The system of cubic curves circumscribing two triangles and apolar to them.

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1. It is very seldom that any properties of a pencil of cubics can be discovered beyond some very general ones. The above case yields, however, some interesting results. It presented itself to me when I was considering the apolar generation of cubic curves which was published in the *Proceedings of the London Mathematical Society* (Ser. 2, Vol. 9, Part 3), and which I shall have frequent occasion to refer to in the present paper.

2. The following definitions will be required :—
The cubic

$$ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + c_2z^2y + 6kxyz = 0 \quad (1)$$

is said to be apolar to the class-cubic

$$Al^3 + Bm^3 + Cn^3 + 3A_2l^2m + 3A_3l^2n + 3B_1m^2l + 3B_3m^2n + 3C_1n^2l + 3C_2n^2m + 6Klmn = 0 \quad (2)$$

if

$$aA + bB + cC + 3a_2A_2 + 3a_3A_3 + 3b_1B_1 + 3b_3B_3 + 3c_1C_1 + 3c_2C_2 + 6kK = 0. \quad (3)$$

The class-cubic may degenerate into three points. If these be taken as the vertices of the triangle of reference, their equation is $lmn = 0$, and the condition that the cubic (1) be apolar to ABC is evidently

$$k = 0. \quad (4)$$

The geometrical definition of the vanishing of k is that the polar conic of any one vertex has the other two vertices as conjugate points.

Two binary cubics

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0 \quad \textit{and} \quad a'x^3 + 3b'x^2y + 3c'xy^2 + d'y^3 = 0$$

are said to be apolar if their invariant $ad' - 3bc' + 3cb' - da'$ *vanishes.*

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3. *Every cubic circumscribing the two triangles ABC and DEF and apolar to them is apolar to all the class-cubics touching the nine lines joining A, B, C and D, E, F and also to their Hessians.*

The circumscribing system of cubics form a one-parameter family, since a cubic can be made to satisfy *nine* independent linear conditions, and to make a cubic pass through A, B, C, D, E, F, and be apolar to the class-cubics consisting of the vertices of the triangles ABC and DEF, is tantamount to *eight* linear conditions.

Let $\Sigma_1 \equiv$ A, B, C in tangential coordinates.

Let $\Sigma_2 \equiv$ D, E, F in tangential coordinates.

Let S be a cubic of the circumscribing system.

I. Then since S is apolar to Σ_1 and also to Σ_2 (by hypothesis).

$\therefore$ S is apolar to all the class-cubics of the system $\Sigma_1 + \lambda\Sigma_2$, since the coefficients of the cubic and class-cubic are linearly involved in (3), which proves the first part of the proposition.

II. To prove the second part, take ABC as triangle of reference.

Let
$$D \equiv (x_1, y_1, z_1)$$
$$E \equiv (x_2, y_2, z_2)$$
$$F \equiv (x_3, y_3, z_3)$$

Then the tangential equation to D, E, F is

$$(lx_1 + my_1 + nz_1)(lx_2 + my_2 + nz_2)(lx_3 + my_3 + nz_3) = 0 \qquad (5)$$

which may be taken in the form (2) where

$$A \equiv x_1x_2x_3,\ B \equiv y_1y_2y_3,\ C \equiv z_1z_2z_3,\ 3A_2 \equiv \Sigma y_1x_2x_3,\ 3A_3 \equiv \Sigma z_1x_2x_3,$$
$$3B_1 \equiv \Sigma x_1y_2y_3,\ 3B_3 \equiv \Sigma z_1y_2y_3,\ 3C_1 \equiv \Sigma x_1z_2z_3,\ 3C_2 \equiv \Sigma y_1z_2z_3,$$
$$6K \equiv \Sigma x_1y_2z_3. \qquad (6)$$

