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CHARACTERIZATION OF UPPER SEMICONTINUOUSLY INTEGRABLE FUNCTIONS

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Abstract

We show that for a Henstock-Kurzweil integrable function f for every $\epsilon > 0$ one can choose an upper semicontinuous gage function δ, used in the definition of the HK-integral if and only if $|f|$ is bounded by a Baire 1 function. This answers a question raised by C. E. Weil.

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1. Introduction

It is known that if f is a Henstock-Kurzweil integrable function then the gage function δ, appearing in the definition of the HK-integral, can be chosen to be nearly upper semicontinuous, that is, δ equals an upper semicontinuous function almost everywhere. This result was obtained for $\mathbb{R}^1$ in Pfeffer [3] and for $\mathbb{R}^m$ in Buczolich [2]. C. E. Weil asked the author whether it is possible to find a characterization of those HK-integrable functions $f : \mathbb{R}^m \to \mathbb{R}$ for which the HK-integral can be defined by using upper semicontinuous gage functions.

In Pfeffer [3] it was shown that if a function is bounded and Lebesgue integrable then δ can be selected so that it is upper semicontinuous, that is, f is upper semicontinuously integrable. Thus boundedness seems to play an important role in characterizations of upper semicontinuously integrable functions. Indeed, in our theorem we verify that f is upper semicontinuously integrable if and only if there exists a Baire 1 function g such that $g > |f|$. This property is also equivalent to the fact that any non-empty closed set has a portion on which f is bounded. The proof of the one-dimensional case is somewhat simpler than the higher dimensional one. The higher dimensional version is related to some interesting combinatorial problems, namely to the chromatic number of maps consisting of certain non-overlapping interval systems in $\mathbb{R}^m$. The

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details of this combinatorial question can be found in the statement of Lemma A.

2. Preliminaries

By $\mathbb{R}^m$ we denote m-dimensional Euclidean space. Given a set $A \subset \mathbb{R}^m$ we denote by $\mathrm{cl}(A)$, $\mathrm{int}A$, ∂A and $|A|$ the closure, the interior, the boundary and the Lebesgue measure of A. If $X \subset Y \subset \mathbb{R}^m$ then we denote the boundary and the interior of X with respect to the subspace topology of Y by $\partial_Y A$ and $\mathrm{int}_Y A$ respectively. The open ball of radius r centered at $x \in \mathbb{R}^m$ is denoted by $B(x, r)$. (In this paper we use the Euclidean metric; some papers use different but equivalent metrics in $\mathbb{R}^m$. The integral defined via any of these metrics is the same.) An m-dimensional interval is a set of the form $[a_1, b_1] \times \cdots \times [a_m, b_m]$. A collection $P = \{(A_i, x_i) : i = 1, \ldots, p\}$ is a subpartition of the interval A if the intervals $A_i \subset A$ are non-overlapping and $x_i \in A_i$. The subpartition P is a partition when $\bigcup_{i=1}^{p} A_i = A$. Given a positive function $\delta : A \to (0, +\infty)$ and a subpartition $P = \{(A_i, x_i) : i = 1, \ldots, p\}$ of A we say that P is δ-*fine* when $A_i \subset B(x_i, \delta(x_i))$.

If $f : A \to \mathbb{R}$ and P is a subpartition of A we put

$$\sigma(f, P) = \sum_{i=1}^{p} f(x_i)|A_i|.$$

DEFINITION. Given an interval $A \subset \mathbb{R}^m$ a function $f : A \to \mathbb{R}$ is HK-*integrable* and its HK-integral, $(\mathrm{HK})\int_A f$, equals $I \in \mathbb{R}$ when for every $\epsilon > 0$ there exists a function $\delta : A \to (0, +\infty)$ such that $|\sigma(f, P) - I| < \epsilon$ holds for any δ-fine partition P of A. In this paper we shall write $\int_A f$ instead of $(\mathrm{HK})\int_A f$.

If $f : A \to \mathbb{R}$ is HK-integrable and $\epsilon > 0$ we denote by $\Delta(f, A, \epsilon)$ the set of those gage functions δ for which $|\sigma(f, P) - I| < \epsilon$ holds for any δ-fine partition P of A.

