, we may assume each x_i has the form

$$\delta_{kj_1}{}^A(a_{j_1})\delta_{kj_2}{}^A(a_{j_2}) \ldots \delta_{kj_t}{}^A(a_{j_t}).$$

Since h^* is a ring homomorphism, and α_0 is a B-algebra homomorphism, $(*)$ implies $\alpha_0(z) = 0$. Thus, $\alpha_0([\operatorname{Im} \mu]) = 0$. Hence, α_0 induces a B-algebra homomorphism

$$\alpha : \Omega_k(B)/[\operatorname{Im} \mu] = \Omega_k(B/A) \to V$$

such that $\alpha \circ \hat{\delta}_q = \lambda_q$ for all $q \geqq 1$. Since $\Omega_k(B/A)$ is generated as a B-algebra by $\{\hat{\delta}_q(b) | b \in B, q \geqq 1\}$, we see that α is necessarily unique.

LEMMA 2. *$\Omega_k(B/A) \cong \Omega_A(B)$ as B-algebras.*

Proof. This lemma follows immediately from Lemma 1 and the uniqueness of the universal object $\Omega_A(B)$.

We have now proven the following theorem:

THEOREM 2. *Let A and B be k-algebras with identity, and let $h : A \to B$ be a k-algebra homomorphism sending the identity of A to the identity of B. Let $\mu : B \otimes_A \Omega_k(A) \to B$ be the induced B-algebra homomorphism. Then*

$$\Omega_A(B) \cong \Omega_k(B)/[\operatorname{Im} \mu]$$

as B-algebras.

We may use the ideas of Theorem 2 to present some information on extensions of k-higher derivations. So as usual, let A and B be k-algebras with identity, and let $h : A \to B$ be as before. We shall say that a k-higher derivation $\lambda = \{\lambda_q\}$ of A into a B-algebra V can be extended to B if there exists $\lambda' = \{\lambda_q'\} \in \mathscr{H}_k(B, V)$ such that for all a in A and $q \geqq 1$, $\lambda_q'(h(a)) = \lambda_q(a)$.

PROPOSITION 1. *If μ is injective, and $\operatorname{Im} \mu$ is an ideal direct summand of $\Omega_k(B)$, then every k-higher derivation of A into a B-algebra V can be extended to a k-higher derivation of B into V. Conversely, if every k-higher derivation of A into a B-algebra V can be extended to a k-higher derivation of B, then μ is injective and, $\operatorname{Im} \mu$ is a B-algebra direct summand of $\Omega_k(B)$.*

Proof. Let us first assume that $\mu : B \otimes_A \Omega_k(A) \to \Omega_k(B)$ is injective, and that $\operatorname{Im} \mu = [\operatorname{Im} \mu]$ is an ideal direct summand of $\Omega_k(B)$. Suppose V is a B-algebra, and $\lambda = \{\lambda_q\} \in \mathscr{H}_k(A, V)$. Then by the U.M.P. of $(\Omega_k(A), \delta_k{}^A)$, there exists a unique A-algebra homomorphism $\alpha : \Omega_k(A) \to V$ such that $\alpha \circ \delta_k{}^A = \lambda$. We may extend α to a B-algebra homomorphism α_1 of $B \otimes_A \Omega_k(A)$ into V in the usual way:

$$\alpha_1(\sum b_i \otimes_A x_i) = \sum b_i \alpha(x_i).$$

Since μ is injective, we may identify $B \otimes_A \Omega_k(A)$ with $\operatorname{Im} \mu$ in $\Omega_k(B)$. By hypothesis, $\operatorname{Im} \mu$ is an ideal in $\Omega_k(B)$, and there exists an ideal L in $\Omega_k(B)$ such that $\operatorname{Im} \mu \oplus L = \Omega_k(B)$. Hence, we may extend α_1 to a B-algebra homomorphism α_2 of $\Omega_k(B)$ into V such that α_2 when restricted to $B \otimes_A \Omega_k(A)$ is α_1. For each $q \geqq 1$, let $\lambda_q' = \alpha_2 \circ \delta_{kq}{}^B$. Clearly $\lambda' = \{\lambda_q'\} \in \mathscr{H}_k(B, V)$. If a is an element of A, then for all $q \geqq 1$, we have

$$\lambda_q'(a) = \alpha_2 \delta_{kq}{}^B(a) = \alpha_2[\mu(1 \otimes_A \delta_{kq}{}^A(a))] = \alpha_1(1 \otimes_A \delta_{kq}{}^A(a)) = \alpha \delta_{kq}{}^A(a) = \lambda_q(a).$$

So we have extended λ to λ' on B.