Now, quoting from Salmon's Higher Plane Curves, the equation to the Hessian of the class-cubic $\Sigma_2 + \lambda\Sigma_1 = 0$ where $\Sigma_1 \equiv 6lmn$ is as follows:—

$$\{AB_1C_1 - A(K+\lambda)^2 + 2A_2A_3(K+\lambda) - B_1A_3^2 - C_1A_2^2\}l^3 + \text{etc.}$$
$$+ \{ABC_1 - 2AB_3(K+\lambda) + AB_1C_2 - BA_3^2 + A_2(K+\lambda)^2$$
$$- B_1C_1A_2 + 2A_2A_3B_3 - C_2A_2^2\}l^2m + \text{etc.}$$
$$+ \{ABC - (AB_3C_2 + BC_1A_3 + CA_2B_1) + 2(K+\lambda)^3$$
$$- 2(K+\lambda)(B_1C_1 + C_2A_2 + A_3B_3) + 3(A_2B_3C_1 + A_3B_1C_2)\}lmn = 0. \qquad (7)$$

Now the condition that

$$S \equiv a_2x^2y + a_3x^2z + b_1y^2x + b_3y^2z + c_1z^2x + c_2z^2y = 0 \qquad (8)$$

(a, b, c and k vanishing, $\because$ S is apolar to ABC and circumscribes it) be apolar to the Hessian (6) is by (3) of Art. 2

$$\Sigma a_2\{ABC_1 - 2AB_3(K+\lambda) + AB_1C_2 - BA_3^2 + A_2(K+\lambda)^2 \\ - B_1C_1A_2 + 2A_2A_3B_3 - C_2A_2^2\} = 0 \quad (9)$$

We wish to show that (9) vanishes for all values of λ.

Now we know that (9) vanishes when $\lambda = 0$, because the Hessian of three points is the three points themselves, and since S is given to be apolar to D, E, F, it is therefore apolar to the Hessian of D, E, F.

Also $\Sigma a_2A_2 = 0$. $\because$ S is apolar to D, E, F. (10)

We have therefore still to prove that $\Sigma a_2AB_3 = 0$. (11)

To prove (11) it will be sufficient to show that *any* two of the cubics of the circumscribing system satisfy (11).

Now the apolar locus of ABC and DEF, *i.e.* the locus of points P such that P(A, B, C) is apolar to P(D, E, F) is one of the circumscribing system, and its equation is in the notation of (6)

$$C_2x^2y + A_3y^2z + B_1z^2x - C_1xy^2 - B_3zx^2 - A_2yz^2 = 0. \quad (12)$$

This evidently satisfies (11).

Furthermore, the curve

$$\Sigma x_1x_2x_3(yz_1 - zy_1)(yz_2 - zy_2)(yz_3 - zy_3) = 0$$

can easily be seen to belong to the circumscribing system, and its equation may be written in accordance with (6)

$$CB_1x^2y + AC_2y^2z + BA_3z^2x - CA_2xy^2 - AB_3yz^2 - BC_1zx^2 = 0 \quad (13)$$

This also satisfies (11).

The proposition enunciated is thus established.

4. The theorem of last article leads at once to the following:—Since the common tangents of a class-cubic and its Hessian are the cuspidal tangents of both curves, and since the nine cuspidal tangents thus defined pass three by three through twelve points forming the vertices of the four cuspidal triangles, and since the vertices of a cuspidal triangle have as their tangential equation an equation of the form

$$\Sigma + \lambda H = 0$$

where Σ is the class-cubic and H its Hessian, we see that *all the cuspidal triangles of the system of class-cubics touching the nine lines joining A, B, C and D, E, F are apolar triangles with respect to the system of cubics circumscribing ABC and DEF and apolar to them.*

5. *To find the three points U, V, W, other than A, B, C and D, E, F, in which the curves of the circumscribing system intersect one another.*

Of all the cubics passing through the intersection of two given cubics, twelve are nodal cubics.

In the above circumscribing system, let us choose a nodal cubic and find where U, V, W are situated on it.