LEMMA (HENSTOCK). *Assume that $A \subset \mathbb{R}^m$ is an interval, f is* HK-*integrable on A, and $\epsilon > 0$ is given. Then there is a gage function $\delta : A \to (0, \infty)$ such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - \int_{A_i} f \right| < \epsilon \tag{1}$$

for all δ-fine subpartitions $P = \{(A_i, x_i) : i = 1, \ldots, p\}$ of A.

For the one-dimensional proofs see Pfeffer [4, Lemma 2.5], or [5, Lemma 3.12]. These one-dimensional proofs can easily be generalized also to the m-dimensional case.

Given $f : A \to \mathbb{R}$, an HK-integrable function, and $\epsilon > 0$, we denote by $\Delta_H(f, A, \epsilon)$ the set of those gage functions δ for which (1) holds for all δ-fine subpartitions of A.

The proof of the Henstock Lemma shows that $\Delta(f, A, \epsilon/3) \subset \Delta_H(f, A, \epsilon)$ (see for example Pfeffer [4]). Therefore the results of Pfeffer [3] and Buczolich [2] imply that if f is HK-integrable on the interval A then for any $\epsilon > 0$ there exists a nearly upper semicontinuous $\delta \in \Delta_H(f, A, \epsilon)$. It follows from the Henstock Lemma that for $m = 1$ the indefinite HK-integral $\int_a^x f$ is a continuous function see for example Pfeffer [5, Proposition 3.13]. It is not difficult to generalize the one-dimensional case to verify that if f is HK-integrable on A_0, $A = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset A_0$, and $A(h_1, \ldots, h_{2m}) = [a_1 + h_1, b_1 + h_2] \times \cdots \times [a_m + h_{2m-1}, a_m + h_{2m}] \subset A_0$ then

$$F(h_1, \ldots, h_{2m}) = \int_{A(h_1, \ldots, h_{2m})} f$$

is a continuous function of $2m$ variables. We shall refer to this property as the continuity of the indefinite HK-integral. (In the literature a more general property used to be called the continuity of an interval function [5] but for this paper the above simple property is sufficient.)

3. Main result

This section is organized as follows. First we state our theorem which is followed by the statement of Lemma A. Then we prove the theorem by using Lemma A. Finally we present a proof of Lemma A.

THEOREM 1. *Suppose that $f : A \to \mathbb{R}$ is* HK-*integrable. Then the following three statements are equivalent.*

(i) *For every $\epsilon > 0$ there exists an upper semicontinuous $\delta \in \Delta(f, A, \epsilon)$.*
(ii) *For every non-empty closed subset $Q \subset A$ there exists an interval I such that $\mathrm{int}_A(I) \cap Q \neq \emptyset$ and $f|_{\mathrm{int}_A(I) \cap Q}$ is bounded.*
(iii) *There exists a Baire 1 function $g : A \to \mathbb{R}$ such that $g(x) > |f(x)|$ for all $x \in A$.*

LEMMA A. *Assume that f is* HK-*integrable on $A \subset \mathbb{R}^m$, $\epsilon > 0$, $\delta \in \Delta_H(f, A, \epsilon)$, $P = \{(A_i, x_i) : i = 1, \ldots, p\}$ is a subpartition of A, and for each x_i there exists a constant K_i and a sequence $x_{i,n} \to x_i$ such that $B(x_{i,n}, \delta(x_{i,n})) \supset A_i$ and $|f(x_{i,n})| < K_i$ for $n = 1, 2, \ldots$ and $i = 1, \ldots, p$. Then there exists a constant $C \geq 1$ depending only on the dimension m such that for each $i = 1, \ldots, p$ there exists an $x_i' \in \{x_{i,1}, x_{i,2}, \ldots\}$ for which*

$$\sum_{i=1}^{p} \left| f(x_i')|A_i| - \int_{A_i} f \right| < C\epsilon.$$

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PROOF OF THE THEOREM. The proof consists of the steps: (i) implies (ii); (iii) implies (ii); (ii) implies (iii); and finally the dificult implication (ii) implies (i).

