BULL. AUSTRAL. MATH. SOC. 34B25, 65F15, 65L15
VOL. 26 (1982), 445-454.

DEFERRED CORRECTION FOR THE ORDINARY DIFFERENTIAL EQUATION EIGENVALUE PROBLEM*

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This paper considers the improvement of approximate eigenvalues and eigenfunctions of ordinary differential equations using the technique of deferred correction. A convergence theorem is proved and a numerical example is given to illustrate the theory.

1. Introduction

The deferred correction technique by Fox [6], Mayers [8], has been generalized and applied to the solution of differential equations, [7], [9], and Fredholm integral equations of the second kind [1]. (See also [10].) In [2], [4], the author extended the technique to apply to integral equation eigenvalue problems. The aim of this paper is to extend the deferred correction technique to deal with ordinary differential equation eigenvalue problems. Only the small eigenvalues and the corresponding eigenfunctions are considered.

In §2, the deferred correction technique for eigenvalue problems is presented and a convergence theorem is proved in §3. Some programming details are described in §4. In §5, a numerical example illustrating the theory is given.

Received 18 June 1982.

* This paper is based on a talk given at the Australian Mathematical Society Applied Mathematics Conference held in Bundanoon, February 7-11, 1982. Other papers delivered at this Conference appear in Volumes 25 and 26.

2. Deferred correction technique for eigenvalue problems

Consider the eigenvalue problem

$$Ly = \lambda y \tag{1}$$

where L is an ordinary differential operator and

$$y \in D = \{f \in C^s[a, b] : f(a) = f(b) = 0\}$$

for some s and finite a, b. Assume the existence of a stable, consistent, qth order discretization L_q of L. (1) is approximated by

$$L_q \mathbf{u} = \nu \mathbf{u} . \tag{2}$$

For a given step-size $h = (b-a)/(n+1)$, L_q is an $n \times n$ matrix and $\mathbf{u}$ an n-vector.

We further assume that λ and ν are simple and

$$|\lambda-\nu|, \ \|r_n y-\mathbf{u}\|_\infty = O(\tau^q(y)) = O(h^q) \tag{3}$$

with

$$\tau^q(y) \triangleq r_n L y - L_q r_n y . \tag{4}$$

$r_n : C(a, b) \to R^n$ is the restriction operator with

$$r_n y = [y(x_1), y(x_2), \ldots, y(x_n)]^T$$

and $x_i = a + ih$, $i = 1, \ldots, n$. $\tau^q(y)$, the discretization error, can usually be represented by a series in powers of h with coefficients involving derivatives of y. It can be estimated by using finite differences.

From (1), (2) and (4), we obtain the perturbed equation

$$L_q \mathbf{y} = \lambda \mathbf{y} - \tau^q(y) , \tag{5}$$

with $\mathbf{y} = r_n y$.

Define the corrections

$$\mu \triangleq \lambda - \nu \tag{6a}$$

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and

$$z \triangleq y - u \,. \tag{6b}$$

Obviously, (3) implies

$$\mu,\ \|z\|_\infty = O(h^q) \,. \tag{7}$$

Rewrite (5), using notation in (6), to

$$L_q(u+z) = (\nu+\mu)(u+z) - \tau^q(y) \,, \tag{8}$$

which is equivalent to

$$(L_q - \nu I_n)z - \mu u = -\tau^q(y) + \mu z \,. \tag{9}$$

An iteration for u and z can be derived from (9) in the following manner.

Assume that $\|y\|_\infty = \|u\|_\infty = 1$ and the first component of u equals 1. Thus

$$u = \left[1,\ \xi_{n-1}^T\right]^T \tag{10}$$

and

$$z = \left[0,\ \eta_{n-1}^T\right]^T \,. \tag{11}$$

(ξ_{n-1} and η_{n-1} are $(n-1)$-vectors.)

Partition L_q in the form

$$L_q = \left[\begin{array}{c|c} l_{11} & d_{n-1}^T \\ \hline c_{n-1} & A_{n-1} \end{array}\right] \,. \tag{12}$$

(A_{n-1} is $(n-1) \times (n-1)$.)

(9) can then be rearranged to

$$T\left[\begin{array}{c} \mu \\ \hline \eta_{n-1} \end{array}\right] = \tau^q(y) - \mu\left[\begin{array}{c} 0 \\ \hline \eta_{n-1} \end{array}\right] \tag{13a}$$

with

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$$T = \left[\begin{array}{c|c} 1 & -d_{n-1}^T \\ \hline \xi_{n-1} & \nu I_{n-1} - A_{n-1} \end{array}\right] . \tag{13b}$$

It can be proved that T is non-singular, if ν is a simple eigenvalue of L_q, and $\|T^{-1}\|_\infty = O(1)$. (See [2] and [4].)