Now let us assume that every k-higher derivation of A into a B-algebra V can be extended to B. We first show that μ is injective. For each $q \geqq 1$, let $p_q : A \to B \otimes_A \Omega_k(A)$ be the k-module homomorphism defined as follows: if a is in A, then $p_q(a) = 1 \otimes_A \delta_{kq}{}^A(a)$. Here 1 of course denotes the identity element of B. One easily checks that $p = \{p_q\}$ is a k-higher derivation of A into $B \otimes_A \Omega_k(A)$. Thus by hypothesis, p can be extended to a k-higher derivation $p' = \{p_q'\}$ of B into $B \otimes_A \Omega_k(A)$. From the U.M.P. of $(\Omega_k(B), \delta_k{}^B)$, there exists a unique B-algebra homomorphism $\Psi : \Omega_k(B) \to B \otimes_A \Omega_k(A)$ such that $\Psi \circ \delta_k{}^B = p'$.

Now let z be an element in the kernel of μ. Since $\Omega_k(A)$ is generated as an A-module by products of the form $\delta_{ki_1}{}^A(a_{i_1})\delta_{ki_2}{}^A(a_{i_2}) \ldots \delta_{ki_n}{}^A(a_{i_n})$, we may assume z has the form

$$z = \sum b_{i_1 \ldots i_n} \otimes_A \delta_{ki_1}{}^A(a_{i_1}) \ldots \delta_{ki_n}{}^A(a_{i_n}), \quad b_{i_1 \ldots i_n} \text{ in } B.$$

Now

$$\begin{aligned} 0 = \Psi(\mu(z)) &= \Psi\{\textstyle\sum b_{i_1 \ldots i_n} \delta_{ki_1}{}^B(a_{i_1}) \ldots \delta_{ki_n}{}^B(a_{i_n})\} \\ &= \textstyle\sum b_{i_1 \ldots i_n} \Psi \circ \delta_{ki_1}{}^B(a_{i_1}) \ldots \Psi \circ \delta_{ki_n}{}^B(a_{i_n}) = \sum b_{i_1 \ldots i_n} p_{i_1}(a_{i_1}) \ldots p_{i_n}(a_{i_n}) \\ &= \textstyle\sum b_{i_1 \ldots i_n}(1 \otimes_A \delta_{ki_1}{}^A(a_{i_1})) \ldots (1 \otimes_A \delta_{ki_n}{}^A(a_{i_n})) = z \end{aligned}$$

Hence, μ is injective. More generally, we have shown that $\Psi \circ \mu$ is the identity map. Thus, $\operatorname{Im} \mu$ is a B-algebra direct summand of $\Omega_k(B)$.

We note that if every k-higher derivation of A into V can be extended to B, then $\operatorname{Im} \mu$ is a B-algebra direct summand of $\Omega_k(B)$. However, $\operatorname{Im} \mu$ need not necessarily be an ideal direct summand of $\Omega_k(B)$. In one important case, we can simplify the statement of Proposition 1.

COROLLARY. *Suppose that $h : A \to B$ is surjective. Then every k-higher derivation of A into a B-algebra V can be extended to B if and only if μ is injective.*

Proof. If h is surjective, then $\mu : B \otimes_A \Omega_k(A) \to \Omega_k(B)$ is easily seen to be surjective. Hence $\operatorname{Im} \mu = \Omega_k(B)$. Thus the result follows from Proposition 1.

3. Localizations: A special case. Let A be a k-algebra with identity, and suppose M is a multiplicatively closed set containing 1 in A. In this section, we further assume that M contains no zero divisors of A. Later we shall remove this second condition on M.

Let A_M denote the localization of A by M. Recall then that A_M is a subring of the total quotient ring of A. Thus, A_M consists of all fractions of the form a/m where a is in A and m is an element of M. Two fractions a/m and a'/m' are equal if $am' = ma'$. Since M contains 1, we may view A as a subring of A_M under the identification of a with $a/1$.

We wish to prove the following theorem:

THEOREM 3. *$A_M \otimes_A \Omega_k(A) \cong \Omega_k(A_M)$ as A_M-algebras.*

We first need two lemmas:

LEMMA 3. $\Omega_A(A_M) = 0$.

Proof. Let $\delta_A^{A_M} = \{\delta_{Aq}^{A_M}\}$ denote the canonical A-higher derivation of A_M into $\Omega_A(A_M)$. Since $\Omega_A(A_M)$ is generated as an A_M-algebra by the set $\{\delta_{Aq}^{A_M}(a/m) | a/m \in A_M, q \geqq 1\}$, it suffices to show that $\delta_{Aq}^{A_M} = 0$ for all q. Now each $\delta_{Aq}^{A_M}$ is zero on A. Hence for a/m in A_M, we have

$$0 = \delta_{Aq}^{A_M}(a) = \delta_{Aq}^{A_M}\left(\frac{a}{m}\cdot\frac{m}{1}\right) = \sum_{i+j=q} \delta_{Ai}^{A_M}(a/m)\cdot\delta_{Aj}^{A_M}\left(\frac{m}{1}\right) =$$

$$\frac{m}{1}\cdot\delta_{Aq}^{A_M}(a/m).$$

Since $m/1$ is a unit in A_M, we get $\delta_{Aq}^{A_M}(a/m) = 0$.

