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Compositio Math. **145** (2009), 1035–1079.

doi:10.1112/S0010437X09003960

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Cluster structures for 2-Calabi–Yau categories and unipotent groups

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Abstract

We investigate cluster-tilting objects (and subcategories) in triangulated 2-Calabi–Yau and related categories. In particular, we construct a new class of such categories related to preprojective algebras of non-Dynkin quivers associated with elements in the Coxeter group. This class of 2-Calabi–Yau categories contains, as special cases, the cluster categories and the stable categories of preprojective algebras of Dynkin graphs. For these 2-Calabi–Yau categories, we construct cluster-tilting objects associated with each reduced expression. The associated quiver is described in terms of the reduced expression. Motivated by the theory of cluster algebras, we formulate the notions of (weak) cluster structure and substructure, and give several illustrations of these concepts. We discuss connections with cluster algebras and subcluster algebras related to unipotent groups, in both the Dynkin and non-Dynkin cases.

I. Introduction

The theory of cluster algebras, initiated by Fomin and Zelevinsky in [FZ02] and further developed in a series of papers including [BFZ05, FZ03, FZ07], has turned out to have interesting connections with many parts of algebra as well as other branches of mathematics. One of these links is with the representation theory of algebras, where a connection was first discovered in [MRZ03]. The philosophy has been to model the main ingredients in the definition of a cluster algebra in a categorical or module-theoretical setting. The cluster categories associated with finite-dimensional hereditary algebras were introduced for this purpose in [BMRRT06] and shown to be triangulated in [Kel05] (see also [CCS06] for the A_n case); the module categories mod Λ for Λ a preprojective algebra of a Dynkin quiver have been used for a similar purpose [GLS06]. This development has both inspired new directions of investigations on the categorical side and provided interesting feedback on the theory of cluster algebras; see, for example, [ABS08, BM06, BMR07, BMR08, BMRT07, CC06, CK06, CK08, GLS08, GLS06, Hub, Iya07a, Iya07b, IT06, KR07, IR08, IY08, KR08, Rin07, Tab07] for material relevant to this paper.

The cluster categories and the stable categories $\underline{\mathrm{mod}}\,\Lambda$ of preprojective algebras are both triangulated Calabi–Yau categories of dimension two (2-CY for short). Both have so-called cluster-tilting objects or subcategories [BMRRT06, IY08, KR07] (referred to as 'maximal 1-orthogonal' in [Iya07a]), which are important since they are the analogs of clusters. The investigation of cluster-tilting objects or subcategories in 2-CY and related categories is

Received 6 November 2007, accepted in final form 21 October 2008, published online 13 May 2009.
2000 Mathematics Subject Classification 18E30 (primary), 16E99, 16G70, 18D99, 18G99 (secondary).
Keywords: preprojective algebras, 2-Calabi–Yau categories, reduced expressions, quiver mutation, cluster algebras, cluster categories, cluster tilting, AR-quivers.
The authors were supported by a STORFORSK grant (no. 167130) from the Norwegian Research Council.

interesting both from the point of view of cluster algebras and in its own right. Hence it is of importance to develop methods for constructing 2-CY categories together with the special objects or subcategories of interest, and this is the main purpose of the first two sections of this paper.

The properties of cluster-tilting objects in (Hom-finite) 2-CY categories that have been important for applications to cluster algebras are: (a) the unique exchange property for indecomposable summands of cluster-tilting objects; (b) the existence of associated exchange triangles; (c) the non-existence of loops or 2-cycles (in the quiver of the endomorphism algebra of a cluster-tilting object); and (d) the fact that when passing from the endomorphism algebra of a cluster-tilting object T to the endomorphism algebra of another cluster-tilting object T^* via an exchange, the change in quivers is given by a Fomin–Zelevinsky mutation. The properties (a) and (b) are known to hold for any 2-CY triangulated category [IY08]; this was proved for cluster categories in [BMRRT06] and for stable categories of preprojective algebras of Dynkin type in [GLS06]. Property (c) does not always hold (see [BIKR08]), hence it is of interest to establish criteria for when it does hold, and this is one of the issues we deal with in this paper. We also show, for any 2-CY category, that if (c) holds, then (d) follows, as previously proved by Palu for algebraic triangulated categories [Pal08]. We construct new 2-CY categories with cluster-tilting objects from old ones via a subfactor construction, extending results from [IY08], with a main focus on how condition (c) behaves under this construction. Associated with this, we introduce the notions of cluster structures and cluster substructures.

Important examples investigated in [GLS06] are the categories $\operatorname{mod} \Lambda$ of finitely generated modules over the preprojective algebra Λ of a Dynkin quiver. We deal with appropriate subcategories of $\operatorname{mod} \Lambda$. The focus of this paper is on the more general case of subcategories of the category $\operatorname{f.l.} \Lambda$ of finite-length modules over the completion of the preprojective algebra of a non-Dynkin quiver with no loops. Our main tool is to extend the tilting theory developed for Λ in the noetherian case in [IR08]. This turns out to give a large class of 2-CY categories associated with elements in the corresponding Coxeter groups. For these categories we construct cluster-tilting objects associated with each reduced expression, and we describe the associated quiver directly in terms of the reduced expression. We prove that this class of 2-CY categories contains all the cluster categories of finite-dimensional hereditary algebras as well as the stable categories $\underline{\operatorname{mod}}\, \Lambda$ for a preprojective algebra Λ of Dynkin type; this allows us to get more information on the latter type.

We illustrate our results with applications to constructing subcluster algebras of cluster algebras, a notion that we shall define here and which is already implicit in the literature. To this end, we introduce (strong) cluster maps, inspired by maps from [CC06, CK08] and [GLS06]. These maps have the property that we can pass from cluster structures and substructures to cluster algebras and subcluster algebras.

In relation to substructures for preprojective algebras of Dynkin type, we discuss examples from $\mathrm{SO}_8(\mathbb{C})$-isotropic Grassmannians and the $Gr_{2,5}$ Schubert variety. For a (non-Dynkin) quiver Q with associated Coxeter group W we can, for each $w \in W$, consider the coordinate ring $\mathbb{C}[U^w]$ of the unipotent cell associated with w in the corresponding Kac–Moody group. We conjecture that this ring has a cluster algebra structure and that it is modelled by our (stably) 2-CY category associated with the same w. As support for this, we verify the conjecture in the $\widehat{A_1}$ case for a word w of length at most four.

The second section is devoted to introducing and investigating the notions of cluster structures and substructures, and to giving sufficient conditions for such structures to occur. Also, the two concrete examples mentioned above are investigated and used to illustrate the connection with cluster algebras and subcluster algebras defined in § IV.2. In § III, we use tilting theory to construct categories whose stable categories are 2-CY, along with natural cluster-tilting objects in these categories. In § IV.2 we present examples for preprojective algebras of Dynkin type, and in § IV.3 we discuss our conjecture.

Part of this work was done independently by Geiss *et al.* in [GLS08]. They used a somewhat different approach to develop, as in our § II, 2-CY categories (in a different language) for the case of subcategories of the form Sub P (or Fac P) for P projective, over a preprojective algebra of Dynkin type. Parallel to our § IV.2, examples arising from Sub P were presented independently in [GLS08]; in this respect, the fourth author was inspired by a lecture of Leclerc in 2005, where cluster algebras associated with Sub P in the A_n case were discussed. Recently, using completely different methods, Geiss *et al.* [GLS07C] have also done work related to our § III in the case of adaptable elements in the Coxeter group.

For general background on representation theory of algebras we refer to [ARS97, ASS06, Rin84, Hap88, AHK07]; for background on Lie theory we refer to [BL00].

Modules in this paper will usually be left modules and, with composition of maps, fg will mean first f, then g.

The second author would like to thank William Crawley-Boevey and Christof Geiss for answering a question about references for the 2-CY property of preprojective algebras. He also thanks Bernard Leclerc for valuable comments.

II. 2-CY categories and substructures

The cluster algebras of Fomin and Zelevinsky have motivated attempts to model the essential ingredients of the definition of a cluster algebra in a categorical or module-theoretical way. In particular, this led to the theory of cluster categories and the investigation of new aspects of the module theory of preprojective algebras of Dynkin type. In § II.1 we give some of the main categorical requirements needed for the modelling, for the cases with and without coefficients; this leads to the notions of weak cluster structure and cluster structure. Like the aforementioned examples, our main examples do have 2-Calabi–Yau-type properties.

We introduce substructures of (weak) cluster structures in § II.2. It is natural to deal with (weak) cluster structures that have so-called coefficients, at least for the substructures. Of particular interest for our applications to cluster algebras is the case of completions of preprojective algebras Λ of a finite connected quiver with no loops over an algebraically closed field K, where the interesting larger category is the stable category $\underline{\mathrm{f.\,l.}}\Lambda$ of the finite-length Λ-modules. For Dynkin quivers this is the stable category $\underline{\mathrm{mod}}\,\Lambda$ of the finitely generated Λ-modules, and in the non-Dynkin case $\underline{\mathrm{f.\,l.}}\Lambda = \mathrm{f.\,l.}\,\Lambda$. The former case is discussed in § II.3, while § III is devoted to the latter case.

II.1 Cluster structures

In this section we introduce the concepts of weak cluster structure and cluster structure for extension-closed subcategories of triangulated categories and for exact categories. These concepts are illustrated with 2-CY categories and other closely related categories, and the main objects we investigate are the cluster-tilting ones. These cases are particularly nice when the quivers of the cluster-tilting subcategories have no loops or 2-cycles. The closely related maximal rigid objects also provide interesting examples.

We start by introducing the notions of weak cluster structure and cluster structure. Throughout this section, all categories are Krull–Schmidt categories over an algebraically closed field K; that is, each object is isomorphic to a finite direct sum of indecomposable objects with local endomorphism ring. The categories we consider are either exact (e.g. abelian) categories or extension-closed subcategories of triangulated categories. Note that an extension-closed subcategory of an exact category is again exact. We refer to [Kel96, Kel90] for the definition and basic properties of exact categories, which behave very much like abelian categories, and also with respect to derived categories and Ext-functors.

We shall often identify a set of indecomposable objects with the additive subcategory consisting of all summands of direct sums of these indecomposable objects. We will also identify an object with the set of indecomposable objects appearing in a direct sum decomposition, and with the subcategory obtained in the above way.

Assume that we have a collection of sets $\underline{x}$ (which may be infinite), called *clusters*, of non-isomorphic indecomposable objects. The union of all indecomposable objects in clusters is called the set of *cluster variables*. Assume also that there is a subset $\underline{p}$ (which may be infinite) of indecomposable objects, called *coefficients*, which are not cluster variables. We denote by T the union of the indecomposable objects in $\underline{x}$ and $\underline{p}$, sometimes viewed as a category with these objects, and call it an *extended cluster*.

We say that the clusters together with the prescribed set of coefficients $\underline{p}$ give a *weak cluster structure* on $\mathcal{C}$ if the following hold.

(a) For each extended cluster T and each cluster variable M which is a summand in T, there is a unique indecomposable object $M^* \not\simeq M$ such that we get a new extended cluster T^* upon replacing M by M^*. We denote this operation, called *exchange*, by $\mu_M(T) = T^*$, and we call (M, M^*) an *exchange pair*.

(b) For each cluster variable M, there are triangles or short exact sequences $M^* \xrightarrow{f} B \xrightarrow{g} M$ and $M \xrightarrow{s} B' \xrightarrow{t} M^*$, where the maps g and t are minimal right $\mathrm{add}(T\backslash\{M\})$-approximations and f and s are minimal left $\mathrm{add}(T\backslash\{M\})$-approximations. These are called *exchange triangles* or *exchange sequences*.

Denote by Q_T the quiver of T, where the vertices correspond to the indecomposable objects in T and the number of arrows $T_i \to T_j$ between two indecomposable objects T_i and T_j is given by the dimension of the space of irreducible maps $\mathrm{rad}(T_i, T_j)/\mathrm{rad}^2(T_i, T_j)$. Here $\mathrm{rad}(\ ,\ )$ denotes the radical in add T, where the objects are finite direct sums of objects in T. For an algebra Λ (where Λ has a unique decomposition as a direct sum of indecomposable objects), the quiver of Λ is then the opposite of the quiver of add Λ.

We say that a quiver $Q = (Q_0, Q_1)$ is an *extended quiver* with respect to a subset of vertices Q_0' if there are no arrows between two vertices in $Q_0\backslash Q_0'$. We regard the quiver Q_T of an extended cluster as an extended quiver by neglecting all arrows between two vertices corresponding to coefficients.

We say that $\mathcal{C}$, with a fixed set of clusters and coefficients, has *no loops* (respectively, *no 2-cycles*) if in the extended quiver of each extended cluster there are no loops (respectively, no 2-cycles). In this case, the extended quiver Q_T is given by the sequences in (b).When $\underline{x}$ is finite, this is the opposite quiver of the factor algebra $\underline{\mathrm{End}}(T)$ of $\mathrm{End}(T)$ by the maps factoring through direct sums of objects from $\underline{p}$.

We say that we have a *cluster structure* if the following additional conditions hold.

(c) There are no loops or 2-cycles. (In other words, for a cluster variable M, any non-isomorphism $u\colon M \to M$ factors through $g\colon B \to M$ and through $s\colon M \to B'$, while any non-isomorphism $v\colon M^* \to M^*$ factors through $f\colon M^* \to B$ and through $t\colon B' \to M^*$, with B and B' having no common indecomposable summand.)

(d) For an extended cluster T, passing from Q_T to Q_{T^*} is given by the Fomin–Zelevinsky mutation at the vertex of Q_T given by the cluster variable M.

Note that (c) is needed for (d) to make sense, but it is still convenient to express this as two separate statements.

We recall that, for an extended quiver Q without loops or 2-cycles and a vertex k in Q_0', the Fomin–Zelevinsky mutation $\mu_k(Q)$ of Q at k is the quiver obtained from Q upon making the following changes [FZ02].

- Reverse all the arrows starting or ending at k.
- Let $s \neq k$ and $t \neq k$ be vertices in Q_0 such that at least one vertex belongs to Q_0'. If in Q we have $n > 0$ arrows from t to k, $m > 0$ arrows from k to s and r arrows from s to t (interpreted as $-r$ arrows from t to s if $r < 0$), then there are $nm - r$ arrows from t to s in the new quiver $\mu_k(Q)$ (interpreted as $r - nm$ arrows from s to t if $nm - r < 0$).

The main known examples of triangulated K-categories with finite-dimensional homomorphism spaces (Hom-finite for short) which have a weak cluster (and usually cluster) structure are the 2-CY categories. These are triangulated K-categories with functorial isomorphisms $D\,\mathrm{Ext}^1(A,B) \simeq \mathrm{Ext}^1(B,A)$ for all A and B in $\mathcal{C}$, where $D = \mathrm{Hom}_K(\ ,K)$. Note that this is called *weakly* 2-CY in [Kel08] (see also [Kel05, § 8]). A Hom-finite triangulated category is 2-CY if and only if it has almost-split triangles with translation τ and $\tau\colon \mathcal{C} \to \mathcal{C}$ is a functor isomorphic to the shift functor [1] (see also [RV02]).

We have the following examples of 2-CY categories.