(i) implies (ii). In fact we verify that the negation of (ii) implies the negation of (i). Suppose that there exists a non-empty closed set $Q \subset A$ such that for every K the set $\{x : |f(x)| > K\}$ is dense in Q. If $x_0 \in Q$ is an isolated point of Q then there exists an interval I such that $Q \cap \mathrm{int}_A(I) = \{x_0\}$ and then $f|_{\mathrm{int}_A(I)\cap Q}$ is bounded by $|f(x_0)|$. Therefore Q is perfect. Proceeding towards a contradiction, suppose that $\delta \in \Delta(f, A, 1)$ is upper semicontinuous. Since $\delta > 0$ by Baire's theorem there exists $d > 0$ and an interval I_0 such that $\mathrm{int}_A(I_0) \cap Q \neq \emptyset$ and $\{x \in Q : \delta(x) > d\}$ is dense in $\mathrm{int}_A(I_0) \cap Q$. We can also assume that d is less than the shortest side of A. Since δ is upper semicontinuous we have $\delta(x) \geq d$ for all $x \in \mathrm{int}_A(I_0) \cap Q$. Applying Baire's theorem again we can find $K_1 > 0$ and an interval $I_1 \subset I_0$ such that $\mathrm{int}_A(I_1) \cap Q \neq \emptyset$ and $\{x : |f(x)| < K_1\}$ is dense in $\mathrm{int}_A(I_1) \cap Q$. Let $K_2 = 2(2\sqrt{m}/d)^m + K_1$. By our assumption $\{x : |f(x)| > K\}$ is dense in Q for any K. Choose a point $x_0 \in \mathrm{int}_A(I_1)\cap Q$ such that $|f(x_0)| > K_2$, and $\delta(x_0) > d$. We also choose a square $B \subset A$ such that $x_0 \in \mathrm{int}_A(B)$, $d/2 < \mathrm{diam}(B) < d$ (here we used the fact that d is sufficiently small, that is, d is less than the shortest side of A and hence the square B will fit into A). Denote by P_0 a fixed δ-fine partition of $\mathrm{cl}(A \setminus B)$. Let $P_1 = P_0 \cup \{(B, x_0)\}$. It is obvious that P_1 is a δ-fine partition of A. Since $\{x : |f(x)| < K_1\}$ is dense in $\mathrm{int}_A(I_1) \cap Q$ there exists an $x_1 \in \mathrm{int}_A(I_1) \cap Q \subset \mathrm{int}_A(I_0) \cap Q$ such that $|f(x_1)| < K_1$ and $x_1 \in B$. Then $P_2 = P_0 \cup \{(B, x_1)\}$ is also a δ-fine partition of A. Thus

$$|\sigma(f, P_1) - \sigma(f, P_2)| = |f(x_0) - f(x_1)| \cdot |B| > \left|2\left(2\sqrt{m}/d\right)^m + K_1 - K_1\right|\left(d/2\sqrt{m}\right)^m = 2.$$

This contradicts the fact that δ was chosen for $\epsilon = 1$.

(iii) implies (ii). If Q has an isolated point then (ii) is valid. If Q is non-empty and perfect then it is well-known that for any Baire 1 function, g, there exists a point $x_0 \in Q$ such that $g|_Q$ is continuous at x_0. Therefore one can easily find an interval I such that $x_0 \in \mathrm{int}_A(I)$ and $g|_{\mathrm{int}_A(I)\cap Q}$ is bounded. Then $f|_{\mathrm{int}_A(I)\cap Q}$ is also bounded.