LEMMA 4. *Every k-higher derivation of A into an A_M-algebra V can be extended to a k-higher derivation of A_M into V.*

Proof. Let V be an A_M-algebra. Then V does not necessarily contain an identity element. So let $V^* = A_M \oplus V$ and define multiplication in V^* as follows:

$$(a_1 + v_1)(a_2 + v_2) = a_1a_2 + a_1v_2 + a_2v_1 + v_1v_2.$$

Then V^* is an A_M-algebra which contains A_M. The identity of A_M is the identity of V^*.

Now let $\lambda = \{\lambda_q\} \in \mathscr{H}_k(A, V)$. Then λ may be viewed as a k-higher derivation of A into V^*. Now by [**2**, Lemma 2], we may extend λ uniquely to a k-higher derivation $\lambda' = \{\lambda_q'\} \in \mathscr{H}_k(A_M, V^*)$. We next show that each λ_q' actually maps A_M into V. We prove this by induction on q. Let a/m be an element of A_M. Then

$$\lambda_1(a) = \lambda_1'\left(\frac{a}{m}\cdot\frac{m}{1}\right) = \frac{m}{1}\lambda_1'(a/m) + \frac{a}{m}\cdot\lambda_1'\left(\frac{m}{1}\right)$$

$$= m\lambda_1'(a/m) + a/m\lambda_1(m).$$

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Thus, $\lambda_1'(a/m) = (1/m)(\lambda_1(a) - (a/m)\lambda_1(m))$ which is an element in V. Assume we have shown that $\lambda_1', \ldots, \lambda_{q-1}'$ map A_M into V. Then

$$\lambda_q(a) = \lambda_q'\left(\frac{a}{m}\cdot\frac{m}{1}\right) = \sum_{i+j=q} \lambda_i'(a/m)\cdot\lambda_j'(m).$$

Thus,

$$\lambda_q'(a/m) = \frac{1}{m}\{\lambda_q(a) - \lambda_1(m)\lambda'_{q-1}(a/m) - \ldots - \lambda_q(m)a/m\}.$$

By the induction hypothesis,

$$(1/m)\{\lambda_q(a) - \lambda_1(m)\lambda_{q-1}'(a/m) - \ldots - \lambda_q(m)a/m\}$$

is an element of V. Hence, $\lambda_q'(a/m)$ is an element of V. Thus, we have extended λ to λ', a k-higher derivation of A_M into V.

We may now proceed with the proof of Theorem 3. By Lemma 4, every k-higher derivation of A into an A_M-algebra V can be extended to a k-higher derivation of A_M into V. Hence by Proposition 1, the A_M-algebra homomorphism $\mu : A_M \otimes_A \Omega_k(A) \to \Omega_k(A_M)$ is injective. Since $\Omega_A(A_M) = 0$, Theorem 2 implies that $[\mathrm{Im}\ \mu] = \Omega_k(A_M)$. Hence, Theorem 3 will follow if we show that $\mathrm{Im}\ \mu$ is an ideal in $\Omega_k(A_M)$.

Let $\delta_k^{A_M} = \{\delta_{kq}^{A_M}\}$ be the canonical k-higher derivation of A_M into $\Omega_k(A_M)$. To show that $\mathrm{Im}\ \mu$ is all of $\Omega_k(A_M)$, it suffices to show that $\delta_{kq}^{A_M}(a/m) \in \mathrm{Im}\ \mu$ for all a/m in A_M and $q \geqq 1$. We prove this by induction on q. For $q = 1$, we have

$$\delta_{k1}^{A_M}(a/m) = \frac{1}{m^2}\{m\delta_{k1}^{A_M}(a) - a\delta_{k1}^{A_M}(m)\} \in \mathrm{Im}\ \mu.$$

By induction,

$$\delta_{kq}^{A_M}(a/m) = \frac{1}{m}\left\{\delta_{kq}^{A_M}(a) - \frac{a}{m}\delta_{kq}^{A_M}(m) - \ldots \delta_{kq-1}^{A_M}(a/m)\delta_{k1}^{A_M}(m)\right\}$$

is an element of $\mathrm{Im}\ \mu$. Therefore, μ is surjective. This completes the proof of Theorem 3.