We can estimate $\tau^q(y)$ by a pth order finite difference operator $\delta^p(y)$ such that

$$\delta^p(y) = \tau^q(y) + O(h^p) , \quad p >> q . \tag{14}$$

An iteration process can then be derived from (13);

$$T\begin{bmatrix} \mu^{(r)} \\ \hline \eta_{n-1}^{(r)} \end{bmatrix} = \delta^p\left(u^{(r-1)}\right) - \mu^{(r-1)} \cdot \begin{bmatrix} 0 \\ \hline \eta_{n-1}^{(r-1)} \end{bmatrix} \tag{15}$$

with

$$u^{(r-1)} = u + \begin{bmatrix} 0 \\ \hline \eta_{n-1}^{(r-1)} \end{bmatrix} ,$$

$\mu^{(0)} = 0$ and $\eta_{n-1}^{(0)} = 0$, for $r = 1, 2, \ldots$.

3. The convergence theorem

On the basis of §2, we now prove the following convergence theorem.

THEOREM 1. *Assume that L_q is of qth order, consistent and stable and (14) holds. In addition we assume that*

A1. *ν is a simple eigenvalue of L_q,*

A2. $\left\|\delta^p(\zeta_1) - \delta^p(\zeta_2)\right\|_\infty = Ch^q \cdot \|\zeta_1 - \zeta_2\|_\infty$ *for* ζ_1 *and* $\zeta_2 \in B = \{\zeta : \|\zeta - y\|_\infty \le \varepsilon\}$ *for a small* ε .

Then the iterates $\mu^{(r)}$, $\eta_{n-1}^{(r)}$ *in* (15) *satisfy*

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$$(16)\qquad |\mu^{(r)}-\mu|,\ \left\|\eta_{n-1}^{(r)}-\eta_{n-1}\right\|_\infty = O\left(h^{\min\left(p,q(r+1)\right)}\right)$$

for $r = 1, 2, \ldots$. (See the theorem in [11, §4].)

Proof. Subtract (13) from (15) to obtain

$$(17)\qquad T\begin{bmatrix}\mu^{(r)}-\mu\\ \hline \eta_{n-1}^{(r)}-\eta_{n-1}\end{bmatrix} = \left\{\delta^p\left(u^{(r-1)}\right)-\tau^q(y)\right\} - \left\{\mu^{(r-1)}\begin{bmatrix}0\\ \hline \eta_{n-1}^{(r-1)}\end{bmatrix}-\mu\begin{bmatrix}0\\ \hline \eta_{n-1}\end{bmatrix}\right\}$$

for $r = 1, 2, \ldots$.

A1 implies that T is non-singular and $\|T^{-1}\|_\infty = O(1)$.

Using A2 and (14), the first term in the right hand side of (17) becomes

$$(18)\qquad \delta^p\left(u^{(r-1)}\right) - \left[\delta^p(y)+O\left(h^p\right)\right] = O\left(h^q\right)\ .\ \|u^{(r-1)}-y\|_\infty + O\left(h^p\right)\ .$$

The second term in the right hand side of (17) can be written as

$$(19)\qquad \left(\mu^{(r-1)}-\mu\right)\ .\ \begin{bmatrix}0\\ \hline \eta_{n-1}\end{bmatrix} + \left[\mu+\left(\mu^{(r-1)}-\mu\right)\right]\ .\ \left\{\begin{bmatrix}0\\ \hline \eta_{n-1}^{(r-1)}\end{bmatrix}-\begin{bmatrix}0\\ \hline \eta_{n-1}\end{bmatrix}\right\}\ .$$

From (17), using (7), (18) and (19), we prove that

$$(20)\qquad \|\omega^{(r)}\|_\infty = O\left(h^q.\|\omega^{(r-1)}\|_\infty\right) + O\left(\|\omega^{(r-1)}\|_\infty^2\right) + O\left(h^p\right)$$

with

$$\omega^{(r)} \triangleq \begin{bmatrix}\mu^{(r)}-\mu\\ \hline \eta_{n-1}^{(r)}-\eta_{n-1}\end{bmatrix} \quad \text{for} \quad r = 1, 2, \ldots .$$

Equations (7) and (20) then imply (16). //

Note that (14) and A2 will involve some extra smoothness conditions on y .

Judging from (16), if a moderate value of n yields reasonable approximations ν and u and τ^q is estimated well enough by δ^p , the deferred correction process in (15) will improve the accuracy of ν and u

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to a great extent. Note that the computation only involves n-dimensional matrices and vectors, a feature of deferred correction. The extra smoothness conditions imposed on y, which enables the estimation of τ^q by the finite difference operator δ^p, may be a disadvantage.

4. Programming details

In this section, we give a few brief notes concerning the actual implementation of the deferred correction process (15) on computers.

(a) The LU-decomposition of T

In general, the kth component of u is the maximum one. T will be a banded matrix with an extra kth column;

$$T = \begin{bmatrix}
x & x & & & & x & & & \\
x & x & x & & O & x & & & \\
 & x & x & x & & x & & & \\
 & & \ddots & & & \vdots & & & \\
 & & & \ddots & & \vdots & & O & \\
 & & & & \ddots & \vdots & & & \\
 & & & & x & x & x & & \\
 & & & & & x & x & x & \\
 & & & & & x & x & x & \\
 & & & & & \vdots & \ddots & & \\
 & & O & & & \vdots & & \ddots & \\
 & & & & & \vdots & & & \ddots \\
 & & & & & x & & x & x \\
 & & & & & x & O & & x \\
 & & & & & x & & & x
\end{bmatrix}$$

(kth column)

assuming a tridiagonal matrix L_q.