(ii) implies (iii). Put $Q_0 = A$. Assume that r is not a limit ordinal and $Q_{r-1} \neq \emptyset$ is a given closed set. By using (ii) choose an interval I_{r-1} such that $\mathrm{int}_A(I_{r-1}) \cap Q_{r-1} \neq \emptyset$ and $f|_{\mathrm{int}_A(I_{r-1})\cap Q_{r-1}}$ is bounded. Put $Q_r = Q_{r-1} \setminus \mathrm{int}_A(I_{r-1})$. Since A is closed in $\mathbb{R}^m$ the set Q_r is also closed in $\mathbb{R}^m$, not only in the subspace topology of A. If r is a limit ordinal put $Q_r = \bigcap_{s<r} Q_s$. It is well-known that any well-ordered decreasing sequence of closed sets in $\mathbb{R}^m$ is of cardinality less than $\aleph_1$. It is also easy to see that the sets $\mathrm{int}_A(I_r) \cap Q_r$ are disjoint and $\bigcup_r(\mathrm{int}_A(I_r) \cap Q_r) = A$. Therefore the sets of the form $\mathrm{int}_A(I_r) \cap Q_r$ can be ordered into a sequence denoted by $H_1, H_2, \ldots, H_n, \ldots$ so that they are pairwise disjoint and $\bigcup_{n=1}^{\infty} H_n = A$. It is also clear that f is bounded on the sets H_n, and the sets H_n are of type F_σ. Choose integer constants K_n such that $K_1 < K_2 < \cdots < K_n < \cdots$ and $|f|$ is bounded on H_n by K_n. Assume that $H_n = \bigcup_{i=1}^{\infty} F_{n,i}$, where the sets $F_{n,i}$ are closed, and $F_{n,1} \subset F_{n,2} \subset \cdots$. Define the function g_j so that $g_j(x) = K_n$ if $x \in F_{n,j}$ and $n \leq j$. Since for a fixed j the closed sets $F_{n,j} \subset H_n$ are disjoint for $n = 1, 2, \ldots$ the function $g_j(x)$ is continuous on $\bigcup_{n=1}^{j} F_{n,j}$. Since $\mathbb{R}^m$ is a normal space by [1, Satz VIII] one can extend g_j to be defined and continuous on $\mathbb{R}^m$. If $x \in A$ then there exists exactly one $n(x)$ such that $x \in H_{n(x)}$ and for $j > j(x)$ we also have $x \in F_{n(x),j}$. Therefore $\lim_{j\to\infty} g_j(x) = K_n$. Thus the function defined by $g = \lim_{j\to\infty} g_j$ is Baire 1 and $g(x) = K_{n(x)} > |f(x)|$ if $x \in H_{n(x)}$.

(ii) implies (i). Assume that $\epsilon > 0$ is given and (ii) holds. Recall that $\Delta_H(f, A, \epsilon_0)$ contains a nearly upper semicontinuous, and hence measurable gage function for any $\epsilon_0 > 0$. Choose a measurable $\delta_H \in \Delta_H(f, A, \epsilon/4C)$, where C is the constant given in Lemma A. Put $W_0 = A$. If W_s is defined for ordinals $s < r < \aleph_1$ and r is a limit ordinal put $W_r = \bigcap_{s<r} W_s$.

If $r = s + 1$ and W_s contains an isolated point w_s put $E_s = \{w_s\} = T_s$, $\eta_s = \delta_H(w_s)/2$ and $W_r = W_s \setminus E_s$. Also choose an interval I_s such that $\mathrm{int}_A(I_s) \cap W_s = E_s$ and $\operatorname{diam} I_s < \eta_s$.

If $r = s + 1$ and W_s is non-empty and perfect then by using (ii) and Baire's theorem we can choose an interval I_s such that $E_s = \mathrm{int}_A(I_s) \cap W_s \neq \emptyset$, $|f|$ is bounded by K_s on E_s, and there exists an $\eta_s > 0$ such that

$$T_s = \{x \in E_s : \delta_H(x) > \eta_s\} \tag{2}$$

is dense in E_s. We can also assume that $\operatorname{diam} I_s < \eta_s$. Put $W_r = W_s \setminus E_s$.

Since a well-ordered strictly decreasing sequence of closed sets of $\mathbb{R}^m$ terminates at an ordinal number $\alpha < \aleph_1$ we can assume that $W_\alpha \neq \emptyset$ and $W_{\alpha+1} = \emptyset$. We remark that the sets $W_s \subset A$ are bounded and closed. Hence if $W_s \neq \emptyset$ for $s < r$ then $\bigcap_{s<r} W_s \neq \emptyset$ and therefore the sequence W_s cannot terminate at a limit ordinal. Clearly $E_\alpha = W_\alpha$. We shall denote by Λ the set of those ordinals which are less than $\alpha + 1$. It is clear that Λ is countable and hence we can find an injective function $j : \Lambda \to \mathbb{N}$.

If $s \in \Lambda$ and $|E_s| = 0$ then put $\kappa_s = \eta_s$, and choose an open set $U_s \supset E_s$ such that

$$|U_s| < \epsilon/(2 \cdot 2^{j(s)} K_s), \tag{3}$$

and $\partial I_s \subset U_s$.