4. Direct sums. As usual, let A be a k-algebra with identity. Suppose there exists two ideals J_1 and J_2 in A such that $J_1 \oplus J_2 = A$ as k-algebras. Then $J_1 + J_2 = A$, and $J_1 \cap J_2 = J_1J_2 = 0$. Let 1 denote the identity of A, and write $1 = e_1 + e_2$ with e_1 in J_1 and e_2 in J_2 respectively. Then each J_i is a k-algebra with identity e_i. Thus, we may form $\Omega_k(J_1)$ and $\Omega_k(J_2)$. Since $\Omega_k(J_i)$ is a J_i-algebra, $\Omega_k(J_1) \oplus \Omega_k(J_2)$ is naturally an $A = J_1 \oplus J_2$-algebra. Specifically if a is an element of A and x and y are elements of $\Omega_k(J_1)$ and $\Omega_k(J_2)$ respectively, then $a(x \oplus y) = a_1x \oplus a_2y$. Here a_1 is in J_1, a_2 in J_2, and $a = a_1 + a_2$. We can now prove the following theorem:

THEOREM 4. *$\Omega_k(A) \cong \Omega_k(J_1) \oplus \Omega_k(J_2)$ as A-algebras.*

Proof. Let $\delta_k^{J_1} = \{\delta_{kq}^{J_1}\}$ and $\delta_k^{J_2} = \{\delta_{kq}^{J_2}\}$ be the canonical k-higher derivations of J_1 into $\Omega_k(J_1)$ and J_2 into $\Omega_k(J_2)$ respectively. Let

$$\hat{\delta}_q : A \rightarrow \Omega_k(J_1) \oplus \Omega_k(J_2)$$

be defined as follows:

For a in A,

$$\hat{\delta}_q(a) = \delta_{kq}{}^{J_1}(ae_1) \oplus \delta_{kq}{}^{J_2}(ae_2).$$

Then one can easily show that $\hat{\delta} = \{\hat{\delta}_q\}$ is a k-higher derivation of A into $\Omega_k(J_1) \oplus \Omega_k(J_2)$. From the U.M.P. of $(\Omega_k(A), \delta_k{}^A)$, we get a unique A-algebra homomorphism $\Psi_1 : \Omega_k(A) \rightarrow \Omega_k(J_1) \oplus \Omega_k(J_2)$ such that $\Psi_1 \circ \delta_k{}^A = \hat{\delta}$. Thus for all a in A,

$$\Psi(\delta_{kq}{}^A(a)) = \delta_{kq}{}^{J_1}(ae_1) \oplus \delta_{kq}{}^{J_2}(ae_2).$$

Let c_1 and c_2 denote the inclusion mappings of J_1 and J_2 into A respectively. Then $\delta_k{}^A \circ c_1 \in \mathscr{H}_k(J_1, \Omega_k(A))$, and $\delta_k{}^A \circ c_2 \in \mathscr{H}_k(J_2, \Omega_k(A))$. Thus, we get a unique J_1-algebra homomorphism $\alpha_1 : \Omega_k(J_1) \rightarrow \Omega_k(A)$ and a unique J_2-algebra homomorphism $\alpha_2 : \Omega_k(J_2) \rightarrow \Omega_k(A)$ such that $\alpha_1 \circ \delta_k{}^{J_1} = \delta_k{}^A \circ c_1$, and $\alpha_2 \circ \delta_k{}^{J_2} = \delta_k{}^A \circ c_2$. We may use α_1 and α_2 to define a map Ψ_2 from $\Omega_k(J_1) \oplus \Omega_k(J_2)$ into $\Omega_k(A)$ as follows: $\Psi_2(x \oplus y) = \alpha_1(x) + \alpha_2(y)$. Here x is an element of $\Omega_k(J_1)$, and y is an element of $\Omega_k(J_2)$. Since $e_1e_2 = 0$, Ψ_2 is an A-algebra homomorphism. If a is any element of A, then for all $q \geqq 1$ we have

$$\begin{aligned}\Psi_2(\hat{\delta}_q(a)) = \Psi_2(\delta_{kq}{}^{J_1} \oplus \delta_{kq}{}^{J_2})(a) &= \alpha_1\delta_{kq}{}^{J_1}(ae_1) + \alpha_2\delta_{kq}{}^{J_2}(ae_2)\\ &= \delta_{kq}{}^A(ae_1) + \delta_{kq}{}^A(ae_2) = \delta_{kq}{}^A(a).\end{aligned}$$

Thus, we have constructed A-algebra homomorphisms

$$\Psi_1 : \Omega_k(A) \rightarrow \Omega_k(J_1) \oplus \Omega_k(J_2) \quad \text{and} \quad \Psi_2 : \Omega_k(J_1) \oplus \Omega_k(J_2) \rightarrow \Omega_k(A).$$