(1) The cluster category $\mathcal{C}_H$ associated with a finite-dimensional hereditary K-algebra H is, by definition, the orbit category $\mathbf{D}^{\mathrm{b}}(H)/\tau^{-1}[1]$, where $\mathbf{D}^{\mathrm{b}}(H)$ is the bounded derived category of finitely generated H-modules and τ is the AR-translation of $\mathbf{D}^{\mathrm{b}}(H)$(see [BMRRT06]). It is a Hom-finite triangulated category [Kel96], and it is 2-CY since $\tau = [1]$.

(2) The stable category of maximal Cohen–Macaulay modules $\underline{\mathrm{CM}}(R)$ over a three-dimensional complete local commutative noetherian Gorenstein isolated singularity R containing the residue field K is 2-CY [Aus78] (see also [Yos90]).

(3) The preprojective algebra Λ associated to a finite connected quiver Q without loops is defined as follows: let $\widetilde{Q}$ be the quiver constructed from Q by adding an arrow $\alpha^*\colon i \to j$ for each arrow $\alpha\colon j \to i$ in Q; then $\Lambda = K\widetilde{Q}/I$, where I is the ideal generated by the sum of commutators $\sum_{\beta \in Q_1}[\beta, \beta^*]$. Note that Λ is uniquely determined up to isomorphism by the underlying graph of Q.
When Λ is the preprojective algebra of a Dynkin quiver over K, the stable category $\underline{\mathrm{mod}}\,\Lambda$ is 2-CY (see [AR96, Propositions 3.1 and 1.2], [CB00] and [Kel05, § 8.5]).

When Λ is the completion of the preprojective algebra of a finite connected quiver without loops which is not Dynkin, the bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{f.\,l.}\,\Lambda)$ of the category $\mathrm{f.\,l.}\,\Lambda$ of the modules of finite length is 2-CY (see [Boc08, CB00, BBK02] and [GLS07b, § 8]).

We shall also use the term 2-CY in more general situations. From now on, we will usually write simply 'category' instead of 'K-category'.

We say that an exact Hom-finite category $\mathcal{C}$ is *derived 2-CY* if the triangulated category $\mathbf{D}^{\mathrm{b}}(\mathcal{C})$ is 2-CY, i.e. if $D\,\mathrm{Ext}^i(A, B) \simeq \mathrm{Ext}^{2-i}(B, A)$ for all A, B in $\mathbf{D}^{\mathrm{b}}(\mathcal{C})$ and all i. Note that when $\mathcal{C}$ is derived 2-CY, then $\mathcal{C}$ has no non-zero projective or injective objects. The category $\mathrm{f.\,l.}\,\Lambda$ where Λ is the completion of the preprojective algebra of a non-Dynkin connected quiver without loops is an important example of a derived 2-CY category.

We say that a category $\mathcal{C}$ is *stably 2-CY* (or sometimes *exact stably 2-CY*) if it is Frobenius, that is, if $\mathcal{C}$ is exact and has enough projectives and injectives, which coincide, and the stable category $\underline{\mathcal{C}}$, which is triangulated [Hap88], is (Hom-finite) 2-CY. Recall that $\mathcal{C}$ is said to have enough projectives if for each X in $\mathcal{C}$ there is an exact sequence $0 \to Y \to P \to X \to 0$ in $\mathcal{C}$ with P projective; having enough injectives is defined in a dual way.

We have the following characterization of stably 2-CY categories.

PROPOSITION II.1.1. *Let $\mathcal{C}$ be a Frobenius category. Then $\mathcal{C}$ is stably 2-CY if and only if $\mathrm{Ext}^1_{\mathcal{C}}(A, B)$ is finite-dimensional and we have functorial isomorphisms $D\,\mathrm{Ext}^1_{\mathcal{C}}(A, B) \simeq \mathrm{Ext}^1_{\mathcal{C}}(B, A)$ for all A and B in $\mathcal{C}$.*

Proof. It is easy to see that $\mathrm{Ext}^1_{\mathcal{C}}(B, A) \simeq \mathrm{Ext}^1_{\underline{\mathcal{C}}}(B, A)$. If $\mathcal{C}$ is stably 2-CY, then $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(B, A)$, and hence $\mathrm{Ext}^1_{\mathcal{C}}(B, A)$, is finite-dimensional, so we have the desired functorial isomorphism. The converse also follows directly. □

Examples of stably 2-CY categories are categories of maximal Cohen–Macaulay modules $\mathrm{CM}(R)$ for a three-dimensional complete local commutative isolated Gorenstein singularity R (containing the residue field K) and $\mathrm{mod}\,\Lambda$ for Λ being the preprojective algebra of a Dynkin quiver. We shall see several further examples later.

We are especially interested in pairs $(\mathcal{C}, \underline{\mathcal{C}})$ of 2-CY categories where $\mathcal{C}$ is a stably 2-CY category. The only difference in indecomposable objects between $\mathcal{C}$ and $\underline{\mathcal{C}}$ is the indecomposable projective objects in $\mathcal{C}$. Note also that given an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, there is an associated triangle $A \to B \to C \to A[1]$ in $\underline{\mathcal{C}}$. Conversely, given a triangle $A \to B \xrightarrow{\underline{g}} C \to A[1]$ in $\underline{\mathcal{C}}$, we lift $\underline{g} \in \mathrm{Hom}_{\underline{\mathcal{C}}}(B, C)$ to $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and obtain an exact sequence $0 \to A \to B \oplus P \to C \to 0$ in $\mathcal{C}$, where P is projective. We then have the following useful fact, which is easy to prove.

PROPOSITION II.1.2. *Let $\mathcal{C}$ be a stably 2-CY category with a set of clusters $\underline{x}$ and a set of coefficients $\underline{p}$ which are the indecomposable projective objects. For the stable 2-CY category $\underline{\mathcal{C}}$, consider the same set of clusters $\underline{x}$ but with no coefficients. Then we have the following.*

(a) *The $(\underline{x}, \underline{p})$ give a weak cluster structure on $\mathcal{C}$ if and only if the $(\underline{x}, \emptyset)$ give a weak cluster structure on $\underline{\mathcal{C}}$.*
(b) *$\mathcal{C}$ has no loops if and only if $\underline{\mathcal{C}}$ has no loops.*
(c) *If $\mathcal{C}$ has a cluster structure, then $\underline{\mathcal{C}}$ has a cluster structure.*

In the examples of 2-CY categories with cluster structure which have been investigated, the extended clusters have been the subcategories T where $\mathrm{Ext}^1(M, M) = 0$ for all $M \in T$ and such that whenever $X \in \mathcal{C}$ satisfies $\mathrm{Ext}^1(M, X) = 0$ for all $M \in T$, then $X \in T$. In [KR07], such a T

is referred to as a *cluster-tilting subcategory* if it is, in addition, functorially finite in the sense of [AS80], which is automatically true when T is finite. Such a T has also been called a maximal 1-orthogonal subcategory in [Iya07a, Iya07b], and an Ext-configuration in [BMRRT06].

We have the following connections between $\mathcal{C}$ and $\underline{\mathcal{C}}$ for a stably 2-CY category $\mathcal{C}$ when using the cluster-tilting subcategories; these follow easily from Propositions II.1.1 and II.1.2.

Lemma II.1.3. *Let $\mathcal{C}$ be a stably 2-CY category, and let T be a subcategory of $\mathcal{C}$ containing all indecomposable projective objects. Then T is a cluster-tilting subcategory in $\mathcal{C}$ if and only if it is the same in $\underline{\mathcal{C}}$.*

Lemma II.1.4. *Let $\mathcal{C}$ be a stably 2-CY category and T a cluster-tilting object in $\mathcal{C}$ with an indecomposable non-projective summand M. Then there is no loop at M for $\mathrm{End}_{\mathcal{C}}(T)$ if and only if there is no loop at M for $\mathrm{End}_{\underline{\mathcal{C}}}(T)$. If $\mathcal{C}$ has no 2-cycles, then there are none for $\underline{\mathcal{C}}$.*

Note that a stably 2-CY category $\mathcal{C}$ with the cluster-tilting subcategories gives a situation where we have a natural set of coefficients, namely the indecomposable projective objects, which clearly belong to all cluster-tilting subcategories, whereas $\underline{\mathcal{C}}$ with the cluster-tilting subcategories gives a case where it is natural to choose no coefficients. We have the following useful observation, which follows from Proposition II.1.2.

Proposition II.1.5. *Let $\mathcal{C}$ be a stably 2-CY category. Then the cluster-tilting subcategories in $\mathcal{C}$, with the indecomposable projectives as coefficients, determine a weak cluster structure on $\mathcal{C}$ if and only if the cluster-tilting subcategories in $\underline{\mathcal{C}}$ determine a weak cluster structure on $\underline{\mathcal{C}}$.*

If $\mathcal{C}$ is triangulated 2-CY, then $\mathcal{C}$ has a weak cluster structure, with the extended clusters being the cluster-tilting subcategories [IY08]. Properties (c) and (d) hold for cluster categories and the stable category $\underline{\mathrm{mod}}\,\Lambda$ of a preprojective algebra of Dynkin type [BMRRT06, BMR08, GLS06]; however, (c) does not hold in general [BIKR08]. Nevertheless, we show that when we have some cluster-tilting subcategory in the 2-CY category $\mathcal{C}$, (d) will hold under the assumption that (c) holds. This was first proved in [Pal08] for when $\mathcal{C}$ is *algebraic*, i.e. by definition the stable category of a Frobenius category, as a special case of a more general result. Our proof is inspired by [IR08, Theorem 7.1].

Theorem II.1.6. *Let $\mathcal{C}$ be triangulated (or stably) 2-CY with some cluster-tilting subcategory. If $\mathcal{C}$ has no loops or 2-cycles, then the cluster-tilting subcategories determine a cluster structure for $\mathcal{C}$.*

Proof. We give a proof for the triangulated 2-CY case, and for simplicity we assume that T is a cluster-tilting object. Using exact sequences instead of triangles, a similar argument works for the stably 2-CY case. Note that in the stably 2-CY case we do not have to consider arrows between projective vertices.

Let $T = \bigoplus_{i=1}^{n} T_i$ for the cluster-tilting object T in $\mathcal{C}$. Fix a vertex $k \in \{1, \ldots, n\}$, and let $T^* = \bigoplus_{i \neq k} T_i \oplus T_k^* = \mu_k(T)$. We have exchange triangles $T_k^* \to B_k \to T_k$ and $T_k \to B'_k \to T_k^*$, which show that when passing from $\mathrm{End}(T)$ to $\mathrm{End}(T^*)$, we reverse all arrows in the quiver of $\mathrm{End}(T)$ starting or ending at k.

We need to consider the situation where we have arrows $j \to k \to i$. By assumption, there is no arrow $i \to k$. Consider the exchange triangles $T_i^* \to B_i \to T_i$ and $T_i \to B'_i \to T_i^*$. Then T_k is not a direct summand of B'_i, and we write $B_i = D_i \oplus T_k^m$ for some $m > 0$, where T_k is not a direct summand of D_i.

https://doi.org/10.1112/S0010437X09003960 Published online by Cambridge University Press

We get the left diagram below by starting with the maps in the upper square and the triangles that they induce, and applying the octahedral axiom. The third row is then a triangle. Using again the octahedral axiom, we get the right diagram of triangles, where the second row is an exchange triangle and the third column is the second column of the left diagram.

$$\begin{tikzcd}
 & (T_k^*)^m[1] \arrow[r, equal] & (T_k^*)^m[1] & \\
T_i[-1] \arrow[r] \arrow[d, equal] & T_i^* \arrow[r] \arrow[u] & D_i \oplus T_k^m \arrow[r] \arrow[u] & T_i \arrow[d, equal] \\
T_i[-1] \arrow[r] & X \arrow[r] \arrow[u] & D_i \oplus B_k^m \arrow[r] \arrow[u] & T_i \\
 & (T_k^*)^m \arrow[r, equal] \arrow[u] & (T_k^*)^m \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (T_k^*)^m[1] \arrow[r, equal] & (T_k^*)^m[1] & \\
T_i \arrow[r] \arrow[d, equal] & B_i' \arrow[r] \arrow[u] & T_i^* \arrow[r] \arrow[u] & T_i[1] \arrow[d, equal] \\
T_i \arrow[r] & Y \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & T_i[1] \\
 & (T_k^*)^m \arrow[r, equal] \arrow[u] & (T_k^*)^m \arrow[u] &
\end{tikzcd}$$

Since T_k is not in add B_i', we have $(B_i', (T_k^*)^m[1]) = 0$ and hence $Y = B_i' \oplus (T_k^*)^m$.

Consider the triangle $X \to D_i \oplus B_k^m \xrightarrow{a} T_i \to X[1]$. Let $\overline{T}^* = (\bigoplus_{t \neq i,k} T_t) \oplus T_k^*$. We now show that $D_i \oplus B_k^m$ is in add $\overline{T}^*$. Note that $D_i \oplus T_k^m = B_i$ is in add T. We know that T_i is not a direct summand of D_i, and T_i is not a direct summand of B_k since there is no arrow from i to k. Furthermore, T_k is not a direct summand of D_i by the choice of D_i, and T_k is not a direct summand of B_k. Hence we see that B_i is in add $\overline{T}^*$.

Next, we want to show that a is a right add $\overline{T}^*$-approximation. It follows from the first commutative diagram that any map $g\colon T_t \to T_i$, where T_t is an indecomposable direct summand of $\overline{T}^*$ not isomorphic to T_k^*, factors through a. Let $f\colon T_k^* \to T_i$ be a map and $h\colon T_k^* \to B_k$ the minimal left add $\overline{T}$-approximation, where $\overline{T} = \bigoplus_{t \neq k} T_t$. Then there is some $s\colon B_k \to T_i$ such that $hs = f$. Thus, by the above, s factors through a since B_k is in add $\overline{T}^*$ (using the fact that T_i is not a direct summand in B_k) and T_k^* is not a direct summand of B_k. It follows that a is a right add $\overline{T}^*$-approximation.

Now consider the triangle $T_i \to B_i' \oplus (T_k^*)^m \xrightarrow{b} X \to T_i[1]$. It is clear that $B_i' \oplus (T_k^*)^m$ is in add $\overline{T}^*$, since T_k is not a direct summand of B_i'. Since T_i is in both T and T^*, we have that $\operatorname{Hom}(\overline{T}^*, T_i[1]) = 0$ and hence b is a right add $\overline{T}^*$-approximation.

By the above, the number of arrows from j to i in the quiver Q_{T^*} is

$$u = \alpha_{D_i \oplus B_k^m}(T_j) - \alpha_{B_i' \oplus (T_k^*)^m}(T_j),$$

where $\alpha_X(T_j)$ denotes the multiplicity of T_j in X. We have

$$u = \alpha_{D_i}(T_j) + m\alpha_{B_k}(T_j) - \alpha_{B_i'}(T_j) = \alpha_{B_i}(T_j) + m\alpha_{B_k}(T_j) - \alpha_{B_i'}(T_j),$$

since $B_i = D_i \oplus T_k^m$. The last expression says that u is equal to the number of arrows from j to i in Q_T, minus the number of arrows from i to j, plus the product of the number of arrows from j to k and from k to i. This is what is required for having the Fomin–Zelevinsky mutation, so we are done. $\square$

We shall use the term 'stably 2-CY' also for certain subcategories of triangulated categories. Let $\mathcal{B}$ be a functorially finite extension-closed subcategory of a triangulated 2-CY category $\mathcal{C}$. We say that $X \in \mathcal{B}$ is *projective* in $\mathcal{B}$ if $\operatorname{Hom}(X, \mathcal{B}[1]) = 0$. In this setting, we shall prove in Theorem II.2.1 that the category $\mathcal{B}$ modulo projectives in $\mathcal{B}$ has a 2-CY triangulated structure. We then say that $\mathcal{B}$ is *stably 2-CY*. Note that $\mathcal{B}$ does not necessarily have enough projectives or injectives, for example, if $\mathcal{B} = \mathcal{C}$.