Let the nodal cubic be

$$x^3+y^3+6xyz=0,$$

i.e. $\qquad x=6t^2 \qquad y=6t \qquad z=-(1+t^3). \qquad (14)$

Then the triangle whose vertices are given by

$$t^3+3pt^2+3qt+r=0$$

will be an apolar triangle if

$$9t_1t_2t_3=(t_1+t_2+t_3)(t_2t_3+t_3t_1+t_1t_2),$$

i.e. if $\qquad r=pq.$

The geometrical interpretation of this result is easily found to be that the tangents at the Hessian points of the triad

$$t^3+3pt^2+3qt+pq=0$$

intersect on the curve.

Let the inscribed apolar triangles ABC and DEF be given by

$$t^3+3pt^2+3qt+pq=0 \qquad (15)$$

and

$$t^3+3rt^2+3st+rs=0 \qquad (16)$$

Now, we have said that the apolar locus of ABC and DEF (*i.e.* the locus of the points P such that P(ABC) is apolar to P(DEF)) belongs to the above circumscribing system of cubics. Hence to find U, V, W we have to find the intersection of the apolar locus with the nodal cubic (14).

This is most easily done by finding the three points t on the nodal cubic (14), which subtend apolar pencils at ABC and DEF.

The line joining the points θ and t on the nodal cubic (14) cuts $z=0$ in the point given by

$$\theta(\theta y-x)t^2-(\theta^2x+y)t-(\theta y-x)=0$$

which may be written

$$\theta t^2-\mu t-1=0 \qquad (17)$$

where $\qquad \mu=\dfrac{\theta^2x+y}{\theta y-x}$

On eliminating t between (15) and (17) we shall get a cubic in μ. Similarly on eliminating t between (16) and (17) we shall get another cubic in μ. We wish to find the condition that the two cubics in μ thus found be apolar.

Eliminating t between (15) and (17), and between (16) and (17), we get

$$pq\mu^3+3q(p^2\theta-1)\mu^2+3p(q\theta-1)^2\mu+\{p^2\theta(q\theta+3)^2-(3q\theta+1)^2\}=0 \quad (18)$$

and

$$rs\mu^3+3s(r^2\theta-1)\mu^2+3r(s\theta-1)^2\mu+\{r^2\theta(s\theta+3)^2-(3s\theta+1)^2\}=0 \quad (19)$$

(18) and (19) will be apolar according to the definition in Art. 2 if

$$(pq-3qr+3ps-rs)(pqrst^3+3qst^2+3prt+1)=0.$$

The circumstances under which $pq-3qr+3ps-rs$ vanishes have been discussed in the paper in the London Mathematical Society referred to in Art. 1. We shall, therefore, suppose here that $pq-3qr+3ps-rs$ does not vanish.

We are left with the three points given by

$$pqrst^3+3qst^2+3prt+1=0. \quad (20)$$

6. *The three points U, V, W form an inscribed apolar triangle with respect to every cubic of the above circumscribing system.*

This is plain from Art. 5 when we consider the coefficients of the equation (20). Being true for *one* nodal cubic of the system, it is true for *all* the nodal cubics, and hence for the whole system.

7. *If the tangents at the Hessian-Points of ABC, DEF, and UVW, intersect on the curve at X, Y, Z respectively, then XYZ are collinear.*

For the Hessian-Points of

$$\text{A, B, C} \equiv t^3+pt^2+3qt+pq=0$$

are given by

$$t^2-q=0. \quad (21)$$

The point X, at which intersect the tangents to (21), has therefore as its parameter $-\dfrac{1}{q}$.

We thus get

$$\text{X} \equiv -\frac{1}{q}$$

$$\text{Y} \equiv -\frac{1}{s}$$

$$\text{Z} \equiv -qs.$$

Now the condition that the points t_1, t_2, t_3 be collinear on a nodal cubic is known to be $t_1t_2t_3 = -1$.

Hence plainly X, Y, Z are collinear.

8. The pencil of lines joining the node to (15) is evidently from (14)

$$x^3 + 3pxy^2 + 3qxy^2 + pqy^3 = 0 \tag{22}$$

The second polars of the nodal tangents (*i.e.* $x=0$ and $y=0$) with respect to the pencil (22) are therefore the same line, viz., $x+py=0$.