Since $(\Psi_2 \circ \Psi_1) \circ \delta_k{}^A = \delta_k{}^A$, the U.M.P. of $(\Omega_k(A), \delta_k{}^A)$ implies that $\Psi_2 \circ \Psi_1 = \text{identity}$. Suppose V is any A-algebra, and $\lambda \in \mathscr{H}_k(A, V)$. Then there exists a unique A-algebra homomorphism $\alpha : \Omega_k(A) \rightarrow V$ such that $\alpha \circ \delta_k{}^A = \lambda$. The composite map $\alpha \circ \Psi_2$ is an A-algebra homomorphism of $\Omega_k(J_1) \oplus \Omega_k(J_2)$ into V such that $(\alpha \circ \Psi_2) \circ \hat{\delta} = \lambda$. We note that $\alpha \circ \Psi_2$ is necessarily the unique A-algebra map for which $(\alpha \circ \Psi_2) \circ \hat{\delta} = \lambda$. Hence, $(\Omega_k(J_1) \oplus \Omega_k(J_2), \hat{\delta})$ has the U.M.P. Thus, $\Psi_1 \circ \Psi_2 = \text{identity}$, and $\Omega_k(J_1) \oplus \Omega_k(J_2)$ is isomorphic to $\Omega_k(A)$ as A-algebras.

5. Residue class formations. Let A be a k-algebra with identity, and let J be an ideal in A. Let $\pi : A \rightarrow A/J$ denote the canonical projection of A onto A/J. Then as we noted before, the induced A-algebra homomorphism $\pi^* : \Omega_k(A) \rightarrow \Omega_k(A/J)$ is surjective. Hence, the induced mapping $\mu : (A/J) \otimes_A \Omega_k(A) \rightarrow \Omega_k(A/J)$ of (A/J)-algebras is surjective. Let N denote the kernel of μ. Then

$$0 \rightarrow N \rightarrow (A/J) \otimes_A \Omega_k(A) \xrightarrow{\mu} \Omega_k(A/J) \rightarrow 0$$

is a short exact sequence of (A/J)-algebras. In this section, we wish to determine the structure of N.

Since $0 \to J \to A \to A/J \to 0$ is exact,

$$J \otimes_A \Omega_k(A) \to A \otimes_A \Omega_k(A) \to (A/J) \otimes_A \Omega_k(A) \to 0$$

is exact. Thus, $(A/J) \otimes_A \Omega_k(A) \cong \Omega_k(A)/J \cdot \Omega_k(A)$ as (A/J)-algebras. Here $J \cdot \Omega_k(A)$ is the image of $J \otimes_A \Omega_k(A)$ under the composite map

$$J \otimes_A \Omega_k(A) \to A \otimes_A \Omega_k(A) \to \Omega_k(A).$$

Clearly $J \cdot \Omega_k(A)$ is an ideal in $\Omega_k(A)$, and $\Omega_k(A)/J \cdot \Omega_k(A)$ is naturally an (A/J)-algebra.

Let $\delta_k{}^A(J)$ denote the ideal in $\Omega_k(A)$ generated by the set $\{\delta_{kq}{}^A(x) | x \text{ in } J, q \geqq 1\}$. Set $\hat{J} = \delta_k{}^A(J) + J \cdot \Omega_k(A)$. Clearly $\hat{J}$ is an ideal in $\Omega_k(A)$.

Now $\mu : (A/J) \otimes_A \Omega_k(A) \to \Omega_k(A/J)$ induces an (A/J)-algebra homomorphism (which we will also call μ) from $\Omega_k(A)/J \cdot \Omega_k(A)$ to $\Omega_k(A/J)$. Since $\pi^*(\hat{J}) = 0$, π^* induces an (A/J)-algebra homomorphism $\bar{\mu}$ from $\Omega_k(A)/\hat{J}$ to $\Omega_k(A/J)$. Specifically, $\bar{\mu}(y + \hat{J}) = \pi^*(y)$. Here y is any element of $\Omega_k(A)$. We shall show that $\bar{\mu}$ is actually an (A/J)-isomorphism of $\Omega_k(A)/\hat{J}$ onto $\Omega_k(A/J)$.

We first define a k-higher derivation $\hat{\delta} = \{\hat{\delta}_q\} \in \mathscr{H}_k(A/J, \Omega_k(A)/\hat{J})$ as follows: For a in A and $q \geqq 1$, set

$$\delta_q(a + J) = \delta_{kq}{}^A(a) + \hat{J}.$$

Since each $\delta_{kq}{}^A$ is additive and $\delta_k{}^A(J) \subset \hat{J}$, each $\hat{\delta}_q$ is a well defined k-linear homomorphism. Since $\delta_k{}^A$ is a k-higher derivation, $\hat{\delta} = \{\hat{\delta}_q\}$ is a k-higher derivation of A/J into $\Omega_k(A)/\hat{J}$. From the U.M.P. of $(\Omega_k(A/J), \delta_k{}^{A/J})$, we get a unique (A/J)-algebra homomorphism

$$\Psi : \Omega_k(A/J) \to \Omega_k(A)/\hat{J}$$

such that $\Psi\delta_{kq}{}^{A/J}(a + J) = \hat{\delta}_q(a + J) = \delta_{kq}{}^A(a) + \hat{J}$ for all $q \geqq 1$ and a in A.