We illustrate this concept with the following example.

EXAMPLE. Let $\mathcal{C}_Q$ be the cluster category of the path algebra KQ, where Q is the quiver $\overset{1}{\cdot}\!\rightarrow\!\overset{2}{\cdot}\!\rightarrow\!\overset{3}{\cdot}$. We have the following AR-quiver for $\mathcal{C}_Q$, where S_i and P_i denote the simple and projective modules associated with vertex i, respectively.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & P_1 \arrow[dr] & & S_3[1] \arrow[dr] & & S_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & P_1/S_3 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & P_2 \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ur] & & S_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_1
\end{tikzcd}$$

Then $\mathcal{B} = \text{mod}\ KQ$ is an extension-closed subcategory of $\mathcal{C}_Q$, and it is easy to see that P_1 is the only indecomposable projective object in $\mathcal{B}$. It is then clear that $\mathcal{B}\,/P_1$ is equivalent to the cluster category $\mathcal{C}_{Q'}$, where Q' is a quiver of type A_2, which is a triangulated 2-CY category. Hence $\mathcal{B}$ is stably 2-CY.

Besides the cluster-tilting objects, the maximal rigid objects have also played an important role in the investigation of 2-CY categories. We now investigate the concepts of cluster structure and weak cluster structure with respect to these objects.

Recall that a subcategory T of a category $\mathcal{C}$ is said to be *rigid* if $\text{Ext}^1(M, M) = 0$ for all M in T, and *maximal rigid* if T is maximal among rigid subcategories [GLS06]. It is clear that any cluster-tilting subcategory is maximal rigid, but the converse is not true [BIKR08]. There always exists a maximal rigid subcategory in $\mathcal{C}$ if the category $\mathcal{C}$ is skeletally small, while the existence of a cluster-tilting subcategory is rather restrictive. It is of interest to obtain sufficient conditions for the two concepts to coincide. For this purpose, the following result is useful; see [BMR07, Iya07a, KR07] for statement (a) and the argument in [GLS06, Corollary 5.2] for statement (b).

PROPOSITION II.1.7. *Let $\mathcal{C}$ be a triangulated (or exact stably) 2-CY category.*

(a) *Let T be a cluster-tilting subcategory. Then for any X in $\mathcal{C}$, there exist triangles (or short exact sequences) $T_1 \to T_0 \to X$ and $X \to T_0' \to T_1'$ with T_i, T_i' in T.*
(b) *Let T be a functorially finite maximal rigid subcategory. Then for any X in $\mathcal{C}$ which is rigid, the same conclusion as in (a) holds.*

We then have the following theorem.

THEOREM II.1.8. *Let $\mathcal{C}$ be an exact stably 2-CY category with some cluster-tilting object. Then:*

(a) *any maximal rigid object in $\mathcal{C}$ (respectively, $\underline{\mathcal{C}}$) is a cluster-tilting object;*
(b) *any rigid subcategory in $\mathcal{C}$ (respectively, $\underline{\mathcal{C}}$) has an additive generator which is a direct summand of a cluster-tilting object;*
(c) *all cluster-tilting objects in $\mathcal{C}$ (respectively, $\underline{\mathcal{C}}$) have the same number of non-isomorphic indecomposable summands.*

Proof. (a) Let N be maximal rigid in $\mathcal{C}$. We only have to show that any $X \in \mathcal{C}$ satisfying $\text{Ext}^1(N, X) = 0$ is contained in add N.

(i) Let M be a cluster-tilting object in $\mathcal{C}$. Since N is maximal rigid and M is rigid, by Proposition II.1.7(b) there exists an exact sequence $0 \to N_1 \to N_0 \to M \to 0$ with $N_i \in \text{add}\ N$. In particular, we have $\text{pd}_{\text{End}(N)}\text{Hom}(N, M) \leq 1$.

(ii) Since M is cluster-tilting, there is, by Proposition II.1.7(a), an exact sequence $0 \to X \to M_0 \to M_1 \to 0$ for X as above, with $M_i \in \text{add}\ M$, obtained by taking the minimal left

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add M-approximation $X \to M_0$. Applying $(N, \ )$, we have an exact sequence $0 \to (N, X) \to (N, M_0) \to (N, M_1) \to \mathrm{Ext}^1(N, X) = 0$. By (i), $\mathrm{pd}_{\mathrm{End}(N)} \mathrm{Hom}(N, X) \leq 1$. Take a projective resolution $0 \to (N, N_1) \to (N, N_0) \to (N, X) \to 0$; then we have a complex

$$0 \to N_1 \to N_0 \to X \to 0 \tag{1}$$

in $\mathcal{C}$. Since $0 \to (P, N_1) \to (P, N_0) \to (P, X) \to 0$ is exact for any projective P in $\mathcal{C}$, it follows from the axioms of Frobenius categories that the complex (1) is an exact sequence in $\mathcal{C}$. Since $\mathrm{Ext}^1(X, N) = 0$, we have $X \in \mathrm{add}\ N$ and hence N is cluster-tilting.

(b) Let M be a cluster-tilting object in $\mathcal{C}$ and N a rigid object in $\mathcal{C}$. By [Iya07b, Lemma 5.3.1], $\mathrm{Hom}(M, N)$ is a partial tilting $\mathrm{End}(M)$-module. In particular, the number of non-isomorphic indecomposable direct summands of N is not greater than that of M. Consequently, any rigid object in $\mathcal{C}$ is a direct summand of some maximal rigid object in $\mathcal{C}$, which is cluster-tilting by (a).

(c) See [Iya07b, Corollary 5.3.3]. □

For a triangulated 2-CY category we also get a weak cluster structure, and sometimes a cluster structure, determined by the maximal rigid objects, if there are any. Note that there are cases where the maximal rigid objects are not cluster-tilting [BIKR08], but we suspect the following.

Conjecture II.1.9. Let $\mathcal{C}$ be a connected triangulated 2-CY category. Then any maximal rigid object without loops or 2-cycles in its quiver is a cluster-tilting object.

Furthermore, we have the next theorem.

Theorem II.1.10. *Let $\mathcal{C}$ be a triangulated 2-CY category (or exact stably 2-CY category) having some functorially finite maximal rigid subcategory.*

(a) *The functorially finite maximal rigid subcategories determine a weak cluster structure on $\mathcal{C}$.*
(b) *If there are no loops or 2-cycles for the functorially finite maximal rigid subcategories, then they determine a cluster structure on $\mathcal{C}$.*

Proof. (a) This follows from [IY08, Theorems 5.1 and 5.3]. The arguments there are stated only for cluster-tilting subcategories; however, they work also for functorially finite maximal rigid subcategories.

(b) The proof of Theorem II.1.6 works in this setting as well. □

There exist triangulated or exact categories with cluster-tilting objects also when the categories are not 2-CY or exact stably 2-CY (see [EH08, Iya07a, KZ08]), but we do not necessarily have even a weak cluster structure in such a situation. To see this, let Λ be a Nakayama algebra that has two simple modules S_1 and S_2 with associated projective covers P_1 and P_2. Assume first that P_1 and P_2 have length three; then in mod Λ we have that $S_1 \oplus P_1 \oplus P_2$, $\substack{S_1 \\ S_2} \oplus P_1 \oplus P_2$ and $\substack{S_2 \\ S_1} \oplus P_1 \oplus P_2$ are the cluster-tilting objects, so we do not have the unique exchange property. If P_1 and P_2 have length four, then the cluster-tilting objects are $S_1 \oplus \substack{S_1 \\ S_2 \\ S_1} \oplus P_1 \oplus P_2$ and $S_2 \oplus \substack{S_2 \\ S_1 \\ S_2} \oplus P_1 \oplus P_2$, so there is no way of exchanging S_1 in the first object to obtain a new cluster-tilting object.

We end this subsection with some information on the endomorphism algebras of cluster-tilting objects in exact stably 2-CY categories, to be used in § III.5. Such algebras are studied as analogs of Auslander algebras in [GLS06, Iya07a, Iya07b, KR07]. We denote by mod $\mathcal{C}$

the category of finitely presented $\mathcal{C}$-modules. If $\mathcal{C}$ has pseudokernels, then mod $\mathcal{C}$ forms an abelian category [Aus66].

Proposition II.1.11. *Let $\mathcal{C}$ be an exact stably 2-CY category. Assume that $\mathcal{C}$ has pseudokernels and the global dimension of* $\operatorname{mod}\mathcal{C}$ *is finite. Let $\Gamma = \operatorname{End}(T)$ for a cluster-tilting object T in $\mathcal{C}$.*

(a) *Γ has finite global dimension.*

(b) *If $\mathcal{C}$ is Hom-finite, then the quiver of Γ has no loops. If, moreover, $\mathcal{C}$ is an extension-closed subcategory of an abelian category closed under subobjects, then the quiver of Γ has no 2-cycles.*

Proof. (a) Let $m = \text{gl.dim}(\text{mod } \mathcal{C})$. For any $X \in \text{mod } \Gamma$, take a projective presentation $(T, T_1) \to (T, T_0) \to X \to 0$. By our assumptions, there exists a complex $0 \to F_m \to \cdots \to F_2 \to T_1 \to T_0$ in $\mathcal{C}$ such that $0 \to (\ , F_m) \to \cdots \to (\ , F_2) \to (\ , T_1) \to (\ , T_0)$ is exact in mod $\mathcal{C}$. Since T is cluster-tilting, we have an exact sequence $0 \to T_1 \to T_0 \to F_i \to 0$, with T_1 and T_0 in add T by Proposition II.1.7. Hence we have $\text{pd}_\Gamma(T, F_i) \leq 1$ and, consequently, $\text{pd}_\Gamma X \leq m + 1$. It follows that Γ has finite global dimension.

(b) For the first assertion, Γ is a finite-dimensional algebra of finite global dimension by part (a). Then, by [Len69, Igu90], the quiver of Γ has no loops. We now show the second assertion; our proof is based on [GLS06, Theorem 6.4]. We start by showing that $\text{Ext}^2_\Gamma(S, S) = 0$ for any simple Γ-module S, which is assumed to be the top of the projective Γ-module (T, M) for an indecomposable summand M of T.

First, we suppose that M is not projective in $\mathcal{C}$. Take exact exchange sequences $0 \to M^* \xrightarrow{f} B \xrightarrow{g} M \to 0$ and $0 \to M \xrightarrow{s} B' \xrightarrow{t} M^* \to 0$. Since Γ has no loops, we have a projective presentation $0 \to (T, M) \xrightarrow{\cdot s} (T, B') \xrightarrow{\cdot tf} (T, B) \xrightarrow{\cdot g} (T, M) \to S \to 0$ of the Γ-module S. Since M is not a summand of B', we have $\text{Ext}^2_\Gamma(S, S) = 0$.

Next, we suppose that M is projective in $\mathcal{C}$. Take a minimal projective presentation $(T, B) \xrightarrow{\cdot g} (T, M) \to S \to 0$ of the Γ-module S. By assumption, $\operatorname{Im} g$ in the abelian category belongs to $\mathcal{C}$. Then $g\colon B \to \operatorname{Im} g$ is a minimal right add T-approximation. By Proposition II.1.7(a), we have that $B' = \operatorname{Ker} g$ belongs to add T. Thus we have a projective resolution $0 \to (T, B') \to (T, B) \xrightarrow{\cdot g} (T, M) \to S \to 0$ of the Γ-module S. Since g is right minimal, B' does not have an injective summand. Thus $\text{Ext}^2_\Gamma(S, S) = 0$.

Since $\text{Ext}^2_\Gamma(S, S) = 0$ in both cases, by [GLS06, Proposition 3.11] we cannot have a 2-cycle. □

II.2 Substructures

For extension-closed subcategories of triangulated or exact categories having a weak cluster structure, we introduce the notion of substructure. Making heavy use of [IY08], we give sufficient conditions for having a substructure when starting with a triangulated 2-CY category or an exact stably 2-CY category and using the cluster-tilting subcategories with the indecomposable projectives as coefficients.

Let $\mathcal{C}$ be an exact or triangulated K-category, and let $\mathcal{B}$ be a subcategory of $\mathcal{C}$ closed under extensions. Assume that both $\mathcal{C}$ and $\mathcal{B}$ have a weak cluster structure. We say that we have a *substructure* of $\mathcal{C}$ induced by an extended cluster T in $\mathcal{B}$ if the following holds.

There is a set A of indecomposable objects in $\mathcal{C}$ such that $\widetilde{T}' = T' \cup A$ is an extended cluster in $\mathcal{C}$ for any extended cluster T' in $\mathcal{B}$ which is obtained by a finite number of exchanges from T.

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Note that for each sequence of cluster variables $M_1, \ldots, M_t$, with M_{i+1} in $\mu_{M_i}(T)$, we have $\mu_{M_t}(\cdots \mu_{M_1}(T)) \cup A = \tilde{\mu}_{M_t}(\cdots \tilde{\mu}_{M_1}(\widetilde{T}))$, where μ denotes the exchange for $\mathcal{B}$ and $\tilde{\mu}$ the exchange for $\mathcal{C}$.

We shall investigate substructures arising from certain extension-closed subcategories of triangulated 2-CY categories and of exact stably 2-CY categories. We begin with the triangulated case, and we first recall some results from [IY08] that are specialized to the setting of 2-CY categories.

For a triangulated category $\mathcal{C}$ and full subcategories $\mathcal{B}$ and $\mathcal{B}'$, let $\mathcal{B}^\perp = \{X \in \mathcal{C} \mid \mathrm{Hom}(\mathcal{B}, X) = 0\}$ and ${}^\perp\mathcal{B} = \{X \in \mathcal{C} \mid \mathrm{Hom}(X, \mathcal{B}) = 0\}$. We denote by $\mathcal{B} * \mathcal{B}'$ the full subcategory of $\mathcal{C}$ consisting of all $X \in \mathcal{C}$ such that there exists a triangle $B \to X \to B' \to B[1]$ with $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$.

We get the following sufficient conditions for constructing 2-CY categories and, hence, categories with weak cluster structures.