If $s \in \Lambda$ and $|E_s| > 0$ then using the fact that δ_H is measurable and positive we can find $0 < \kappa_s \leq \eta_s$ such that letting $V_s = \{x \in E_s : \delta_H(x) \leq \kappa_s\}$ we have $|V_s| < \epsilon/(16 \cdot 2^{j(s)} K_s)$. The measurability of δ_H also implies the existence of a non-empty open set $U_s \supset V_s$ such that

$$|U_s| < \epsilon/(16 \cdot 2^{j(s)} K_s), \tag{4}$$

and we can also assume that U_s contains ∂I_s.

If $s \in \Lambda$ and $x \in U_s \cap E_s$ put

$$\delta(x) = \min\{\operatorname{dist}(x, \partial_A I_s),\ \operatorname{dist}(x, \mathbb{R}^m \setminus U_s), 1/j(s)\}.$$

It is obvious that

$$B(x, \delta(x)) \cap A \subset U_s \cap I_s \quad \text{for} \quad x \in U_s \cap E_s. \tag{5}$$

If $x \in E_s \setminus U_s$ put

$$\delta(x) = \min\{\operatorname{dist}(x, \partial_A I_s), 1/j(s), \kappa_s\}.$$

Now δ is defined for all $s \in \Lambda$ and it is clear that δ is positive. Observe that if $y \in \partial_A I_s$ then

$$\delta(x) \le \operatorname{dist}(x, \partial_A I_s) \le \operatorname{dist}(x, y). \tag{6}$$

Since W_s is closed and U_s contains the boundary of I_s the set $E_s \setminus U_s$ is closed. It is obvious that δ is upper semicontinuous on the set $E_s \setminus U_s$. It is also clear from its definition that δ is upper semicontinuous on the set $E_s \cap U_s$, and if $x \notin E_s \cap U_s$, $x_n \to x$, $x_n \in E_s \cap U_s$ $(n = 1, 2, \dots)$ then $\delta(x_n) \to 0$.

(α) The above properties imply that δ is upper semicontinuous on E_s.

(β) Assume that $x_n \to y$, and for any $s \in \Lambda$ we have $x_n \in E_s$ for only finitely many indices n. For any $\nu > 0$ there exists only finitely many indices $s \in \Lambda$ for which $1/j(s) \ge \nu$. Therefore from $\delta(x) \le 1/j(s)$ when $x \in E_s$, it follows that $\delta(x_n) \ge \nu$ can hold only for finitely many n's and hence $\delta(x_n) \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} \delta(x_n) = 0 < \delta(y)$.

(γ) If $x_n \in E_s = \operatorname{int}_A(I_s) \cap W_s$ for $n = 1, 2, \dots$, and $x_n \to y \notin E_s$ then using that W_s is closed we obtain $y \in \partial_A I_s$ and (6) implies that $\delta(x_n) \le \operatorname{dist}(x_n, y)$.

If $y \in A$ then there exists exactly one $r \in \Lambda$ for which $y \in E_r$. Assume that $x_n \to y$. For any natural number n there exists exactly one $s(n) \in \Lambda$ for which $x_n \in E_{s(n)}$. For $s \in \Lambda$ put $N_s = \{n : x_n \in E_s\}$. Put $M_1 = N_r$, $M_2 = \{n : s(n) \ne r \text{ and } N_{s(n)} \text{ is finite}\}$, and $M_3 = \mathbb{N} \setminus (M_1 \cap M_2) = \{n : s(n) \ne r \text{ and } N_{s(n)} \text{ is infinite}\}$. If M_2 is infinite then (β) is applicable to the subsequence $\{x_n : n \in M_2\}$. If $n' \in M_3$ then (γ) is applicable to the subsequence $\{x_n : n \in N_{s(n')}\}$ and we obtain $\delta(x_{n'}) \le \operatorname{dist}(x_{n'}, y)$. This implies $\lim_{n\to\infty,\ n\in M_3} \delta(x_n) = 0$. Thus by splitting $\{x_n\}$ into at most three subsequences and using (α), (β), and (γ) we obtain that $\limsup_{n\to\infty} \delta(x_n) \le \delta(y)$ proving that δ is upper semicontinuous on A.