The (A/J)-algebra homomorphism $\bar{\mu} : \Omega_k(A)/\hat{J} \to \Omega_k(A/J)$ is clearly surjective. Using Ψ, we can now show the $\bar{\mu}$ is injective. Suppose z is an element in the kernel of $\bar{\mu}$. Then z has the form

$$z = \sum a_{i_1 \ldots i_n} \delta_{k i_1}{}^A(a_{i_1}) \ldots \delta_{k i_n}{}^A(a_{i_n}) + \hat{J}$$

where $a_{i_1 \ldots i_n}$ and $a_{i_1}, \ldots, a_{i_n}$ are elements of A. Let $\bar{a}$ denote the image of an element a (in A) in A/J. Then we have

$$\begin{aligned} 0 = \Psi(\bar{\mu}(z)) &= \Psi\{\sum \bar{a}_{i_1 \ldots i_n} \delta_{k i_1}{}^{A/J}(\bar{a}_{i_1}) \ldots \delta_{k i_n}{}^{A/J}(\bar{a}_{i_n})\} \\ &= \sum \bar{a}_{i_1 \ldots i_n} \Psi\delta_{k i_1}{}^{A/J}(\bar{a}_{i_1}) \ldots \Psi\delta_{k i_n}{}^{A/J}(\bar{a}_{i_n}) \\ &= \sum \bar{a}_{i_1 \ldots i_n} \hat{\delta}_{i_1}(\bar{a}_{i_1}) \ldots \hat{\delta}_{i_n}(\bar{a}_{i_n}) \\ &= \sum \bar{a}_{i_1 \ldots i_n} (\delta_{k i_1}{}^A(a_{i_1}) + \hat{J}) \ldots (\delta_{k i_n}{}^A(a_{i_n}) + \hat{J}) = z. \end{aligned}$$

Hence, μ is injective.

We have now proven that $\Omega_k(A)/\delta_k{}^A(J) + J \cdot \Omega_k(A)$ is isomorphic to $\Omega_k(A/J)$. Thus, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & N & \to & (A/J) \otimes_A \Omega_k(A) & \to & \Omega_k(A/J) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \dfrac{\delta_k{}^A(J) + J \cdot \Omega_k(A)}{J \cdot \Omega_k(A)} & \to & \dfrac{\Omega_k(A)}{J \cdot \Omega_k(A)} & \to & \Omega_k(A/J) & \to & 0. \end{array}$$

The middle mapping is the (A/J)-algebra isomorphism sending

$$\sum \bar{a}_i \otimes x_i \to \sum a_i x_i + J \cdot \Omega_k(A).$$

The commutativity of the above diagram implies that

$$N \cong \delta_k{}^A(J) + J \cdot \Omega_k(A)/J \cdot \Omega_k(A)$$

as (A/J)-algebras. Hence, we have proven the following theorem:

THEOREM 5. *Let A be a k-algebra with identity and J an ideal of A. Let $\delta_k{}^A(J)$ be the ideal in $\Omega_k(A)$ generated by the set $\{\delta_{kq}{}^A(x) | x \in J, q \geqq 1\}$. Then if N is the kernel of the map $\mu : (A/J) \otimes_A \Omega_k(A) \to \Omega_k(A/J)$,*

$$N = \{\sum \bar{a}_i \otimes_A x_i | \bar{a}_i \in A/J, x_i \in \delta_k{}^A(J)\}.$$

6. Localizations. The general case. In this section, we shall prove Theorem 3 without the added assumption that M consist of nonzero divisors. So let A be a k-algebra with identity, and let M be any multiplicatively closed set containing the identity 1 in A. Let $n = \{x \in A | mx = 0$ for some m in $M\}$. Let $\pi : A \to A/n$ be the natural projection of A onto A/n. Then $\pi(M) = M^*$ consists of nonzero divisors in A/n, and $(A/n)_{M^*} = A_M$. We wish to prove the following theorem:

THEOREM 6. $A_M \otimes_A \Omega_k(A) \cong \Omega_k(A_M)$ *as A_M-algebras.*

Proof. Let N denote the kernel of $\mu : A/n \otimes_A \Omega_k(A) \to \Omega_k(A/n)$. Then

$$0 \to N \to A/n \otimes_A \Omega_k(A) \to \Omega_k(A/n) \to 0$$

is an exact sequence of (A/n)-algebras. Tensoring with $\otimes_A A_M (= \otimes_{A/n} (A/n)_{M^*})$ which is exact, we obtain that

$$0 \to N \otimes_A A_M \to (A/n) \otimes_A \Omega_k(A) \otimes_A A_M \to \Omega_k(A/n) \otimes_A A_M \to 0$$

is exact. Now $(A/n) \otimes_A \Omega_k(A) \otimes_A A_M \cong (A/n) \otimes_A \Omega_k(A) \otimes_{(A/n)} (A/n)_{M^*} \cong A_M \otimes_A \Omega_k(A)$ and $\Omega_k(A/n) \otimes_A A_M \cong \Omega_k(A/n) \otimes_{A/n} (A/n)_{M^*}$ which by Theorem 3 is isomorphic to $\Omega_k(A_M)$. Hence, we have

$$0 \to N \otimes_A A_M \to A_M \otimes_A \Omega_k(A) \to \Omega_k(A_M) \to 0$$

is exact. Thus, the result will follow if we show $N \otimes_A A_M = 0$.