THEOREM II.2.1. *Let $\mathcal{C}$ be a triangulated 2-CY category and $\mathcal{B}$ a functorially finite extension-closed subcategory of $\mathcal{C}$.*

(a) *$\mathcal{B}^\perp$ and ${}^\perp\mathcal{B}$ are functorially finite extension-closed subcategories of $\mathcal{C}$. Moreover, the equalities $\mathcal{B} * \mathcal{B}^\perp = \mathcal{C} = {}^\perp\mathcal{B} * \mathcal{B}$ and ${}^\perp(\mathcal{B}^\perp) = \mathcal{B} = ({}^\perp\mathcal{B})^\perp$ hold.*

(b) *Let $\mathcal{D} = \mathcal{B} \cap {}^\perp\mathcal{B}[1]$. Then $\mathcal{B} / \mathcal{D}$ is a triangulated 2-CY category, and so $\mathcal{B}$ is a stably 2-CY category. Moreover, $\mathcal{B} \subseteq (\mathcal{D} * \mathcal{B}[1]) \cap (\mathcal{B}[-1] * \mathcal{D})$ holds, and $\mathcal{D}$ is a functorially finite rigid subcategory of $\mathcal{C}$.*

(c) *Let $\mathcal{D}$ be a functorially finite rigid subcategory of $\mathcal{C}$ and let $\mathcal{B}' = {}^\perp\mathcal{D}[1]$. Then $\mathcal{B}'$ is a functorially finite extension-closed subcategory of $\mathcal{C}$ and $\mathcal{B}'/\mathcal{D}$ is a triangulated 2-CY category. Moreover, there exists a one-to-one correspondence between cluster-tilting (respectively, maximal rigid or rigid) subcategories of $\mathcal{C}$ containing $\mathcal{D}$ and cluster-tilting (respectively, maximal rigid or rigid) subcategories of $\mathcal{B}'/\mathcal{D}$. It is given by $T \mapsto T/\mathcal{D}$.*

Proof. (a) Since $\mathcal{B}^\perp = {}^\perp\mathcal{B}[2]$ holds by the 2-CY property, the assertion follows from [IY08, Proposition 2.3].

(b) Clearly, $\mathcal{B} / \mathcal{D}$ is Hom-finite since $\mathcal{C}$ is. To show that $\mathcal{B} / \mathcal{D}$ is a triangulated 2-CY category, we need only check, using [IY08, Theorem 4.2], that $\mathcal{B} \subseteq (\mathcal{D} * \mathcal{B}[1]) \cap (\mathcal{B}[-1] * \mathcal{D})$. Let Z be in $\mathcal{B}$. Since $\mathcal{B}$, and hence $\mathcal{B}[1]$, is functorially finite in $\mathcal{C}$, it follows from (a) that we have a triangle $X \to Y \to Z \to X[1]$ with Y in ${}^\perp\mathcal{B}[1]$ and $X[1]$ in $\mathcal{B}[1]$. Since $\mathcal{B}$ is extension-closed, Y is in $\mathcal{B}$, and consequently Y is in $\mathcal{B} \cap {}^\perp\mathcal{B}[1] = \mathcal{D}$. It follows that Z is in $\mathcal{D} * \mathcal{B}[1]$ and, similarly, in $\mathcal{B}[-1] * \mathcal{D}$.

To see that $\mathcal{D}$ is functorially finite in $\mathcal{C}$, we only have to show that $\mathcal{D}$ is functorially finite in $\mathcal{B}$. For any $Z \in \mathcal{B}$, take the above triangle $X \to Y \xrightarrow{f} Z \to X[1]$ with Y in $\mathcal{D}$ and $X[1]$ in $\mathcal{B}[1]$. Since $(\mathcal{D}, X[1]) = 0$, we have that f is a right $\mathcal{D}$-approximation. Thus $\mathcal{D}$ is contravariantly finite in $\mathcal{B}$ and, similarly, covariantly finite in $\mathcal{B}$.

(c) See [IY08, 4.9]. $\square$

The example from the previous section of the cluster category $\mathcal{C}$ of the path algebra KQ, where Q is of type A_3, illustrates part of this theorem. Let $\mathcal{D} = \mathrm{add}\ P_1$, then $\mathcal{B}' = {}^\perp\mathcal{D}[1] = \mathrm{mod}\ KQ$ and $\mathcal{B}' / \mathcal{D} = \mathcal{C}_{KQ'}$, where Q' is a quiver of type A_2. The cluster-tilting objects in $\mathcal{C}$ containing P_1 are $P_1 \oplus S_3 \oplus P_2$, $P_1 \oplus P_2 \oplus S_2$, $P_1 \oplus S_2 \oplus P_1/S_3$, $P_1 \oplus P_1/S_3 \oplus S_1$ and$P_1 \oplus S_1 \oplus S_3$, which are in one-to-one correspondence with the cluster-tilting objects in $\mathcal{B}' / \mathcal{D}$.

In order to get sufficient conditions for having a substructure, we investigate cluster-tilting subcategories in $\mathcal{B}$. For this, the following lemma is useful.

Lemma II.2.2. *Let $\mathcal{C}$ be a triangulated 2-CY category. For any functorially finite and thick subcategory $\mathcal{C}_1$ of $\mathcal{C}$, there exists a functorially finite and thick subcategory $\mathcal{C}_2$ of $\mathcal{C}$ such that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$.*

Proof. Let $\mathcal{C}_2 = \mathcal{C}_1^{\perp}$. Then we have $\mathcal{C}_2 = \mathcal{C}_1^{\perp} = {}^{\perp}\mathcal{C}_1[2] = {}^{\perp}\mathcal{C}_1$ by Serre duality, using the fact that $\mathcal{C}_1$ is triangulated. We only have to show that any object in $\mathcal{C}$ is a direct sum of objects in $\mathcal{C}_1$ and $\mathcal{C}_2$. For any $X \in \mathcal{C}$, there exists a triangle $A_1 \to X \to A_2 \xrightarrow{f} A_1[1]$ in $\mathcal{C}$ with A_1 in $\mathcal{C}_1$ and A_2 in $\mathcal{C}_2 = \mathcal{C}_1^{\perp}$, by Theorem II.2.1(a). Since $f = 0$, we have $X \simeq A_1 \oplus A_2$. Thus $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$. □

Using Lemma II.2.2, we get the following decomposition of triangulated categories.

Proposition II.2.3. *Let $\mathcal{C}$ be a triangulated 2-CY category and $\mathcal{B}$ a functorially finite extension-closed subcategory of $\mathcal{C}$. Let $\mathcal{D} = \mathcal{B} \cap {}^{\perp}\mathcal{B}[1]$ and $\mathcal{B}' = {}^{\perp}\mathcal{D}[1]$.*

(a) *There exists a functorially finite and extension-closed subcategory $\mathcal{B}''$ of $\mathcal{C}$ such that $\mathcal{D} \subseteq \mathcal{B}'' \subseteq \mathcal{B}'$ and $\mathcal{B}' / \mathcal{D} = \mathcal{B} / \mathcal{D} \times \mathcal{B}'' / \mathcal{D}$ as a triangulated category.*

(b) *There exists a one-to-one correspondence between pairs consisting of cluster-tilting (respectively, maximal rigid or rigid) subcategories of $\mathcal{B}$ and of $\mathcal{B}''$ and cluster-tilting (respectively, maximal rigid or rigid) subcategories of $\mathcal{B}'$. It is given by $(T, T'') \mapsto T \oplus T''$.*

Proof. (a) We know from parts (b) and (c) of Theorem II.2.1 that $\mathcal{D}$ is a functorially finite rigid subcategory and that $\mathcal{B} / \mathcal{D}$ and $\mathcal{B}' / \mathcal{D}$ are both triangulated 2-CY categories. The inclusion functor $\mathcal{B} / \mathcal{D} \to \mathcal{B}' / \mathcal{D}$ is a triangle functor by the construction of their triangulated structures in [IY08, Theorem 4.2]. In particular, $\mathcal{B} / \mathcal{D}$ is a thick subcategory of $\mathcal{B}' / \mathcal{D}$; hence we have a decomposition by Lemma II.2.2.

(b) This follows by Theorem II.2.1(c). □

We then obtain the following.

Corollary II.2.4. *Let $\mathcal{C}$ be a 2-CY algebraic triangulated category with a cluster-tilting object, and let $\mathcal{B}$ be a functorially finite extension-closed subcategory of $\mathcal{C}$. Then we have the following.*

(a) *The stably 2-CY category $\mathcal{B}$ also has some cluster-tilting object. Any maximal rigid object in $\mathcal{B}$ is a cluster-tilting object in $\mathcal{B}$.*

(b) *There is some rigid object A in $\mathcal{C}$ such that $T \oplus A$ is a cluster-tilting object in $\mathcal{C}$ for any cluster-tilting object T in $\mathcal{B}$.*

(c) *Any cluster-tilting object T in $\mathcal{B}$ determines a substructure for the weak cluster structures on $\mathcal{B}$ and $\mathcal{C}$ given by cluster-tilting objects.*

Proof. (a) Let $\mathcal{D} = {}^{\perp}\mathcal{B}[1]$ and $\mathcal{B}' = {}^{\perp}\mathcal{D}[1]$. Since $\mathcal{C}$ is algebraic, by Theorem II.1.8 we have a cluster-tilting object T in $\mathcal{C}$ containing $\mathcal{D}$. By Proposition II.2.3, we have the decompositions $\mathcal{B}' / \mathcal{D} = \mathcal{B} / \mathcal{D} \times \mathcal{B}'' / \mathcal{D}$, for some subcategory $\mathcal{B}''$ of $\mathcal{B}'$, and $T = T_1 \oplus T_2$, with cluster-tilting objects T_1 in $\mathcal{B}$ and T_2 in $\mathcal{B}''$. Thus $\mathcal{B}$ has a cluster-tilting object.

Now we show the second assertion in (a). Let M be maximal rigid in $\mathcal{B}$. By Proposition II.2.3(b), $M \oplus T_2$ is maximal rigid in $\mathcal{C}$. From Theorem II.1.8 it follows that $M \oplus T_2$ is cluster-tilting in $\mathcal{C}$ and, by Proposition II.2.3(b), we have that M is cluster-tilting in $\mathcal{B}$.

(b) We only have to take $A = T_2$.

(c) This follows from part (b). □

It is curious to note that by combining Proposition II.2.3 with Theorem II.2.1, we obtain a kind of classification of functorially finite extension-closed subcategories of a triangulated 2-CY category in terms of functorially finite rigid subcategories, analogous to results from [AR91].

THEOREM II.2.5. *Let $\mathcal{C}$ be a 2-CY triangulated category. Then the functorially finite extension-closed subcategories $\mathcal{B}$ of $\mathcal{C}$ are all obtained as preimages under the functor $\pi\colon \mathcal{C} \to \mathcal{C}\,/\,\mathcal{D}$ of the direct summands of ${}^{\perp}\mathcal{D}[1]/\,\mathcal{D}$ as a triangulated category, for functorially finite rigid subcategories $\mathcal{D}$ of $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ be a functorially finite rigid subcategory in $\mathcal{C}$. Then $\mathcal{B}' = {}^{\perp}\mathcal{D}[1]$ is functorially finite extension-closed in $\mathcal{C}$, by Theorem II.2.1(a). The preimage under $\pi\colon \mathcal{C} \to \mathcal{C}\,/\,\mathcal{D}$ of any direct summand of $\mathcal{B}'\,/\,\mathcal{D}$ as a triangulated category is therefore functorially finite and extension-closed in $\mathcal{C}$.

Conversely, let $\mathcal{B}$ be a functorially finite extension-closed subcategory of $\mathcal{C}$ and let $\mathcal{D} = \mathcal{B} \cap {}^{\perp}\mathcal{B}[1]$. By Proposition II.2.3, we have that $\mathcal{B}\,/\,\mathcal{D}$ is a direct summand of ${}^{\perp}\mathcal{D}[1]/\,\mathcal{D}$. □

We now investigate substructures for exact categories which are stably 2-CY. We have the following main result.

THEOREM II.2.6. *Let $\mathcal{C}$ be an exact stably 2-CY category, and let $\mathcal{B}$ be a functorially finite extension-closed subcategory of $\mathcal{C}$. Then $\mathcal{B}$ has enough projectives and injectives and is an exact stably 2-CY category.*

Proof. We know that $\mathcal{B}$ is an exact category and that $\mathcal{D} = \mathcal{B} \cap {}^{\perp}\mathcal{B}[1]$ is the subcategory of projective injective objects. Since $\mathcal{B} \subseteq \mathcal{B}[-1] * \mathcal{D}$ holds by Theorem II.2.1(b), for any $X \in \mathcal{B}$ there exists a triangle $X \to Y \to Z \to X[1]$ with $Y \in \mathcal{D}$ and $Z \in \mathcal{B}$. This is induced from an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$. Thus $\mathcal{B}$ has enough injectives. Dually, then, $\mathcal{B}$ has enough projectives, which coincide with the injectives. Hence $\mathcal{B}$ is a Frobenius category, and consequently $\mathcal{B}$ is exact stably 2-CY. □

We have the following interesting special case as a consequence of Theorem II.2.6 and Corollary II.2.4. For X in $\mathcal{C}$, we denote by $\operatorname{Sub} X$ the subcategory of $\mathcal{C}$ whose objects are subobjects of finite direct sums of copies of X.

COROLLARY II.2.7. *Let $\mathcal{C}$ be a Hom-finite abelian stably 2-CY category, and let X be an object in $\mathcal{C}$ with $\operatorname{Ext}^1(X, X) = 0$ and $\operatorname{id} X \leq 1$. Then the following hold.*

(a) $\operatorname{Sub} X$ *is a functorially finite extension-closed subcategory of $\mathcal{C}$ and is exact stably 2-CY.*
(b) *If $\mathcal{C}$ has a cluster-tilting object, then so does* $\operatorname{Sub} X$, *and any cluster-tilting object in* $\operatorname{Sub} X$ *determines a substructure of the cluster structure for $\mathcal{C}$.*
(c) *If $\mathcal{C}$ is abelian, then* $\operatorname{Sub} X$ *has no loops or 2-cycles.*

Proof. One can show that $\operatorname{Sub} X$ is extension-closed by using the assumption that $\operatorname{id} X \leq 1$. It is functorially finite by [AS80], so we can use Theorem II.2.6. Furthermore, (b) follows from Corollary II.2.4 and (c) follows from Proposition II.1.11. □

In order to see when we have cluster structures, we would like to give sufficient conditions for algebraic triangulated (or stably) 2-CY categories not to have loops or 2-cycles.

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Proposition II.2.8. *Let $\mathcal{C}$ be an algebraic triangulated (or exact stably) 2-CY category with a cluster-tilting object, and let $\mathcal{B}$ be a functorially finite extension-closed subcategory.*

(a) *If $\mathcal{C}$ has no 2-cycles, then $\mathcal{B}$ also has no 2-cycles.*

(b) *If $\mathcal{C}$ has no loops, then $\mathcal{B}$ also has no loops.*

Proof. We give a proof for the algebraic triangulated 2-CY case. A similar argument works for the exact stably 2-CY case.

(a) Let $\mathcal{D} = \mathcal{B} \cap^{\perp} \mathcal{B}[1]$ and $\mathcal{B}' = {}^{\perp}\mathcal{D}[1]$. Since cluster-tilting objects in $\mathcal{B}'$ are exactly cluster-tilting objects in $\mathcal{C}$ which contain $\mathcal{D}$, our assumption implies that $\mathcal{B}'$ has no 2-cycles.

We now show that $\mathcal{B}$ has no 2-cycles. Let T be a cluster-tilting object in $\mathcal{B}$. By Corollary II.2.4(b), there exists $T' \in \mathcal{B}'$ such that $T \oplus T'$ is a cluster-tilting object in $\mathcal{C}$. We already observed that $T \oplus T'$ has no 2-cycles. If T has a 2-cycle, then at least one arrow in the 2-cycle represents a morphism $f\colon X \to Y$ which factors through an object in T'. We write f as a composition of $f_1\colon X \to Z$ and $f_2\colon Z \to Y$ with $Z \in T'$. Since $\mathcal{B}\,/\,\mathcal{D}$ is a direct summand of $\mathcal{B}'\,/\,\mathcal{D}$ by Proposition II.2.3, any morphism between T and T' factors through $\mathcal{D}$. Thus we can write f_1 (respectively, f_2) as a composition of $g_1\colon X \to W_1$ and $h_1\colon W_1 \to Z$ (respectively, of $g_2\colon Z \to W_2$ and $h_2\colon W_2 \to Y$) with $W_1 \in \mathcal{D}$ (respectively, $W_2 \in \mathcal{D}$). We have $f = f_1 f_2 = g_1(h_1 g_2)h_2$, where $h_1 g_2$ is in $\operatorname{rad}\mathcal{B}$ and at least one of h_2 and g_1 is in $\operatorname{rad}\mathcal{B}$, since at least one of X and Y is not in $\mathcal{D}$. So f cannot be irreducible in add T, which is a contradiction.