Next we have to verify that $\delta \in \Delta(f, A, \epsilon)$. For an $x \in A$ denote by $r(x)$ the index in Λ for which $x \in E_{r(x)}$.

Assume that $\{(A_i, x_i) : i = 1, \dots, p\}$ is a δ-fine partition of A. Then

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - \int_{A_i} f \right| = \sum_{s\in\Lambda} \sum_{\{i : x_i \in E_s\}} \left| f(x_i)|A_i| - \int_{A_i} f \right| = \Psi_1$$

where the 'transfinite' sum $\sum_{s\in\Lambda}$ can be defined since there are only finitely many indices s for which there exists x_i such that $r(x_i) = s$. By definition, the empty sum and the transfinite sum of zeros equal zero.

Put

$$\Psi_2 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\cap U_s\}} \left| f(x_i)|A_i| - \int_{A_i} f \right|,$$

and

$$\Psi_3 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\setminus U_s\}} \left| f(x_i)|A_i| - \int_{A_i} f \right|.$$

Obviously $\Psi_1 = \Psi_2+\Psi_3$. If $x_i \in E_s \setminus U_s$ then $|E_s| > 0$ and $x_i \notin V_s$; therefore $\delta_H(x_i) > \kappa_s \geq \delta(x_i)$. Hence the partition in A defined by $\{(A_i, x_i) : \exists s \in \Lambda$ such that $x_i \in E_s \setminus U_s\}$ is δ_H-fine. Thus $\Psi_3 < \epsilon/4C < \epsilon/2$ where we used the fact that $\delta_H \in \Delta_H(f, A, \epsilon/4C)$ and $C \geq 1$.

To estimate Ψ_2 assume in this paragraph that an $s \in \Lambda$ is given. The set T_s, introduced during the definition of E_s, is dense in E_s. By (5) for any $x \in E_s \cap U_s$, we have $B(x, \delta(x)) \cap A \subset I_s$. Since diam $I_s < \eta_s$ by using (2) we have $B(x, \delta_H(x)) \supset I_s \supset B(y, \delta(y)) \cap A$ for any $x \in T_s$, $y \in E_s \cap U_s$. Thus for any $x_i \in E_s \cap U_s$ there exists a sequence $x_{i,n} \to x_i$ such that $x_{i,n} \in T_s$ and hence $B(x_{i,n}, \delta_H(x_{i,n})) \supset I_s \supset B(x_i, \delta(x_i)) \cap A \supset A_i$, for $n = 1, 2, \dots$ and $|f(x_{i,n})| < K_s$.

The above argument is valid for any $s \in \Lambda$. Observe that Lemma A is applicable with $f = f$, $A = A$, $\epsilon = \epsilon/4C$, $\delta = \delta_H$, $P = \{(A_i, x_i) : i \in \{1, \dots, p\}, \exists s \in \Lambda,\ x_i \in E_s \cap U_s\}$, and for $x_i \in E_s \cap U_s$ choose $K_i = K_s$ and $x_{i,n} = x_{i,n}$ ($n = 1, 2, \dots$). We obtain that for any i for which $x_i \in E_s \cap U_s$ there exists an $x_i' \in T_s$ such that

$$\Psi_4 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\cap U_s\}} \left| f(x_i')|A_i| - \int_{A_i} f \right| < \epsilon/4. \tag{7}$$

Recall that $|f|$ is bounded by K_s on E_s. For $x_i \in E_s \cap U_s$, by (5) we have $A_i \subset B(x_i, \delta(x_i)) \cap A \subset U_s$. Therefore using (4) we obtain

$$\sum_{\{i:x_i\in E_s\cap U_s\}} |f(x_i)||A_i| < K_s \cdot \sum_{\{i:x_i\in E_s\cap U_s\}} |A_i| < K_s|U_s| < K_s \frac{\epsilon}{16\cdot 2^{j(s)}K_s} = \frac{\epsilon}{16\cdot 2^{j(s)}}.$$

Thus we have

$$\Psi_5 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\cap U_s\}} |f(x_i)|\cdot|A_i| < \sum_{j=1}^{\infty} \frac{\epsilon}{16\cdot 2^j} = \epsilon/16. \tag{8}$$