By Theorem 5, $N = \{\sum \bar{a}_i \otimes x_i | \bar{a}_i \in A/n, x_i \in \delta_k{}^A(n)\}$. Consider any generator of $N \otimes_A A_M$ of the form

$$(*) \qquad \bar{a}_1 \otimes_A y\delta_{k1}{}^A(x) \otimes_A \bar{a}/\bar{m}.$$

Here $\bar{a}_1$ is an element of A/n, y an element of $\Omega_k(A)$, x an element of n and $\bar{a}/\bar{m}$ a representative of an element in A_M. Since x is in n, there exists an m' in M such that $m'x = 0$. Now $\bar{a}m'/\bar{m}m'$ is a well defined element of A_M, and $\bar{a}/\bar{m} = \bar{a}m'/\bar{m}m'$. Hence,

$$\bar{a}_1 \otimes_A y\delta_{k1}{}^A(x) \otimes_A (\bar{a}/\bar{m}) = \bar{a}_1 \otimes_A ym'\delta_{k1}{}^A(x) \otimes_A (\bar{a}/\bar{m}m').$$

Now, $0 = \delta_{k1}{}^A(m'x) = m'\delta_{k1}{}^A(x) + x\delta_{k1}{}^A(m')$. Thus,

$$\bar{a}_1 \otimes_A ym'\delta_{k1}{}^A(x) \otimes_A \frac{\bar{a}}{\bar{m}m'} = -\bar{a}_1 \otimes_A yx\delta_{k1}{}^A(m') \otimes_A \frac{\bar{a}}{\bar{m}m'} =$$

$$-x\bar{a}_1 \otimes_A y\delta_{k1}{}^A(m') \otimes_A \frac{\bar{a}}{\bar{m}m'} = 0.$$

The last term is zero since $x\bar{a}_1 = 0$. Hence, any generator of the form $(*)$ is zero. The same proof shows that any generator of the form

$$\bar{a}_1 \otimes_A \delta_{k1}{}^A(x) \otimes_A \bar{a}/\bar{m}$$

is zero also.

Let us assume we have proven that any generators of $N \otimes_A A_M$ of the form $\bar{a}_1 \otimes_A y\delta_{ki}{}^A(x) \otimes_A \bar{a}/\bar{m}$ or $\bar{a}_1 \otimes_A \delta_{ki}{}^A(x) \otimes_A \bar{a}/\bar{m}$ are zero for $i = 1, \ldots, q-1$. Consider a generator of the form

$$z = \bar{a}_1 \otimes_A y\delta_{kq}{}^A(x) \otimes_A \bar{a}/\bar{m}.$$

Since x is an element of n, we can find an m' in M such that $m'x = 0$. Then

$$z = \bar{a}_1 \otimes_A ym'\delta_{kq}{}^A(x) \otimes_A \bar{a}/\bar{m}m'.$$

Now, $0 = \delta_{kq}{}^A(m'x) = \sum_{i+j=q} \delta_{ki}{}^A(m')\delta_{kj}{}^A(x)$. Thus,

$$z = -\bar{a}_1 \otimes_A y(\delta_{k1}{}^A(m')\delta_{kq-1}{}^A(x) + \delta_{k2}{}^A(m')\delta_{kq-2}{}^A(x) + \ldots$$
$$x\delta_{kq}{}^A(m')) \otimes_A \bar{a}/\bar{m}m' = 0$$

by the induction hypothesis. A similar proof shows that

$$\bar{a}_1 \otimes_A \delta_{kq}{}^A(x) \otimes_A \bar{a}/\bar{m} = 0.$$

Thus, for all $q \geqq 1$ any generator of the form

$$\bar{a}_1 \otimes_A y\delta_{kq}{}^A(x) \otimes_A \bar{a}/\bar{m} \quad \text{or} \quad \bar{a}_1 \otimes_A \delta_{kq}{}^A(x) \otimes_A \bar{a}/\bar{m}$$

is zero. Since any element of $N \otimes_A A_M$ is a linear combination of generators of these two types, we get $N \otimes_A A_M = 0$. Thus,

$$A_M \otimes_A \Omega_k(A) \cong \Omega_k(A_M).$$

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7. Tensor products. In this last section involving the functorial properties of $\Omega_k(A)$, we shall present a tensor product relationship. Let A and B be k-algebras with identities. Then $B \otimes_k A$ is a k-algebra with identity and, we can construct $\Omega_B(B \otimes_k A)$. We can also consider $B \otimes_k \Omega_k(A)$ as a $(B \otimes_k A)$-algebra. These two $(B \otimes_k A)$-algebras are isomorphic.