(b) This is proved in a similar way. □

Note that the quiver Q_T may have 2-cycles between coefficients. For example, let $\mathcal{C} = \operatorname{mod}\ \Lambda$ for the preprojective algebra of a Dynkin quiver and let $\mathcal{B}$ be the subcategory add Λ. Then there are no 2-cycles for $\mathcal{C}$, but there are 2-cycles for $\mathcal{B}$ since Λ is the only cluster-tilting object in $\mathcal{B}$.

II.3 Preprojective algebras of Dynkin type

In this subsection we specialize our general results from § II.2 to the case of finitely generated modules over a preprojective algebra of Dynkin type; we illustrate this with three concrete examples. The same examples will be used in § IV to show how to use this theory to construct subcluster algebras of cluster algebras.

The category $\mathcal{C} = \operatorname{mod}\ \Lambda$ for Λ preprojective of Dynkin type is a Hom-finite Frobenius category. By [GLS06] (see also § III.2), a rigid Λ-module is cluster-tilting if and only if the number of non-isomorphic indecomposable summands is the number of positive roots, i.e. $(n(n+1))/2$ for A_n, $n(n-1)$ for D_n, 36 for E_6, 63 for E_7 and 120 for E_8.

Let $\mathcal{B}$ be an extension-closed functorially finite subcategory of $\mathcal{C}$. We know that $\mathcal{B}$ is exact stably 2-CY by Theorem II.2.6. It is known, too, that $\mathcal{C}$ and $\underline{\mathcal{C}}$ have no loops or 2-cycles for the cluster-tilting objects (see [GLS06]; this also follows from Proposition II.1.11). Then, by Proposition II.2.8, there are also no loops or 2-cycles for $\mathcal{B}$ and the subcategory $\underline{\mathcal{B}}$ of $\underline{\mathcal{C}}$. Note that $\underline{\mathcal{B}}$ is not the stable category of $\mathcal{B}$ since $\mathcal{B}$ may have more projectives than $\mathcal{C}$.

We then have the following result.

Theorem II.3.1. *Let $\mathcal{B}$ be an extension-closed functorially finite subcategory of the category $\mathcal{C} = \operatorname{mod}\ \Lambda$ for the preprojective algebra Λ of a Dynkin quiver. Then the following hold.*

(a) *The exact stably 2-CY category $\mathcal{B}$ has a cluster-tilting object, and any maximal rigid object in $\mathcal{B}$ is a cluster-tilting object that can be extended to a cluster-tilting object for $\mathcal{C}$ and which gives rise to a substructure.*

(b) *The category $\underline{\mathcal{B}}$ is a triangulated 2-CY category with no loops or 2-cycles for the cluster-tilting objects; hence it has a cluster structure.*

Proof. (a) This follows from Theorem II.1.8 and Corollary II.2.7.

(b) This follows from the above comments and Theorem II.1.6. $\square$

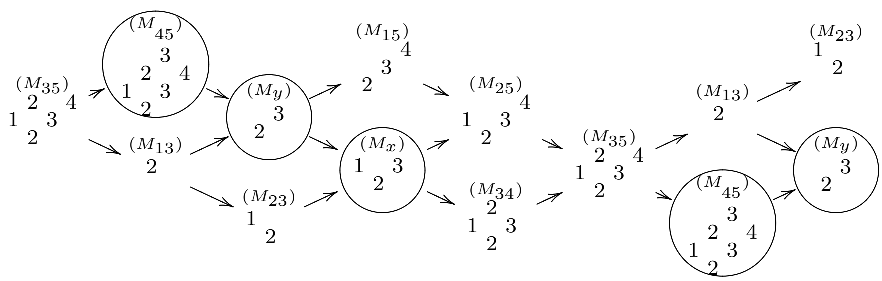
We now give some concrete examples of weak cluster structures and substructures. In § IV, these examples will be revisited and used to model cluster algebras and subcluster algebras.

We let P_i denote the indecomposable projective module associated to vertex i, J the radical of a ring, and S_i the simple top of P_i. Usually, we represent a module M by its radical filtration, with the numbers in the first row representing the indices of the simples in M/JM and the numbers in the ith row representing the indices of the simples in $J^{i-1}M/J^iM$. For example, $\begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}$ represents the indecomposable projective module P_2 for the preprojective algebra of type A_3, which has quiver $1 \rightleftarrows 2 \rightleftarrows 3$.

Example 1. Let Λ be the preprojective algebra of a Dynkin quiver A_4. This algebra has quiver $1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4$. Consider the modules P_3 and $M = JP_3$; these are represented by their radical filtrations

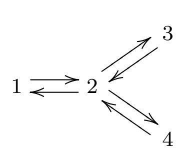
$$\begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}$$

Let $\mathcal{C}' = \operatorname{Sub} P_3$ and $\mathcal{B} = \{X \in \mathcal{C}' \mid \operatorname{Ext}^1(M, X) = 0\}$. The AR-quiver of $\mathcal{C}'$ is given below. The indexing of the indecomposables will be explained in § IV.2.



From the AR-quiver we see that the indecomposable projectives in $\mathcal{C}'$ are M_{45}, M_{34}, M_{23} and M_{15}. The indecomposables in $\mathcal{B}$ are obtained from $\mathcal{C}'$ by deleting the encircled indecomposable objects, and $\mathcal{B}$ is extension-closed by definition. Thus $T = M_{34} \oplus M_{23} \oplus M_{13} \oplus M_{15} \oplus M_{35}$ is a cluster-tilting object in $\mathcal{B}$, which has a cluster structure, with coefficients the indecomposable projectives M_{35}, M_{34}, M_{23} and M_{15}. It is easy to see that $T' = T \oplus M_{45}$ is a cluster-tilting object in $\mathcal{C}'$; hence $\mathcal{C}'$ has a cluster structure, with coefficients the indecomposable projectives, such that we have a substructure for $\mathcal{B}$ induced by T. One can also show that the cluster-tilting object T' in $\mathcal{C}'$ can be extended to a cluster-tilting object $\widetilde{T} = T' \oplus P_1 \oplus P_2 \oplus P_4 \oplus Z$ in mod Λ, where Z is the Λ-module with radical filtration $\begin{smallmatrix} 1 & & \\ & 2 & \\ & & 3 \end{smallmatrix}$, by first showing that $\operatorname{Ext}^1(\widetilde{T}, \widetilde{T}) = 0$ and then using the fact that $\widetilde{T}$ has the correct number $10 = (4 \cdot 5)/2$ of indecomposable direct summands.

Example 2. Let Λ be the preprojective algebra of a Dynkin quiver D_4:



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Using Corollary II.2.7, we see that the subcategory $\mathcal{B} = \operatorname{Sub} P_2$ is extension-closed. By Theorem II.3.1, $\mathcal{B}$ has a cluster-tilting object that can be extended to a cluster-tilting object for $\mathcal{C} = \operatorname{mod} \Lambda$.

The indecomposable submodules of P_2 for which we need to construct a cluster-tilting object have the following radical filtrations. The indexing will be explained in § IV.

P_2	$\begin{smallmatrix} &&2&& \\ 1&&3&&4 \\ &2&&2& \\ 1&&3&&4 \\ &&2&& \end{smallmatrix}$	M_{16}	$\begin{smallmatrix} 3&&4 \\ &2& \end{smallmatrix}$	M_{24}	$\begin{smallmatrix} 1&&3 \\ &2& \end{smallmatrix}$	M_{25}	$\begin{smallmatrix} 1&&4 \\ &2& \end{smallmatrix}$	M_{26}	$\begin{smallmatrix} 1&3&4 \\ &2& \end{smallmatrix}$
M_{68}	$\begin{smallmatrix} 1&&3&&4 \\ &2&&2& \\ 1&&3&&4 \\ &&2&& \end{smallmatrix}$	M_{18}	$\begin{smallmatrix} &1& \\ &2& \\ 3&&4 \\ &2& \end{smallmatrix}$	M_-	$\begin{smallmatrix} &4& \\ &2& \\ 1&&3 \\ &2& \end{smallmatrix}$	M_+	$\begin{smallmatrix} &3& \\ &2& \\ 1&&4 \\ &2& \end{smallmatrix}$		

Let T be the sum of the indecomposables in the above table and let $\widetilde{T} = T \oplus P_1 \oplus P_3 \oplus P_4$. It is easy to see that $\widetilde{T}$ is a cluster-tilting object in $\mathcal{C}$ and hence that T is a cluster-tilting object in $\mathcal{B}$. It follows that $\mathcal{B}$ has a substructure of the cluster structure of $\mathcal{C}$.

III. Preprojective algebras for non-Dynkin quivers

In this section we deal with completions of preprojective algebras of a finite connected quiver Q with no oriented cycles, mainly those which are not Dynkin. In this case, the modules of finite length coincide with the nilpotent modules over the preprojective algebra. These algebras Λ are known to be derived 2-CY (see [Boc08, CB00, BBK02, GLS07b]). Tilting Λ-modules of projective dimension at most one were investigated in [IR08] for when the quiver Q is a (generalized) extended Dynkin quiver. It was shown that such tilting modules are exactly the ideals in Λ which are finite products of two-sided ideals $I_i = \Lambda(1 - e_i)\Lambda$, where $e_1, \ldots, e_n$ correspond to the vertices of the quiver, and that they are in one-to-one correspondence with the elements of the corresponding Weyl group, where $w = s_{i_1} \cdots s_{i_k}$ corresponds to $I_w = I_{i_1} \cdots I_{i_k}$. Here we generalize some of the results from [IR08] beyond the noetherian case. In particular, we show that any finite product of ideals of the form I_i is a tilting module and, moreover, that there is a bijection between cofinite tilting ideals and elements of the associated Coxeter group W.

For any descending chain of tilting ideals of the form $\Lambda \supseteq I_{i_1} \supseteq I_{i_1} I_{i_2} \supseteq I_{i_1} I_{i_2} \cdots I_{i_k} \supseteq \cdots$ we show that for $\Lambda_m = \Lambda / I_{i_1} \cdots I_{i_m}$, the categories $\operatorname{Sub} \Lambda_m$ and $\underline{\operatorname{Sub}}\, \Lambda_m$ are, respectively, stably 2-CY and 2-CY with nice cluster-tilting objects. In this way we get, for any $w \in W$, a stably 2-CY category $\mathcal{C}_w = \operatorname{Sub}(\Lambda / I_w)$ and, for any reduced expression $w = s_{i_1} \cdots s_{i_k}$, a cluster-tilting object $\bigoplus_{j=1}^{k} \Lambda / I_{s_{i_1} \cdots s_{i_j}}$ in $\mathcal{C}_w$. We also construct cluster-tilting subcategories of the derived 2-CY category f. l. Λ. Thus we obtain many examples of weak cluster structures without loops or 2-cycles, which are then cluster structures by Theorem II.1.6. We also get many examples of substructures. In particular, any cluster category and the stable category $\underline{\operatorname{mod}}\, \Lambda$ of a preprojective algebra of Dynkin type occur amongst this class. We give a description of the quivers of the cluster-tilting objects or subcategories in terms of the associated reduced expressions. For example, the quiver of the preprojective component of the hereditary algebra with additional arrows from X to τX occurs in this manner. In § IV.3, a conjectural connection with coordinate rings of unipotent cells having a cluster algebra structure is given.

We refer to [Iya] for corresponding results for d-CY algebras.

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III.1 Tilting modules over preprojective algebras

Let Q be a finite connected quiver without oriented cycles which is not Dynkin. Let K be an algebraically closed field and Λ the completion of the associated preprojective algebra. In [IR08], the tilting Λ-modules of projective dimension at most one were investigated in the noetherian case, that is, when Q is extended Dynkin [BGL87] (and also the generalized case with loops). In this section we generalize some of these results to the non-noetherian case, concentrating on the aspects that will be needed later for our construction of new 2-CY categories with cluster-tilting objects or subcategories. Note that since Λ is complete, the Krull–Schmidt theorem holds for finitely generated projective Λ-modules.

We say that a finitely presented Λ-module T is a *tilting module* if: (i) there exists an exact sequence $0 \to P_n \to \cdots \to P_0 \to \Lambda \to 0$ with finitely generated projective Λ-modules P_i; (ii) $\operatorname{Ext}^i_\Lambda(T,T) = 0$ for any $i > 0$; and (iii) there exists an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_n \to 0$ with T_i in add T.

We say that $T \in \mathbf{D}(\operatorname{Mod}\Lambda)$ is a *tilting complex* [Ric89] if: (i′) T is quasi-isomorphic to an object in the category $\mathbf{K}^{\mathrm{b}}(\operatorname{pr}\Lambda)$ of bounded complexes of finitely generated projective Λ-modules $\operatorname{pr}\Lambda$; (ii′) $\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod}\ \Lambda)}(T, T[i]) = 0$ for any $i \neq 0$; and (iii′) T generates $\mathbf{K}^{\mathrm{b}}(\operatorname{pr}\Lambda)$.

A tilting module is none other than a module which is a tilting complex, since the condition (iii) can be replaced by (iii′). A *partial tilting complex* is a direct summand of a tilting complex. A *partial tilting module* is a module which is a partial tilting complex.

Let $1, \ldots, n$ denote the vertices in Q, and let $e_1, \ldots, e_n$ be the corresponding idempotents. For each i we denote by I_i the ideal $\Lambda(1 - e_i)\Lambda$. Then $S_i = \Lambda/I_i$ is a simple Λ-module and Λ^{op}-module, since by assumption there are no loops in the quiver. We shall show that each I_i, and any finite product of such ideals, is a tilting ideal in Λ, and we will give some information about how the different products are related. But first we present some preliminary results where new proofs are needed since, in contrast to [IR08], we do not assume Λ to be noetherian.

Lemma III.1.1. *Let T be a partial tilting Λ-module of projective dimension at most one, and let S be a simple Λ^{op}-module. Then at least one of the statements $S \otimes_\Lambda T = 0$ and $\operatorname{Tor}^\Lambda_1(S, T) = 0$ holds.*

Proof. We only have to show that there is a projective resolution $0 \to P_1 \to P_0 \to T \to 0$ such that P_0 and P_1 do not have a common summand. This is done as in [HU05, Lemma 1.2]. □

Recall that for rings Λ and Γ, we call an object T in $\mathbf{D}(\operatorname{Mod}\Lambda \otimes_{\mathbb{Z}} \Gamma^{\mathrm{op}})$ a *two-sided tilting complex* if T is a tilting complex in $\mathbf{D}(\operatorname{Mod}\Lambda)$ and $\operatorname{End}_{\mathbf{D}(\operatorname{Mod}\Lambda)}(T) \simeq \Gamma$ naturally.

The following result is useful (see [Ric89] and [Yek99, Corollary 1.7]).