Since $x_i' \in T_s \subset E_s$ implies $|f(x_i')| < K_s$ one can obtain similarly to the previous estimation

$$\Psi_6 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\cap U_s\}} |f(x_i')|\cdot|A_i| < \epsilon/16. \tag{9}$$

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From (7) and (9) we obtain that

$$\Psi_7 = \sum_{s\in\Lambda} \sum_{\{i:x_i\in E_s\cap U_s\}} \left|\int_{A_i} f\right| < \frac{\epsilon}{4} + \Psi_6 < \frac{\epsilon}{4} + \frac{\epsilon}{16}. \tag{10}$$

Finally using (8) and (10) we infer

$$\Psi_2 < \Psi_5 + \Psi_7 < \frac{\epsilon}{16} + \frac{\epsilon}{4} + \frac{\epsilon}{16} < \frac{\epsilon}{2}.$$

Therefore $\Psi_1 = \Psi_2 + \Psi_3 < \epsilon/2 + \epsilon/2 = \epsilon$. This proves $\delta \in \Delta_H(f, A, \epsilon) \subset \Delta(f, A, \epsilon)$ and concludes the proof of the theorem.

PROOF OF LEMMA A. First we prove the one-dimensional case. If we order the intervals A_i according to the ordering of the real line, then taking every second interval A_i the set $\{1, \dots, p\}$ can be split into two subsets, N_1 and N_2 such that $N_1 \cup N_2 = \{1, \dots, p\}$, $N_1 \cap N_2 = \emptyset$ and the (closed) intervals $\{A_i : i \in N_j\}$ are pairwise disjoint for $j = 1, 2$.

If $x_i \in \mathrm{int}_A(A_i)$ then using the assumption of Lemma A one can choose an $x_{i,n} \in \mathrm{int}_A(A_i)$. Put $x_i' = x_{i,n}$, and $A_i' = A_i$.

If x_i is one of the endpoints of A_i then using the fact that the indefinite HK-integral is a continuous function by enlarging slightly the intervals A_i one can obtain the intervals A_i' such that

(i) $x_i \in \mathrm{int}_A(A_i')$ and hence there exists an $n(i)$ such that $x_i' = x_{i,n(i)}$ belongs to A_i' and $A_i' \subset B(x_i', \delta(x_i'))$,

(ii)
$$\left|\int_{A_i} f - \int_{A_i'} f\right| < \frac{\epsilon}{2p},$$

(iii)
$$K_i \left| |A_i'| - |A_i| \right| < \frac{\epsilon}{2p},$$

(iv) the intervals $\{A_i' : i \in N_j\}$ are non-overlapping for $j = 1, 2$.

By (i) we have $A_i' \subset B(x_i', \delta(x_i'))$. This and (iv) implies that the subpartitions $\{(A_i', x_i') : i \in N_j\}$ are δ-fine in A for $j = 1, 2$. Thus

$$\sum_{i\in N_j} \left| f(x_i')|A_i'| - \int_{A_i'} f\right| < \epsilon$$

holds for $j = 1, 2$. Therefore

$$\sum_{i=1}^{p} \left| f(x_i')|A_i'| - \int_{A_i'} f \right| < 2\epsilon.$$

Finally

$$\begin{aligned}
&\sum_{i=1}^{p} \left| f(x_i')|A_i| - \int_{A_i} f \right| \\
&\quad \leq \sum_{i=1}^{p} \left| f(x_i')|A_i| - \int_{A_i} f - f(x_i')|A_i'| + \int_{A_i'} f \right| + \sum_{i=1}^{p} \left| f(x_i')|A_i'| - \int_{A_i'} f \right| \\
&\quad \leq \sum_{i=1}^{p} |f(x_i')| \cdot \Big| |A_i'| - |A_i| \Big| + \sum_{i=1}^{p} \left| \int_{A_i} f - \int_{A_i'} f \right| + 2\epsilon < 3\epsilon
\end{aligned}$$

where at the last inequality we used (ii) and (iii). Thus when $m = 1$ we can choose $C = 3$ and this concludes the proof of the one-dimensional case.

To generalize the proof of the one-dimensional case the following statement would be useful.