THEOREM 7. *Let A and B be two k-algebras with identities. Then*

$$\Omega_B(B \otimes_k A) \cong B \otimes_k \Omega_k(A)$$

as $(B \otimes_k A)$-algebras.

Proof. We begin by defining a B-higher derivation of $B \otimes_k A$ into $B \otimes_k \Omega_k(A)$ as follows: For all $q \geqq 1$ set

$$\hat{\delta}_q\left(\sum_{i=1}^{n} b_i \otimes_k a_i\right) = \sum_{i=1}^{n} b_i \otimes \delta_{kq}{}^A(a_i).$$

Here $b_1, \ldots, b_n$ are in B, and $a_1, \ldots, a_n$ are in A. Then one can readily verify that $\hat{\delta} = \{\hat{\delta}_q\} \in \mathscr{H}_B(B \otimes_k A, B \otimes_k \Omega_k(A))$. From the U.M.P. of $(\Omega_B(B \otimes_k A), \delta_B{}^{B \otimes_k A})$, there exists a unique $(B \otimes_k A)$-algebra homomorphism $\Psi_1 : \Omega_B(B \otimes_k A) \to B \otimes_k \Omega_k(A)$ such that $\Psi_1 \circ \delta_B{}^{B \otimes_k A} = \hat{\delta}$.

Let $\alpha : A \to B \otimes_k A$ be the k-algebra homomorphism given by $\alpha(a) = 1 \otimes a$. Then $\delta_B{}^{B \otimes_k A} \circ \alpha$ is a k-higher derivation of A into $\Omega_B(B \otimes_k A)$. Hence by the U.M.P. of $(\Omega_k(A), \delta_k{}^A)$, there exists a unique A-algebra homomorphism $\Psi_2' : \Omega_k(A) \to \Omega_B(B \otimes_k A)$ such that $\Psi_2' \circ \delta_k{}^A = \delta_B{}^{B \otimes_k A} \circ \alpha$. We may extend Ψ_2' in the usual way to a $(B \otimes_k A)$-algebra homomorphism $\Psi_2 : B \otimes_k \Omega_k(A) \to \Omega_B(B \otimes_k A)$. Specifically, $\Psi_2(\sum b_i \otimes x_i) = \sum b_i \Psi_2'(x_i)$. Here the b_i are in B, and the x_i are in $\Omega_k(A)$.

Thus, we have $(B \otimes_k A)$-algebra homomorphisms

$$\Psi_1 : \Omega_B(B \otimes_k A) \to B \otimes_k \Omega_k(A) \quad \text{and} \quad \Psi_2 : B \otimes_k \Omega_k(A) \to \Omega_B(B \otimes_k A).$$

It remains to show that these maps are inverses of each other. This is a somewhat laborious but straight forward computation and will be omitted.

8. An example. In this last section, we compute $\Omega_k(A)$ in the case that A is a polynomial ring.

Let $x_1, \ldots, x_n$ denote a set of indeterminates, and let $A = k[x_1, \ldots, x_n]$ be the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in k.

Let $\{u_{iq} | i = 1, \ldots, n, q = 1, \ldots, \infty\}$ be a second set of indeterminates over A. Let $A\langle u_{iq}\rangle$ denote the ring of polynomials in the u_{iq} with coefficients in A but without constant term.

We may define a k-higher derivation $\delta = \{\delta_q\}$ of A into $A\langle u_{iq}\rangle$ by defining

$$\delta_q(x_i) = u_{iq}, \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad q = 1, \ldots, \infty$$

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and then extending by Leibniz's rule [**3**, Proposition 2]. Clearly $A\langle u_{iq}\rangle$ is generated as an A-algebra by the set $\{\delta_q(x_i)|i = 1, \ldots, n, q \geqq 1\}$.

Suppose $\lambda = \{\lambda_q\} \in \mathscr{H}_k(A, V)$ where V is an A-algebra. We can define an A-algebra homomorphism $\Psi : A\langle u_{iq}\rangle \to V$ by setting $\Psi(u_{iq}) = \lambda_q(x_i)$. Then Ψ is the unique A-algebra homomorphism which satisfies $\Psi \circ \delta = \lambda$. Hence, $\Omega_k(k[x_1, \ldots, x_n]) = k[x_1, \ldots, x_u]\langle u_{iq}\rangle$.

Added in proof. It has recently come to my attention that P. Ribenboim in *Higher derivations of rings. I*, Rev. Roumaine Math. Pures Appl. 24 (1971), 77–110, has also constructed a universal object using different techniques than appear here.

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