Lemma III.1.2. *Let $T \in \mathbf{D}(\operatorname{Mod}\Lambda \otimes_{\mathbb{Z}} \Gamma^{\mathrm{op}})$ be a two-sided tilting complex.*

(a) *For any tilting complex (respectively, partial tilting complex) U of Γ, we have a tilting complex (respectively, partial tilting complex) $T \overset{\mathbf{L}}{\otimes}_\Gamma U$ of Λ such that $\operatorname{End}_{\mathbf{D}(\operatorname{Mod}\Lambda)}(T \overset{\mathbf{L}}{\otimes}_\Gamma U) \simeq \operatorname{End}_{\mathbf{D}(\operatorname{Mod}\Gamma)}(U)$.*

(b) *$\mathbf{R}\operatorname{Hom}_\Lambda(T, \Lambda)$ and $\mathbf{R}\operatorname{Hom}_{\Gamma^{\mathrm{op}}}(T, \Gamma)$ are two-sided tilting complexes and are isomorphic in $\mathbf{D}(\operatorname{Mod}\Gamma \otimes_{\mathbb{Z}} \Lambda^{\mathrm{op}})$.*

We collect some basic information on preprojective algebras.

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Proposition III.1.3. *Let Λ be the completion of the preprojective algebra of a finite connected non-Dynkin diagram without loops.*

(a) *Let Γ be the completion of $\Lambda \otimes_{KQ_0} \Lambda^{\mathrm{op}}$ with respect to the ideal $J \otimes_{KQ_0} \Lambda^{\mathrm{op}} + \Lambda \otimes_{KQ_0} J^{\mathrm{op}}$, where J is the radical of Λ. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & P_2 & \xrightarrow{f_2} & P_1 & \xrightarrow{f_1} & P_0 & \to & \Lambda & \to & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \to & \mathrm{Hom}_\Gamma(P_0,\Gamma) & \xrightarrow{f_1\cdot} & \mathrm{Hom}_\Gamma(P_1,\Gamma) & \xrightarrow{f_2\cdot} & \mathrm{Hom}_\Gamma(P_2,\Gamma) & \to & \Lambda & \to & 0 \end{array}$$

of exact sequences of Γ-modules such that each P_i is a finitely generated projective Γ-module and $P_0 \simeq P_2 \simeq \Gamma$.

(b) *There exists a functorial isomorphism $\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\,\Lambda)}(X, Y[1]) \simeq D\,\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\,\Lambda)}(Y, X[1])$ for any $X \in \mathbf{D}^{\mathrm{b}}(\mathrm{f.\,l.}\,\Lambda)$ and $Y \in \mathbf{K}^{\mathrm{b}}(\mathrm{pr}\,\Lambda)$.*

(c) $\mathrm{f.\,l.}\,\Lambda$ *is derived 2-CY and* $\mathrm{gl.\,dim}\,\Lambda = 2$*. In particular, any left ideal I of Λ satisfies* $\mathrm{pd}_{\,\Lambda} I \le 1$.

(d) $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0$ *for $i \neq 2$ and $\mathrm{Ext}^2_\Lambda(X, \Lambda) \simeq DX$ for any $X \in \mathrm{f.\,l.}\,\Lambda$.*

Proof. (a) See [GLS06, § 8] and [BBK02, § 4.1].

(b) This follows from (a) and [Boc08, Theorem 4.2].

(c) and (d) These follow immediately from (a) and (b). □

We are now ready to show that each I_i, and a finite product of such ideals, is a tilting module.

Proposition III.1.4. *I_i is a tilting Λ-module of projective dimension at most one and $\mathrm{End}_\Lambda(I_i) = \Lambda$.*

Proof. We have $\mathrm{Ext}^n_\Lambda(S_i, \Lambda) \simeq D\,\mathrm{Ext}^{2-n}_\Lambda(\Lambda, S_i) = 0$ for $n = 0, 1$ by Proposition III.1.3. Applying $\mathrm{Hom}_\Lambda(\ ,\Lambda)$ to the exact sequence $0 \to I_i \to \Lambda \to S_i \to 0$, we get $\mathrm{Hom}_\Lambda(I_i, \Lambda) = \Lambda$. Applying $\mathrm{Hom}_\Lambda(I_i,\ )$, we get an exact sequence $0 \to \mathrm{End}_\Lambda(I_i) \to \mathrm{Hom}_\Lambda(I_i, \Lambda) \to \mathrm{Hom}_\Lambda(I_i, S_i)$. Since $\mathrm{Hom}_\Lambda(I_i, S_i) = 0$, we have $\mathrm{End}_\Lambda(I_i) = \mathrm{Hom}_\Lambda(I_i, \Lambda) = \Lambda$.

Applying $\otimes_\Lambda S_i$ to the exact sequence in Proposition III.1.3(a) we have a projective resolution

$$0 \to \Lambda e_i \xrightarrow{g} P \xrightarrow{f} \Lambda e_i \to S_i \to 0 \tag{2}$$

with $\mathrm{Im}\, f = I_i e_i$ and $P \in \mathrm{add}\ \Lambda(1 - e_i)$. In particular, $I_i = \mathrm{Im}\, f \oplus \Lambda(1 - e_i)$ is a finitely presented Λ-module with $\mathrm{pd}\, I_i \le 1$.

We have $\mathrm{Ext}^1_\Lambda(I_i, I_i) \simeq \mathrm{Ext}^2_\Lambda(S_i, I_i) \simeq D\,\mathrm{Hom}_\Lambda(I_i, S_i) = 0$. Using (2), we have an exact sequence

$$0 \to \Lambda \to P \oplus \Lambda(1 - e_i) \to I_i e_i \to 0$$

such that the middle and the right terms belong to add I_i. Thus I_i is a tilting Λ-module. □

Proposition III.1.5. *Let T be a tilting Λ-module of projective dimension at most one.*

(a) *If $\mathrm{Tor}^\Lambda_1(S_i, T) = 0$, then $I_i \overset{\mathbf{L}}{\otimes}_\Lambda T = I_i \otimes_\Lambda T = I_i T$ is a tilting Λ-module of projective dimension at most one.*

(b) *$I_i T$ is always a tilting Λ-module of projective dimension at most one, and $\mathrm{End}_\Lambda(I_i T) \simeq \mathrm{End}_\Lambda(T)$.*

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Proof. (a) $\mathrm{Tor}_1^\Lambda(I_i, T) = \mathrm{Tor}_2^\Lambda(S_i, T) = 0$ because $\mathrm{pd}\, T \le 1$, so we have $I_i \overset{\mathbf{L}}{\otimes}_\Lambda T = I_i \otimes_\Lambda T$. Since there is an exact sequence

$$0 = \mathrm{Tor}_1^\Lambda(S_i, T) \to I_i \otimes_\Lambda T \to \Lambda \otimes_\Lambda T \to S_i \otimes_\Lambda T \to 0,$$

we have $I_i \otimes_\Lambda T = I_i T$. Thus $I_i \overset{\mathbf{L}}{\otimes}_\Lambda T = I_i T$ is a tilting Λ-module by Lemma III.1.2 and Proposition III.1.4. Since $\mathrm{pd}\, T \le 1$ and $\mathrm{pd}\, T/I_iT \le 2$ by Proposition III.1.3, we have $\mathrm{pd}\, I_iT \le 1$.

(b) By Lemma III.1.1, either $S_i \otimes_\Lambda T = 0$ or $\mathrm{Tor}_1^\Lambda(S_i, T) = 0$. If $S_i \otimes_\Lambda T = 0$, then $I_iT = T$ holds. If $\mathrm{Tor}_1^\Lambda(S_i, T) = 0$, then we apply (a). For the rest we use Lemma III.1.2. ◻

A left ideal I of Λ is said to be *cofinite* if $\Lambda/I \in \mathrm{f.\,l.}\Lambda$ and *tilting* (respectively, *partial tilting*) if it is a tilting (respectively, partial tilting) Λ-module. A cofinite, tilting or partial tilting right ideal of Λ can be defined similarly. An ideal I of Λ is called *cofinite tilting* if it is cofinite tilting as a left and as a right ideal. We denote by $\langle I_1, \dots, I_n\rangle$ the ideal semigroup generated by $I_1, \dots, I_n$. Then we have the following result.

Theorem III.1.6.

(a) *Any $T \in \langle I_1, \dots, I_n\rangle$ is a cofinite tilting ideal and satisfies* $\mathrm{End}_\Lambda(T) = \Lambda$.

(b) *Any cofinite tilting ideal of Λ belongs to $\langle I_1, \dots, I_n\rangle$.*

(c) *Any cofinite partial tilting left or right ideal of Λ is a cofinite tilting ideal.*

(d) *If two cofinite tilting ideals are isomorphic as left or as right Λ-modules, then they coincide.*

Proof. (a) This is a direct consequence of Propositions III.1.4 and III.1.5.

(b) and (c) Let T be a cofinite partial tilting left ideal of Λ. If $T \ne \Lambda$, then there exists a simple submodule S_i of Λ/T. Since $\mathrm{Hom}_\Lambda(S_i, \Lambda) = 0$, we have $\mathrm{Ext}_\Lambda^1(S_i, T) \ne 0$. Therefore $\mathrm{Tor}_1^\Lambda(S_i, T) \simeq D\,\mathrm{Ext}_\Lambda^1(T, S_i) \simeq \mathrm{Ext}_\Lambda^1(S_i, T) \ne 0$. From Lemma III.1.1 it follows that $S_i \otimes_\Lambda T = 0$.

Now put $U = \mathbf{R}\,\mathrm{Hom}_\Lambda(I_i, T)$. By Lemma III.1.2, we have that $U \simeq \mathbf{R}\,\mathrm{Hom}_\Lambda(I_i, \Lambda) \overset{\mathbf{L}}{\otimes}_\Lambda T$ is a partial tilting complex of Λ. Because $\mathrm{pd}\, I_i \le 1$ and $\mathrm{Ext}_\Lambda^1(I_i, T) \simeq \mathrm{Ext}_\Lambda^2(S_i, T) \simeq D\,\mathrm{Hom}_\Lambda(T, S_i) \simeq S_i \otimes_\Lambda T = 0$, we have $U = \mathrm{Hom}_\Lambda(I_i, T)$, which is a partial tilting Λ-module. Since we have a commutative diagram

$$\begin{array}{ccccccccc}
0 = \mathrm{Hom}_\Lambda(S_i, \Lambda) & \to & \Lambda & \to & \mathrm{Hom}_\Lambda(I_i, \Lambda) & \to & \mathrm{Ext}_\Lambda^1(S_i, \Lambda) = 0 & & \\
 & & \cup & & \cup & & & & \\
 & & T & \to & \mathrm{Hom}_\Lambda(I_i, T) & \to & \mathrm{Ext}_\Lambda^1(S_i, T) & \to & 0
\end{array}$$

of exact sequences, U is a cofinite partial tilting left ideal of Λ containing T properly such that U/T is a direct sum of copies of S_i. By $S_i \otimes_\Lambda T = 0$, we have $T = I_iU$. Thus $T \in \langle I_1, \dots, I_n\rangle$ by induction on the length of Λ/T.

(d) Assume that $f\colon T \to U$ is an isomorphism of Λ-modules for $T, U \in \langle I_1, \dots, I_n\rangle$, and let $g = f^{-1}$. Since $\mathrm{Ext}_\Lambda^1(\Lambda/T, U) = 0$, we can extend f and g to morphisms $f, g\colon \Lambda \to \Lambda$. Since $(fg)|_T = \mathrm{id}_T$ and $(gf)|_T = \mathrm{id}_U$, one can easily check that $fg = \mathrm{id}_\Lambda = gf$. Thus there exists an invertible element $x \in \Lambda$ such that f is right multiplication with x. We then have $U = f(T) = Tx = T$. ◻

We pose the following question, for which there is a positive answer in the extended Dynkin case [IR08].

Question III.1.7. For any tilting Λ-module T of projective dimension at most one, does there exist some U in $\langle I_1, \dots, I_n\rangle$ such that $\mathrm{add}\, T = \mathrm{add}\, U$?

We have some stronger statements concerning products of the ideals I_i, which generalize results for the noetherian case from [IR08].

Proposition III.1.8. *The following equalities hold for multiplication of ideals.*

(a) $I_i^2 = I_i$.

(b) $I_iI_j = I_jI_i$ *if there is no arrow between* i *and* j *in* Q.

(c) $I_iI_jI_i = I_jI_iI_j$ *if there is precisely one arrow between* i *and* j *in* Q.

Proof. The equality in (a) is obvious.

Parts (b) and (c) are proved in [IR08, Proposition 6.12] for module-finite 2-CY algebras. Here we give a direct proof for an arbitrary preprojective algebra Λ associated with a finite quiver Q without loops. Let $I_{i,j} = \Lambda(1 - e_i - e_j)\Lambda$. Then any product of ideals I_i and I_j contains $I_{i,j}$. If there is no arrow from i to j, then $\Lambda/I_{i,j}$ is semisimple. Thus I_iI_j and I_jI_i are contained in $I_{i,j}$, and we have $I_iI_j = I_{i,j} = I_jI_i$.

If there is precisely one arrow from i to j, then $\Lambda/I_{i,j}$ is the preprojective algebra of type A_2. Hence there are two indecomposable projective $\Lambda/I_{i,j}$-modules, whose Loewy series are $\binom{i}{j}$ and $\binom{j}{i}$. Thus $I_iI_jI_i$ and $I_jI_iI_j$ are contained in $I_{i,j}$, and we have $I_iI_jI_i = I_{i,j} = I_jI_iI_j$. □

Now let W be the Coxeter group associated to the quiver Q; so W has generators $s_1, \ldots, s_n$ with relations $s_i^2 = 1$, $s_is_j = s_js_i$ if there is no arrow between i and j in Q, and $s_is_js_i = s_js_is_j$ if there is a precisely one arrow between i and j in Q.

Theorem III.1.9. *There exists a bijection* $W \to \langle I_1, \ldots, I_n\rangle$. *It is given by* $w \mapsto I_w = I_{i_1}I_{i_2}...I_{i_k}$ *for any reduced expression* $w = s_{i_1}s_{i_2}...s_{i_k}$.

Proof. The corresponding result was proved in [IR08] for the noetherian case, using a partial order of tilting modules. Here we use, instead, properties of Coxeter groups.

First, we show that the map is well-defined. Take two reduced expressions $w = s_{i_1}s_{i_2}...s_{i_k} = s_{j_1}s_{j_2}...s_{j_k}$. By [BB05, Theorem 3.3.1(ii)], two words $s_{i_1}s_{i_2}...s_{i_k}$ and $s_{j_1}s_{j_2}...s_{j_k}$ can be connected by a sequence made up of the following operations: (i) replace s_is_j by s_js_i (there is no arrow from i to j); (ii) replace $s_is_js_i$ by $s_js_is_j$ (there is precisely one arrow from i to j). Consequently, by Proposition III.1.8 parts (b) and (c), we have $I_{i_1}I_{i_2}...I_{i_k} = I_{j_1}I_{j_2}...I_{j_k}$. Thus the map is well-defined.

Next, we show that the map is surjective. For any $I \in \langle I_1, \ldots, I_n\rangle$, take an expression $I = I_{i_1}I_{i_2}...I_{i_k}$ with a minimal number k. Let $w = s_{i_1}s_{i_2}...s_{i_k}$. By [BB05, Theorem 3.3.1(i)], a reduced expression of w is obtained from the word $s_{i_1}s_{i_2}...s_{i_k}$ by a sequence made up of the operations (i) and (ii) above along with: (iii) remove s_is_i. By Proposition III.1.8, the operation (iii) cannot appear since k is minimal. Thus $w = s_{i_1}s_{i_2}...s_{i_k}$ is a reduced expression, and we have $I = I_w$.