Let this line meet the curve again in L.

Hence $\qquad \mathrm{L} \equiv -p \qquad$ parametrically.

Similarly $\qquad \mathrm{M} \equiv -r$

and $\qquad \mathrm{N} \equiv -\dfrac{1}{pr}$

Hence L, M, N are collinear.

9. It is therefore plain that ABC, DEF, UVW form a perfectly symmetrical system of triangles.

10. It is a known theorem that in a system of cubics passing through the nine intersections of two given cubics, twelve cubics of the system are nodal, and the polar line of each node is the same with respect to all the cubics of the system. *We wish to identify the nodal polars in the above system.*

Let any cubic passing through the nine points A, B, C, D, E, F, U, V, W (other than the given nodal cubic) be

$$ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y + 6kxyz = 0 \tag{23}$$

If we substitute from (14) in (23), we shall find that (23) cuts the nodal cubic in points given by

$$216at^6 + 216bt^3 - c(1+t^3)^3 + 648a_2t^5 - 108a_3t^4(1+t^3) + 648b_1t^4 - 108b_3t^2(1+t^3) + 18c_1(1+t^3)^2t^2 + 18c_2(1+t^3)^2t - 216k(1+t^3)t^3 = 0. \tag{24}$$

But we know that the cubic (23) is going to cut the nodal cubic in points given by

$$(t^3+3pt^2+3qt+pq)(t^3+3rt^2+3st+rs)(pqrst^3+3qst^2+3prt+1) = 0. \tag{25}$$

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Hence identifying the coefficients of t^9, t^6, t in (24) and (25) we get

$$\frac{-c}{pqrs}=\frac{18c_1}{3qs+3(p+r)pqrs}=\frac{18c_2}{3p^2qr^2s+3qs(p+r)}$$

i.e. $$\frac{-6c}{6pr}=\frac{6c_1}{1+pr(p+r)}=\frac{6c_2}{p^2r^2+(p+r)}. \tag{26}$$

Now the polar line of the nodal point with respect to the cubic (23) is

$$c_1x+c_2y+cz=0$$

i.e. $$(1+pr\,\overline{p+r})x+(p^2r^2+\overline{p+r})y-6prz=0,$$

which meets the cubic in the three points given by

$$(1+pr\,\overline{p+r})t^2+(p^2r^2+\overline{p+r})t+pr(1+t^3)=0,$$

i.e. $$(t+p)(t+r)(prt+1)=0,$$

i.e. L, M, N as defined in Art. 8.

11. We can find the position of U, V, W in still another way.

We shall require the following theorem, proved in my paper on the Focal Circles of Circular Cubics, read before the Edinburgh Mathematical Society during last session, (1910–1911).

If the tangent at T cut the cubic again at O, and if P and P' be fixed points on the cubic such that P, O, P' are collinear; if also A and A' be variable points such that A, T, A' are collinear, then a one-to-one algebraic correspondence exists between PA and $P'A'$.

Corollary.

If $P(ABC)$ be apolar to $P(DEF)$, then $P'(A'B'C')$ is apolar to $P'(D'E'F')$.

Now, reverting to the problem under consideration, we have seen in Art. 3 that the apolar locus of ABC and DEF passes through U, V, W. Thus U(ABC) is apolar to U(DEF), and so for V and W.

Hence if U, V, W be the three common points of intersection of the system of cubics circumscribing ABC and DEF and apolar to them; and if one cubic of the system be chosen and A, B, C; D, E, F be projected through a point T of this cubic on to the cubic again (thus obtaining A', B', C'; D', E', F'); if, furthermore, U, V, W be projected through O (the point where the tangent at T meets the cubic) on to the cubic again (thus obtaining U', V', W'), then are U', V', W' the common points of intersection of the system of cubics circumscribing $A'B'C'$ and $D'E'F'$ and apolar to them.