STATEMENT. There exists a constant K depending only on the dimension such that given any finite collection $\{A_i : i = 1, \dots, p\}$ of non-overlapping intervals in $\mathbb{R}^m$, they can be colored by K colors, that is, there exists a function $h : \{1, \dots, p\} \to \{1, \dots, K\}$, such that intervals of the same color, that is, $\{A_i : h(i) = j\}$, are disjoint for all $j \in \{1, \dots, K\}$.

For $m = 1, 2$ the statement is true, although the best possible value of K for $m = 2$ seems to be unknown. When $m \geq 3$ then it is not known whether the statement is true or not.

If $\rho \geq 1$ is fixed and we are looking at ρ-regular coverings, that is, the ratio of the longest and shortest side of each A_i is bounded by ρ, then the statement is true. As B. Kirchheim pointed out to the author the ideas used in the proof of Ziemer [6, 1.3.5. Theorem, p. 9-12] can be generalized to this case.

Fortunately to prove Lemma A it is not necessary to prove the statement. We shall show that we can shrink the intervals A_i slightly to obtain the intervals A_i'' for which the statement and Lemma A holds. Using the continuity of the HK-integral for $i = 1, \dots p$ choose $0 < \gamma_i < 1$ such that if we denote by A_i'' the image of A_i under the affine transformation $T_i(x) = x_i + \gamma_i(x - x_i)$ then

$$K_i \Big| |A_i| - |A_i''| \Big| < \epsilon/2p, \quad \text{and} \quad \left| \int_{A_i} f - \int_{A_i''} f \right| < \epsilon/2p$$

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holds for $i = 1, \ldots, p$. It is clear that if there exists C_0 depending only on m and $x_i' \in \{x_{i,1}, x_{i,2}, \ldots\}$ such that

$$\sum_{i=1}^{p} \left| f(x_i')|A_i''| - \int_{A_i''} f \right| < C_0\epsilon \tag{11}$$

then

$$\sum_{i=1}^{p} \left| f(x_i')|A_i| - \int_{A_i} f \right| < C_0\epsilon + \sum_{i=1}^{p} K_i \cdot \left| |A_i| - |A_i''| \right| + \sum_{i=1}^{p} \left| \int_{A_i} f - \int_{A_i''} f \right| < (C_0+1)\epsilon$$

and Lemma A holds with $C = C_0 + 1$.

For ease of notation we work out the two-dimensional case; the higher dimensional ones are similar. Divide the index set $\{1, \ldots, p\}$ into 9 disjoint subsets $N_1, N_2, \ldots, N_9$. If $x_i \in \mathrm{int}_A(A_i)$ then let $i \in N_1$. If x_i is not a vertex of A_i and x_i is on the lower side of A_i then let i be in N_2. The index sets N_3, N_4, and N_5 are defined like N_2 by using the right, upper, and left sides of A_i in the definition respectively. If x_i is the lower left (lower right, upper right, upper left) vertex of A_i then i is in N_6, (N_7, N_8, N_9) respectively.

Since the intervals A_i are non-overlapping it is obvious that $\{(A_i'', x_i) : i \in N_1\}$ consists of disjoint intervals, and a moment's reflection shows that each of the sub-subpartitions $\{(A_i'', x_i) : i \in N_j\}$ also consists of disjoint intervals when $j = 2, \ldots, 9$. Thus with $K = 9$ the conclusion of the statement holds for the intervals $\{A_i'' : i = 1, \ldots, p\}$.

Assume now that $j \in \{1, 2, \ldots, 9\}$ is fixed. Since the intervals $\{A_i'' : i \in N_j\}$ are disjoint one can enlarge the intervals A_i'' slightly and obtain the intervals A_i' such that $x_i \in \mathrm{int}(A_i')$ and properties (i)-(iv) at the beginning of the proof of the one-dimensional case of Lemma A hold with A_i'' instead of A_i and for $j = 1, \ldots, 9$. Replacing A_i by A_i'' in the rest of the computations used for the one-dimensional case one can obtain that

$$\sum_{i=1}^{p} \left| f(x_i')|A_i''| - \int_{A_i''} f \right| < 10\epsilon$$

and hence (11) holds with $C_0 = 10$. This concludes the proof for $m = 2$. When $m \geq 3$ then a similar separation of cases according to the location of x_i on the boundary of A_i can be used. This concludes the proof of Lemma A.

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