Finally, we show that the map is injective, using a similar argument as in [IR08]. Let $\mathcal{E} = \mathbf{K}^{\mathrm{b}}(\mathrm{pr}\ \Lambda)$. For any i, we have an autoequivalence $I_i \overset{\mathbf{L}}{\otimes}_\Lambda$ of $\mathcal{E}$ and an automorphism $[I_i \overset{\mathbf{L}}{\otimes}_\Lambda\ ]$ of the Grothendieck group $K_0(\mathcal{E})$. By [IR08, proof of Theorem 6.6], we have the action $s_i \mapsto [I_i \overset{\mathbf{L}}{\otimes}_\Lambda\ ]$ of W on $K_0(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{C}$, which is known to be faithful [BB05, Theorem 4.2.7].

For any reduced expression $w = s_{i_1}s_{i_2}...s_{i_k}$ we have $I_w = I_{i_1} \overset{\mathbf{L}}{\otimes}_\Lambda \ldots \overset{\mathbf{L}}{\otimes}_\Lambda I_{i_k}$, by Proposition III.1.5(a) and the minimality of k. Thus, the action of w on $K_0(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{C}$ coincides with

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$[I_w \overset{\mathbf{L}}{\otimes}_\Lambda\ ]$. In particular, if $w, w' \in W$ satisfy $I_w = I_{w'}$, then the actions of w and w' on $K_0(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{C}$ coincide, so we have $w = w'$ by the faithfulness of the action. $\square$

We denote by $l(w)$ the length of $w \in W$. We say that an infinite expression $s_{i_1} s_{i_2} \cdots s_{i_k} \cdots$ is *reduced* if the expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ is reduced for any k.

PROPOSITION III.1.10. *Let $w \in W$ and $i \in \{1, \ldots, n\}$. If $l(ws_i) > l(w)$, then we have $I_w I_i = I_{ws_i} \subsetneq I_w$. If $l(ws_i) < l(w)$, then we have $I_w I_i = I_w \subsetneq I_{ws_i}$.*

Proof. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression. If $l(ws_i) > l(w)$, then $ws_i = s_{i_1} \cdots s_{i_k} s_i$ is a reduced expression, so the assertion follows from Theorem III.1.9. If $l(ws_i) < l(w)$, then $u = ws_i$ satisfies $l(us_i) > l(u)$, so $I_u I_i = I_{us_i} = I_w \subsetneq I_u$. $\square$

Let $s_{i_1} s_{i_2} \cdots s_{i_k} \cdots$ be a (finite or infinite) expression such that $i_k \in \{1, \ldots, n\}$. Let

$$w_k = s_{i_1} s_{i_2} \cdots s_{i_k}, \quad T_k = I_{w_k} = I_{i_1} I_{i_2} \cdots I_{i_k} \quad \text{and} \quad \Lambda_k = \Lambda / T_k.$$

We have a descending chain

$$\Lambda = T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$$

of cofinite tilting ideals of Λ, as well as a chain

$$\Lambda_1 \leftarrow \Lambda_2 \leftarrow \Lambda_3 \leftarrow \cdots$$

of surjective ring homomorphisms. The chains have the following properties.

PROPOSITION III.1.11.

(a) *If $T_{m-1} \neq T_m$, then Λ_m differs from Λ_{m-1} in exactly one indecomposable summand $\Lambda_m e_{i_m}$.*

(b) *Let $k \leq m$. Then $\Lambda_k e_{i_k}$ is a projective Λ_m-module if and only if $i_k \notin \{i_{k+1}, i_{k+2}, \ldots, i_m\}$.*

(c) *$T_1 \supsetneq T_2 \supsetneq T_3 \supsetneq \cdots$ holds if and only if $s_{i_1} s_{i_2} \cdots$ is reduced.*

Proof. (a) This follows from $T_m(1 - e_{i_m}) = T_{m-1} I_{i_m} (1 - e_{i_m}) = T_{m-1}(1 - e_{i_m})$.

(b) If $i_k \notin \{i_{k+1}, \ldots, i_m\}$, then $\Lambda_k e_{i_k}$ is a summand of Λ_m by (a), so it is a projective Λ_m-module. Otherwise, take the smallest k' with $k < k' \leq m$ that satisfies $i_k = i_{k'}$. Then we have $\Lambda_k e_{i_k} = \Lambda_{k'-1} e_{i_k}$ and that $\Lambda_k e_{i_k}$ is a proper factor module of $\Lambda_{k'} e_{i_k}$, by (a). Hence $\Lambda_k e_{i_k}$ is not a projective Λ_m-module.

(c) This follows from Proposition III.1.10. $\square$

Our next goal is to show that $\mathrm{Ext}^1_\Lambda(T_k, T_m) = 0$ for $k \leq m$. For this the following result will be useful.

LEMMA III.1.12. *Let the notation and assumptions be as above. Then ${}^{\perp_{>0}}T_{m-1} \subseteq {}^{\perp_{>0}}T_m$, where ${}^{\perp_{>0}}T = \{X \in \mathrm{mod}\ \Lambda \mid \mathrm{Ext}^i_\Lambda(X, T) = 0$ for all $i > 0\}$.*

Proof. We can assume that $T_{m-1} \neq T_m$. Then we have that $T_{m-1} \otimes_\Lambda S_{i_m} \neq 0$. Hence $\mathrm{Tor}^\Lambda_1(T_{m-1}, S_{i_m}) = 0$ by Lemma III.1.1, and so $T_{m-1} \otimes_\Lambda I_{i_m} = T_{m-1} I_{i_m} = T_m$ by Proposition III.1.5. Let $0 \to P_1 \to P_0 \to I_{i_m} \to 0$ be a projective resolution. We have $\mathrm{Tor}^\Lambda_1(T_{m-1}, I_{i_m}) \simeq \mathrm{Tor}^\Lambda_2(T_{m-1}, S_{i_m}) = 0$. Applying $T_{m-1} \otimes_\Lambda\ $, we have an exact sequence $0 \to T_{m-1} \otimes_\Lambda P_1 \to T_{m-1} \otimes_\Lambda P_0 \to T_m \to 0$. This immediately implies that ${}^{\perp_{>0}}T_{m-1} \subseteq {}^{\perp_{>0}}T_m$. $\square$

We now have the following consequence.

PROPOSITION III.1.13. *With the above notation and assumptions, we have $\mathrm{Ext}^1_\Lambda(T_k, T_m) = 0$ for $k \leq m$.*

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Proof. By Lemma III.1.12 we have ${}^{\perp_{>0}}T_k \subseteq {}^{\perp_{>0}} T_{k+1} \subseteq \cdots \subseteq {}^{\perp_{>0}} T_m$. Since T_k is in ${}^{\perp_{>0}}T_k$, we then have that T_k is in ${}^{\perp_{>0}}T_m$. Hence $\operatorname{Ext}^1_\Lambda(T_k, T_m) = 0$ for $k \leq m$. □

Later we will use the following observation.

Lemma III.1.14. *Assume that the expression $s_{i_1}s_{i_2}\cdots$ is reduced. Let $T_{k,m} = I_{i_k}\cdots I_{i_m}$ if $k \leq m$ and $T_{k,m} = \Lambda$ otherwise. Then we have $\operatorname{Hom}_\Lambda(T_k, T_m) \simeq T_{k+1,m} = \{x \in \Lambda \mid T_k x \subseteq T_m\}$ and $\operatorname{Hom}_\Lambda(\Lambda_k, \Lambda_m) \simeq T_{k+1,m}/T_m$.*

Proof. Let $U = \{x \in \Lambda \mid T_k x \subseteq T_m\} \supseteq T_{k+1,m}$.

If $k \geq m$, then clearly $U = \Lambda = T_{k+1,m}$ holds and, by Theorem III.1.6, $\operatorname{Hom}_\Lambda(T_k, T_m) \subseteq \operatorname{End}_\Lambda(T_k) \simeq \Lambda$. Thus $\operatorname{Hom}_\Lambda(T_k, T_m) \simeq \Lambda$.

Now assume that $k < m$. Since $T_m = T_k \overset{\mathbf{L}}{\otimes}_\Lambda T_{k+1,m}$ holds by Proposition III.1.5(a) and Lemma III.1.2, we have $\mathbf{R}\operatorname{Hom}_\Lambda(T_k, T_m) = \mathbf{R}\operatorname{Hom}_\Lambda(T_k, (T_k \overset{\mathbf{L}}{\otimes}_\Lambda T_{k+1,m})) = \mathbf{R}\operatorname{Hom}_\Lambda(T_k, T_k) \overset{\mathbf{L}}{\otimes}_\Lambda T_{k+1,m} = \Lambda \overset{\mathbf{L}}{\otimes}_\Lambda T_{k+1,m} = T_{k+1,m}$. In particular, $\operatorname{Hom}_\Lambda(T_k, T_m) = T_{k+1,m}$. On the other hand, there is a commutative diagram

$$\begin{array}{ccll} \Lambda & \to & \operatorname{Hom}_\Lambda(T_k, \Lambda) & \\ \cup & & \quad\cup & \\ U & \to & \operatorname{Hom}_\Lambda(T_k, T_m) & \simeq T_{k+1,m} \end{array}$$

where the horizontal map is given by $x \mapsto (\cdot x)$ for any $x \in \Lambda$, which is injective. Thus we have $U \subseteq T_{k+1,m}$, and so $U = T_{k+1,m}$.

To show the second equality, note that for any $f \in \operatorname{Hom}_\Lambda(\Lambda_k, \Lambda_m)$, there exists a unique element $x \in \Lambda_m$ such that $f(y) = yx$ for any $y \in \Lambda$. Since $T_k x \subseteq T_m$ holds, we have $x \in U$. Thus $\operatorname{Hom}_\Lambda(\Lambda_k, \Lambda_m) \simeq U/T_m = T_{k+1,m}/T_m$. □

III.2 Cluster-tilting objects for preprojective algebras

Again, take Λ to be the completion of the preprojective algebra of a finite connected non-Dynkin quiver without loops over the field K. We show that for a large class of cofinite tilting ideals I in Λ, Λ/I is a finite-dimensional K-algebra which is Gorenstein of dimension at most one, and the categories $\operatorname{Sub}\Lambda/I$ and $\underline{\operatorname{Sub}}\,\Lambda/I$ are stably 2-CY and 2-CY, respectively. We describe some cluster-tilting objects in these categories, by using tilting ideals. We also describe cluster-tilting subcategories in the derived 2-CY abelian category $\operatorname{f.l.}\Lambda$ which have an infinite number of nonisomorphic indecomposable objects. Hence we get examples of cluster structures with infinite clusters (see [KR08] for other examples).

We start by investigating Λ/T for our special cofinite tilting ideals T as a module over Λ and over the factor ring Λ/U for a cofinite tilting ideal U contained in T.

Lemma III.2.1. *Let T and U' be cofinite tilting ideals in Λ, and let $U = TU'$. Then $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/U) = 0 = \operatorname{Ext}^1_\Lambda(\Lambda/U, \Lambda/T)$.*

Proof. Consider the exact sequence $0 \to U \to \Lambda \to \Lambda/U \to 0$. Applying $\operatorname{Hom}_\Lambda(\Lambda/T, \ )$, we have an exact sequence

$$\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda) \to \operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/U) \to \operatorname{Ext}^2_\Lambda(\Lambda/T, U).$$

By Proposition III.1.3, $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda) = 0$. It follows from Corollary III.1.13 that $\operatorname{Ext}^1_\Lambda(T, U) = 0$. Since $\operatorname{Ext}^2_\Lambda(\Lambda/T, U) \simeq \operatorname{Ext}^1_\Lambda(T, U) = 0$, we have $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/U) = 0$. Because Λ is derived 2-CY, it follows that also $\operatorname{Ext}^1_\Lambda(\Lambda/U, \Lambda/T) = 0$. □

Using this lemma, we can obtain more information on Λ/T.

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Proposition III.2.2.

(a) *For a cofinite ideal T in Λ with $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/T) = 0$, the algebra Λ/T is Gorenstein of dimension at most one.*

(b) *For a cofinite tilting ideal T in Λ, the factor algebra Λ/T is Gorenstein of dimension at most one.*

Proof. (a) Consider the exact sequence $0 \to \Omega_\Lambda(D(\Lambda/T)) \to P \to D(\Lambda/T) \to 0$ with a projective Λ-module P. Using Lemma III.2.1 and [CE56], we obtain $\operatorname{Tor}_1^\Lambda(\Lambda/T, D(\Lambda/T)) \simeq D\operatorname{Ext}^1_{\Lambda^{\mathrm{op}}}(\Lambda/T, \Lambda/T) = 0$. Applying $\Lambda/T \otimes_\Lambda$ to the above exact sequence, we get the exact sequence $0 \to \Lambda/T \otimes_\Lambda \Omega_\Lambda(D(\Lambda/T)) \to \Lambda/T \otimes_\Lambda P \to \Lambda/T \otimes_\Lambda D(\Lambda/T) \to 0$. The Λ/T-module $\Lambda/T \otimes_\Lambda P$ is projective. To see that $\Lambda/T \otimes_\Lambda \Omega_\Lambda(D(\Lambda/T))$ is also a projective Λ/T-module, we show that the functor $\operatorname{Hom}_{\Lambda/T}(\Lambda/T \otimes_\Lambda \Omega_\Lambda(D(\Lambda/T)),\ ) \simeq \operatorname{Hom}_\Lambda(\Omega_\Lambda(D(\Lambda/T)),\ )$ is exact on $\operatorname{mod} \Lambda/T$. This follows from the functorial isomorphisms

$$\begin{aligned}\operatorname{Ext}^1_\Lambda(\Omega_\Lambda(D(\Lambda/T)),\ ) &\simeq \operatorname{Ext}^2_\Lambda(D(\Lambda/T),\ )\\ &\simeq D\operatorname{Hom}_\Lambda(\ , D(\Lambda/T))\\ &\simeq D\operatorname{Hom}_{\Lambda/T}(\ , D(\Lambda/T)) \simeq \mathrm{id}_{\operatorname{mod} \Lambda/T}.\end{aligned}$$

Hence we conclude that $\operatorname{pd}_{\Lambda/T} D(\Lambda/T) \leq 1$. It is then well-known and easy to verify that $\operatorname{pd}_{(\Lambda/T)^{\mathrm{op}}} D(\Lambda/T) \leq 1$; so, by definition, Λ/T is Gorenstein of dimension at most one.

(b) This is a direct consequence of (a) and Lemma III.2.1. □

When Λ/T is Gorenstein of dimension at most one, the category of Cohen–Macaulay modules is the category $\operatorname{Sub} \Lambda/T$ of first syzygy modules (see [AR91, Hap91]). It is known that $\operatorname{Sub} \Lambda/T$ is a Frobenius category, with $\operatorname{add} \Lambda/T$ being the category of projective and injective objects, and that the stable category $\underline{\operatorname{Sub}}\, \Lambda/T$ is triangulated [Hap88]. Moreover, by Corollary II.2.7, $\operatorname{Sub} \Lambda/T$ is an extension-closed subcategory of $\operatorname{mod} \Lambda/T$, since $\operatorname{id}_{\Lambda/T} \Lambda/T \leq 1$ and $\operatorname{Ext}^1_{\Lambda/T}(\Lambda/T, \Lambda/T) = 0$. However, to show that the stably 2-CY property can be deduced from $\operatorname{f.l.} \Lambda$ being derived 2-CY, we need $\operatorname{Sub} \Lambda/T$ to be extension-closed also in $\operatorname{f.l.} \Lambda$.