12. The following theorems will also be required. They are proved in the paper referred to in Art. 1.

If ABC be an apolar triangle inscribed in a cubic curve, and if A, B, C be projected through a point of the curve on to the curve again, giving A′, B′, C′, then A′B′C′ is also an inscribed apolar triangle.

One might be inclined to regard any three collinear points situated on the cubic as an apolar triad, since the polar conic of each harmonically separates the other two. It is found, however, that *if three collinear points of the cubic form an apolar triad, the tangents at the three points are concurrent. In other words, if three collinear points of the cubic form an apolar triad, the line joining them is a tangent to the Cayleyan.*

Any inscribed apolar triangle may be regarded as the projection through a point of the cubic on to the cubic again of the three points in which a tangent to the Cayleyan cuts the cubic curve.

13. Suppose now that ABC and DEF are two inscribed apolar triangles on a given cubic curve.

We wish to find UVW as above defined.

Let T be a point on the cubic curve, such that if AT, etc., meet the curve again in A′, etc., then A′, B′, C′, D′, E′, F′ lie on the same conic.

It is plain that this is possible by elliptic functions. For let A, B, etc., be the elliptic parameters of the corresponding points.

$$\therefore \quad A + T + A' = \text{period}.$$
$$B + T + B' = \text{period},$$
$$\text{etc.}$$

Hence, adding and remembering that

$$A' + B' + C' + D' + E' + F' = \text{period},$$

if these points lie on a conic, we see that T is given by

$$A + B + C + D + E + F + 6T = \text{period}.$$

By Art. 11, we have to find U′V′W′ corresponding to A′B′C′ and D′E′F′. Since A′B′C′D′E′F′ lie on the same conic, and since A′B′C′D′E′F′U′V′W′ are the nine points of intersection of a system of cubics, therefore U′V′W′ is a straight line.

One cubic of the circumscribing system of A′B′C′ and D′E′F′ is the given cubic.

Another is the conic A′B′C′D′E′F′ and a certain line. To find this line take A′B′C′ as triangle of reference, and regard the conic through A′B′C′ as the polar conic of the point (x', y', z') with respect to the triangle A′B′C′.

The equation to the conic will therefore be

$$x'yz + y'zx + z'xy = 0. \qquad (27)$$

Let the line

$$lx + my + nz = 0 \qquad (28)$$

form, along with the conic (27), the cubic

$$(x'yz + y'zx + z'xy)(lx + my + nz) = 0,$$

which will have A′B′C′ as an inscribed apolar triangle, provided that by Art. (2)

$$lx' + my' + nz' = 0.$$

Hence a triangle inscribed in a conic is apolar to the cubic made up of that conic and any line through the pole of the conic with respect to the triangle.

14. The following result will now be plain :—

Two cubics circumscribing A′B′C′ and D′E′F′ and apolar to them are (1) the given cubic and (2) the cubic consisting of the conic A′B′C′D′E′F′ and the line joining the poles of this conic with respect to the triangles A′B′C′ and D′E′F′ respectively.

Hence the line joining the poles of the conic A′B′C′D′E′F′ with respect to the triangles A′B′C′ and D′E′F′ passes through U′V′W′.

15. Let the tangent at T cut the cubic again in O, and project U′V′W′ through O into UVW. Thus the points UVW are determined.

16. Since U′V′W′ is an inscribed apolar triad with respect to all the cubics circumscribing A′B′C′ and D′E′F′ and apolar to them by Art. 6, and since U′V′W′ is a straight line, we obtain that *U′V′W′ touches the Cayleyan of every cubic of the above circumscribing system.*

17. Let the line U′V′W′ meet the conic A′B′C′D′E′F′ in G′ and H′.

Let the tangents at U′V′W′ to a cubic of the apolar system circumscribing A′B′C′ and D′E′F′ meet in P′.

The locus of P′ is a straight line which is the polar line of either G′ or H′ with respect to all the cubics of the above circumscribing system.