The extra kth column of T can be shifted to the first (last) column to form $\tilde{T}$ if $k \le n/2$ $(> n/2)$; for example

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$$\tilde{T} = \begin{bmatrix}
x & x & x & & & & & & & & & & \\
x & x & x & x & & & & & & & & & \\
x & 0 & x & x & x & & & & & & & & \\
. & & 0 & . & . & . & & & & & & & \\
. & & & . & . & . & . & & & & O & & \\
. & & & & . & . & . & . & & & & & \\
x & & & & & . & x & x & x & & & & \\
x & & & & & & 0 & x & x & 0 & & & \\
. & & & & & & & 0 & x & x & 0 & & \\
. & & & O & & & & & 0 & x & x & 0 & \\
. & & & & & & & & & x & x & x & . \\
. & & & & & & & & & & . & . & . \\
. & & & & & & & & & & & . & . \\
x & & & & & & & & & & & & . \\
x & & & & & & & & & & & & \\
x & & & & & & & & & & & &
\end{bmatrix}$$

kth column

if $k \leq n/2$. The two super-diagonals (sub-diagonals) can then be eliminated and the resulting operation counts for both the LU-decomposition and back-substitution are of $O(n)$. The strategy of shifting the kth column to the first or last column, depending on the value of k , improves the operation counts slightly because the outermost super-diagonal (sub-diagonal) will contain more zeros.

A computer program implementing the above method is contained in [3], for the LU-decomposition and back-substitution of tridiagonal matrices with an extra full column. It has been used in the solution of Sturm-Liouville problems, using the deferred correction process.

(b) Estimation of τ^q

The process of differencing in δ^p should be carried out with care in computer programs. Multiprecision accumulation of differences of opposite signs should be employed to avoid the overflow of round-off error.

In [6] and [8], the differences in δ^p are calculated until all significant terms are included. Note also that any non-convergence of differences in δ^p indicates non-smoothness in y .

Alternatively, an iterated deferred correction technique can be applied. More differences are included as the iteration goes on. (See [9].)

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(c) $\|T^{-1}\|_\infty$

The size of $\|T^{-1}\|_\infty$ should be monitored. A large $\|T^{-1}\|_\infty$ implies that ν is close or equal to another eigenvalue of L_q, that is, λ is a multiple eigenvalue, and thus invalidates the theory. The technique in [5] for estimating condition numbers can be used.

5. A numerical example

Assume that the Sturm-Liouville eigenvalue problem in (1) is discretized using second order central differences. Thus $q = 2$ in (3). For δ^p in (14), pth order central differences are used for points near the centre of $[a, b]$ and onesided differences near the edges $x = a, b$. We required $y \in C^{p+2}[a, b]$ in order to have (14) satisfied. The highest order of the differences used in δ^p will be $(p+1)$. The main result (16) becomes

$$\|\omega^{(r)}\|_\infty = O\left(h^{\min(p,2r+2)}\right) \tag{21}$$

for $r = 1, 2, \ldots$.

We give the following example. Consider Mathieu's equation

$$D^2y + (\lambda - 2\cos 2x)y = 0$$

with $y(0) = y(\pi) = 0$. The smallest eigenvalue $\lambda_1 = -0.1102488168$, with the corresponding eigenfunction

$$y_1 = \sum_{r=1}^{\infty} A_{2r+1} \sin(2r+1)x .$$

Table 1 contains $E^p_{n,r} = |\lambda - (\nu + \mu^{(r)})|$, the errors of the improved eigenvalues for various values of n and r, with $p = 5$.

Table 2 contains the ratios $E^p_{m,r}/E^p_{2m+1,r}$, which indicate the numerical rate of convergence.

As expected the method produces good results. The theoretical rate in (21) is in good agreement with the numerically estimated one. Recall that

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TABLE 1. $E^5_{n,r}$

r \ n	15	31	63
0	.612E-2	.159E-2	.396E-3
1	.168E-3	.579E-5	.352E-6
2	.152E-3	.171E-6	.330E-8
3	.127E-3	.235E-6	.260E-8

TABLE 2. Ratios

r	$E^5_{15,r}/E^5_{31,r}$	$E^5_{31,r}/E^5_{63,r}$	Rate of Convergence	
			Numerical	Theoretical (21)
0	3.85	4.01	$O(h^{2.0})$	$O(h^2)$
1	29.03	16.43	$O(h^{4.0})$	$O(h^4)$
2	712.01	51.73	$O(h^{5.7})$	$O(h^5)$
3	541.42	90.54	$O(h^{6.5})$	$O(h^5)$

$(p+1)$th order central differences are used in δ^p for most points in $[a, b]$ except those near the edges $x = a, b$. Thus the numerically estimated rate of convergence is usually better than the theoretical one.

The behaviour of the approximations to the eigenfunctions is similar to that of the eigenvalues.

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