Take any line through U′, viz., U′P′.

A cubic of the above system touching U′P′ at U′ is uniquely defined. Let the tangent at V′ meet U′P′ at P′. Then a one-to-one algebraic correspondence exists between U′P′ and V′P′, where plainly U′V′ corresponds to V′U′. Thus the locus of P′ is a straight line.

Furthermore, since the conic A′B′C′D′E′F′ together with the line G′H′ form a nodal cubic of the given system having G′ and H′ as nodes, therefore the polar line of G′ (or H′) with respect to every cubic of the system is the same line.

Now, part of the polar conic of P′ with respect to any cubic of the system is the line U′V′W′.

Hence, since the polar conic of P′ passes through G′, therefore the polar line of G′ passes through P′.

18. *If R′ and S′ be the Hessian-Points of the triad U′V′W′, then the conic A′B′C′D′E′F′ harmonically separates R′ and S′.*

Take the line U′V′W′ as $z=0$, and R′, S′ as the intersections of $z=0$ with $x=0$ and $y=0$ respectively.

Consider a cubic S of the circumscribing system passing through U′, V′, W′. The tangents at these points are concurrent, and hence it will be found that its equation $S=0$ lacks the term in xyz.

Now all the cubics of the system A′B′C′D′E′F′U′V′W′ pass through the intersections of $S=0$ and the conic $A'B'C'D'E'F' \equiv C$, and hence have as their general equation

$$S + \lambda z C = 0 \qquad (29)$$

If the equation (29) lacks the term in xyz, C must lack the term in xy, *i.e.* C harmonically separates R′ and S′.

19. The three triangles ABC, DEF, UVW of Art. 5 can also be regarded as obtained in the following way :—

Let a plane π cut a Twisted Cubic in the points XYZ. Let O be a point in π. Let the points of contact of the two tangents meeting OX be X_1, X_2. Let x be the chord of the Twisted Cubic joining X_1 and X_2. Let Y_1, Y_2, y : Z_1, Z_2, z be similarly defined.

Now since the points X_1, X_2; Y_1, Y_2; Z_1, Z_2 form three pairs of an involution by a known theorem, it follows that a quadric Γ can be described containing the Twisted Cubic and having x, y, z as generators.

Let p, q, r be any three generators of Γ of the opposite system meeting the Twisted Cubic at P, Q, R where P, Q, R are coplanar with O.

Then the polar planes of the points px, qy, rz (or any such similar combination) with respect to the Null-Complex of the given Twisted Cubic cut the Twisted Cubic in three triads of points which, on projection from O on to any plane, are of the above species. (30)

For let the double points of the involution X_1, X_2; Y_1, Y_2; Z_1, Z_2 be taken as the vertices A_0, A_3 of the tetrahedron of reference $A_0A_1A_2A_3$. Let the Twisted Cubic be

$$x_0 = \theta^3,\ x_1 = 3\theta^2,\ x_2 = 3\theta,\ x_3 = 1.$$

Then the equation to Γ (which has the tangents at A_0 and A_3 as generators) must be $x_1x_2 = 9x_0x_3$. (31)

Furthermore, the plane through the triad of points on the Twisted Cubic $\theta^3 + 3p\theta^2 + 3q\theta + pq = 0$ has as its pole the point $(pq,\ -3q,\ 3p,\ -1)$ which lies on Γ by (31), and is in fact the intersection of the generators

$$3x_0 + px_1 = 0 \qquad (32)$$
$$x_2 + 3px_3 = 0$$

and

$$3x_0 = qx_2 \qquad (33)$$
$$x_1 = 3qx_3$$

The first of these generators, viz. (32), meets the cubic in the point given by $\theta + p = 0.$ (34)

The second of these generators, viz. (33), meets the cubic in the points given by $\theta^2 - q = 0.$ (35)

We thus see that the p and q of Arts. 7 and 8 are interpreted, and the collinearities of Arts. 7 and 8 must be derived by a projection from the point O in space, as stated in (30) of the present article.