Proposition III.2.3. *Let T be a cofinite ideal with $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/T) = 0$ (e.g. a cofinite tilting ideal).*

(a) $\operatorname{Ext}^1_\Lambda(\Lambda/T, X) = 0 = \operatorname{Ext}^1_\Lambda(X, \Lambda/T)$ *for all X in* $\operatorname{Sub} \Lambda/T$.

(b) $\operatorname{Sub} \Lambda/T$ *is an extension-closed subcategory of* $\operatorname{f.l.} \Lambda$.

(c) $\operatorname{Sub} \Lambda/T$ *and* $\underline{\operatorname{Sub}}\, \Lambda/T$ *are stably 2-CY and 2-CY, respectively.*

Proof. (a) For X in $\operatorname{Sub} \Lambda/T$ we have an exact sequence $0 \to X \to P \to Y \to 0$ with Y in $\operatorname{Sub} \Lambda/T$ and P in $\operatorname{add} \Lambda/T$. Upon applying $\operatorname{Hom}_\Lambda(\Lambda/T,\ ) \simeq \operatorname{Hom}_{\Lambda/T}(\Lambda/T,\ )$, the sequence does not change. Since $\operatorname{Ext}^1_\Lambda(\Lambda/T, \Lambda/T) = 0$, we conclude that $\operatorname{Ext}^1_\Lambda(\Lambda/T, X) = 0$. Hence $\operatorname{Ext}^1_\Lambda(X, \Lambda/T) = 0$ by the derived 2-CY property of $\operatorname{f.l.} \Lambda$.

(b) Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\operatorname{f.l.} \Lambda$, with X and Z in $\operatorname{Sub} \Lambda/T$. Then we have a monomorphism $X \to P$, with P in $\operatorname{add} \Lambda/T$. Since $\operatorname{Ext}^1_\Lambda(Z, P) = 0$ by (a), we have a commutative diagram of exact sequences as follows.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d, hook] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & P \arrow[r] & P \oplus Z \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

Thus Y is a submodule of $P \oplus Z \in \operatorname{Sub} \Lambda/T$, and we have $Y \in \operatorname{Sub} \Lambda/T$.

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(c) Since $\operatorname{Sub}\Lambda/T$ is extension-closed in $\mathrm{f.l.}\,\Lambda$, we have $\operatorname{Ext}^1_{\operatorname{Sub}\Lambda/T}(X,Y)=\operatorname{Ext}^1_\Lambda(X,Y)$. As $\operatorname{Sub}\Lambda/T$ is Frobenius, it follows from Proposition II.1.1 that $\operatorname{Sub}\Lambda/T$ is stably 2-CY because $\mathrm{f.l.}\,\Lambda$ is derived 2-CY, and so $\underline{\operatorname{Sub}}\,\Lambda/T$ is 2-CY. □

We now want to investigate the cluster-tilting objects in $\operatorname{Sub}\Lambda/T$ and $\underline{\operatorname{Sub}}\,\Lambda/T$ for certain tilting ideals T; later we shall also investigate the cluster-tilting subcategories of $\mathrm{f.l.}\,\Lambda$. The next observation will be useful.

Lemma III.2.4. *Let Δ be a finite-dimensional algebra and M a Δ-module which is a generator. Let $\Gamma=\operatorname{End}_\Delta(M)$, and assume $\operatorname{gl.dim}\Gamma\le 3$ and $\operatorname{pd}_\Gamma D(M)\le 1$. Then for any X in $\operatorname{mod}\Delta$ there is an exact sequence $0\to M_1\to M_0\to X\to 0$, with M_0 and M_1 in $\operatorname{add} M$.*

Proof. Let X be in $\operatorname{mod}\Delta$, and consider the exact sequence $0\to X\to I_0\to I_1$ where I_0 and I_1 are injective. Apply $\operatorname{Hom}_\Delta(M,\ )$ to get an exact sequence $0\to\operatorname{Hom}_\Delta(M,X)\to\operatorname{Hom}_\Delta(M,I_0)\to\operatorname{Hom}_\Delta(M,I_1)$. Since, by assumption, $\operatorname{pd}_\Gamma\operatorname{Hom}_\Delta(M,I_i)\le 1$ for $i=0,1$ and $\operatorname{gl.dim}\Gamma\le 3$, we obtain $\operatorname{pd}_\Gamma\operatorname{Hom}_\Delta(M,X)\le 1$. Hence we have an exact sequence $0\to P_1\to P_0\to\operatorname{Hom}_\Delta(M,X)\to 0$ in $\operatorname{mod}\Gamma$ with P_0 and P_1 projective. This sequence is the image under the functor $\operatorname{Hom}_\Delta(M,\ )$ of the complex $0\to M_1\to M_0\to X\to 0$ in $\operatorname{mod}\Delta$, with M_0 and M_1 in $\operatorname{add}M$. Since M is assumed to be a generator, this complex must be exact, and we have our desired exact sequence. □

Now let $\Lambda=T_0\supsetneq T_1\supsetneq T_2\supsetneq\cdots$ be a strict descending chain of tilting ideals corresponding to a (finite or infinite) reduced expression $s_{i_1}s_{i_2}s_{i_3}\cdots$. We wish to describe some natural cluster-tilting objects for the algebras $\Lambda_m=\Lambda/T_m$. Let

$$\Lambda_k=\Lambda/T_k,\qquad M_m=\bigoplus_{k=0}^{m}\Lambda_k$$

and $\Gamma=\operatorname{End}_{\Lambda_m}(M_m)$. The following proposition will be essential.

Proposition III.2.5. *With the above notation, the following properties hold.*

(a) *For X in $\operatorname{mod}\Lambda_m$ there is an exact sequence $0\to N_1\to N_0\to X\to 0$ in $\operatorname{mod}\Lambda_m$, with N_i in $\operatorname{add}M_m$ for $i=1,2$.*

(b) $\operatorname{gl.dim}\Gamma\le 3$.

Proof. We prove (a) and (b) by induction on m. Assume first that $m=1$. Then $\Lambda_1=\Lambda/T_1$, which is a simple Λ_1-module. Since $M_1=\Lambda/T_1$, (a) and (b) are trivially satisfied in this case.

Assume now that $m>1$ and that (a) and (b) have been established for $m-1$. Let us prove that (b) holds for m. Note that since there are no loops for Λ, we have $T_{m-1}J\subseteq T_m$, where J is the Jacobson radical of Λ so that $J\Lambda_m$ is a Λ_{m-1}-module $(*)$. For an indecomposable object X in $\mathcal{M}_m=\operatorname{add}M_m$, let $f\colon C_0\to X$ be a minimal right almost-split map in $\mathcal{M}_m$.

Suppose that X is not a projective Λ_m-module. Then f must be surjective. An indecomposable object which is in $\mathcal{M}_m$ but not in $\mathcal{M}_{m-1}$ is a projective Λ_m-module, so we can write $C_0=C_0'\oplus P$ where $C_0'\in\mathcal{M}_{m-1}$ and P is a projective Λ_m-module. Since f is right minimal, we have $\operatorname{Ker}f\subseteq C_0'\oplus JP$ so that $\operatorname{Ker}f$ is a Λ_{m-1}-module by $(*)$. It follows by the induction hypothesis that there is an exact sequence $0\to C_2\to C_1\to\operatorname{Ker}f\to 0$ with C_1 and C_2 in $\mathcal{M}_{m-1}$. Hence we have an exact sequence $0\to C_2\to C_1\to C_0\to X\to 0$. Applying $\operatorname{Hom}_\Lambda(M_m,\ )$ gives an exact sequence

$$0\to\operatorname{Hom}_\Lambda(M_m,C_2)\to\operatorname{Hom}_\Lambda(M_m,C_1)\to\operatorname{Hom}_\Lambda(M_m,C_0)\to\operatorname{Hom}_\Lambda(M_m,X)\to S\to 0.$$

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Then the module S, which is a simple module in the top of $\mathrm{Hom}_\Lambda(M_m, X)$ in mod Γ, has projective dimension at most three.

Suppose now that X is a projective Λ_m-module. Then, by $(*)$, we have that JX is in mod Λ_{m-1}. By the induction hypothesis, there is then an exact sequence $0 \to C_1 \to C_0 \to JX \to 0$ with C_0 and C_1 in $\mathcal{M}_{m-1}$. Hence we have an exact sequence $0 \to C_1 \to C_0 \to X$. Applying $\mathrm{Hom}_\Lambda(M_m, \ )$ gives the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M_m, C_1) \to \mathrm{Hom}_\Lambda(M_m, C_0) \to \mathrm{Hom}_\Lambda(M_m, X) \to S \to 0,$$

where S is the simple top of $\mathrm{Hom}_\Lambda(M_m, X)$, and hence $\mathrm{pd}_\Gamma S \leq 2$. It now follows that $\mathrm{gl.dim}\, \Gamma_m \leq 3$.

Finally, we show that (a) holds for m. By Proposition III.2.2 we have an exact sequence $0 \to P_1 \to P_0 \to D(\Lambda_m) \to 0$ in mod Λ_m, where P_0 and P_1 are projective Λ_m-modules. From Lemma III.2.1 we have $\mathrm{Ext}^1_\Lambda(M_m, \Lambda_m) = 0$. Applying $\mathrm{Hom}_\Lambda(M_m, \ )$ gives the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M_m, P_1) \to \mathrm{Hom}_\Lambda(M_m, P_0) \to \mathrm{Hom}_\Lambda(M_m, D(\Lambda_m)) \to 0.$$

Since $\mathrm{Hom}_\Lambda(M_m, D(\Lambda_m)) \simeq D(M_m)$, we have $\mathrm{pd}_{\Gamma_m} D(M_m) \leq 1$. Now our desired result follows from Lemma III.2.4. □

We can now describe some cluster-tilting objects in $\mathrm{Sub}\, \Lambda_m$ and $\underline{\mathrm{Sub}}\, \Lambda_m$.

Theorem III.2.6. *With the above notation, M_m is a cluster-tilting object in* $\mathrm{Sub}\, \Lambda_m$ *and in* $\underline{\mathrm{Sub}}\, \Lambda_m$.

Proof. We already have that $\mathrm{Ext}^1_\Lambda(M_m, M_m) = 0$ by Lemma III.2.1, so $\mathrm{Ext}^1_{\Lambda_m}(M_m, M_m) = 0$. Note that $\mathrm{Sub}\, \Lambda_m = \{X \in \mathrm{mod}\ \Lambda_m \mid \mathrm{Ext}^1_{\Lambda_m}(X, \Lambda_m) = 0\}$, because Λ_m is a cotilting module with $\mathrm{id}\, \Lambda_m \leq 1$. Since Λ_m is a summand of M_m, we have that M_m is in $\mathrm{Sub}\, \Lambda_m$. Assume then that $\mathrm{Ext}^1_{\Lambda_m}(X, M_m) = 0$ for X in mod Λ_m. By Proposition III.2.5(a), there is an exact sequence $0 \to C_1 \to C_0 \to X \to 0$ with C_1 and C_0 in add M_m, which must split by our assumption. Hence X is in add M_m, and it follows that M_m is a cluster-tilting object in $\mathrm{Sub}\, \Lambda_m$. It then follows, as usual, that it is a cluster-tilting object also in $\underline{\mathrm{Sub}}\, \Lambda_m$. □

We have now obtained a large class of 2-CY categories $\mathrm{Sub}\, \Lambda/I_w$ and $\underline{\mathrm{Sub}}\, \Lambda/I_w$ defined via elements w of the associated Coxeter group W, along with cluster-tilting objects associated with reduced expressions of elements in W. We call these *standard cluster-tilting objects* for $\mathrm{Sub}\, \Lambda/I_w$ or $\underline{\mathrm{Sub}}\, \Lambda/I_w$. We can also describe cluster-tilting subcategories with an infinite number of non-isomorphic indecomposable objects in the categories f. l. Λ.

Theorem III.2.7. *With the above notation, assume that each i occurs an infinite number of times in $i_1, i_2, \ldots$. Then $\mathcal{M} = \mathrm{add}\{\Lambda_m \mid 0 \leq m\}$ is a cluster-tilting subcategory of* f. l. Λ.

Proof. We already know that $\mathrm{Ext}^1_\Lambda(\Lambda_k, \Lambda_m) = 0$ for all k and m. Now let X be indecomposable in f. l. Λ. Then X is a Λ/J^k-module for some k. We have $J = I_1 \cap \cdots \cap I_n \supseteq I_1 \cdots I_n$, where $1, \ldots, n$ are the vertices in the quiver. By our assumptions, we have $J^k \supseteq T_m$ for some m so that X is a Λ_m-module. Consider the exact sequence $0 \to C_1 \to C_0 \to X \to 0$ in mod Λ_m, with C_1 and C_0 in add M_m, obtained from Proposition III.2.5. Assume that $\mathrm{Ext}^1_\Lambda(X, \mathcal{M}) = 0$. Since also $\mathrm{Ext}^1_{\Lambda_m}(X, \mathcal{M}_m) = 0$, the sequence splits, so X is in $\mathcal{M}_m$ and hence in $\mathcal{M}$.

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It only remains to show that $\mathcal{M}$ is functorially finite; so let X be in $\mathrm{f.\,l.}\,\Lambda$. Using the above exact sequence $0 \to C_1 \to C_0 \to X \to 0$, we get the exact sequence

$$0 \to (C, C_1) \to (C, C_0) \to (C, X) \to \mathrm{Ext}^1_\Lambda(C, C_1) = 0$$

for C in $\mathcal{M}$. Hence $\mathcal{M}$ is contravariantly finite.

For X in $\mathrm{f.\,l.}\,\Lambda$, take the left $(\mathrm{Sub}\,\mathcal{M})$-approximation $X \to Y$ and choose m such that $Y \in \mathrm{Sub}\,\Lambda_m$. For any $Y \in \mathrm{Sub}\,\Lambda_m$, there exists an exact sequence $0 \to Y \to C_0 \to C_1 \to 0$ with $C_i \in \mathcal{M}_m$ by Proposition II.1.7(a). Then the composition $X \to Y \to C_0$ is a left $\mathcal{M}$-approximation since $\mathrm{Ext}^1_\Lambda(C_1, \mathcal{M}) = 0$; therefore $\mathcal{M}$ is also covariantly finite. □

Summarizing our results, we state the following.

Theorem III.2.8.

(a) *For any $w \in W$, we have a stably 2-CY category $\mathcal{C}_w = \mathrm{Sub}\,\Lambda/I_w$.*

(b) *For any reduced expression $w = s_{i_1} \cdots s_{i_m}$ of $w \in W$, we have a cluster-tilting object $\bigoplus_{k=1}^m \Lambda/I_{s_{i_1} \cdots s_{i_k}}$ in $\mathcal{C}_w$. In particular, the number of non-isomorphic indecomposable summands in any cluster-tilting object is $l(w)$.*

(c) *For any infinite reduced expression $s_{i_1} s_{i_2} \cdots$ such that each i occurs an infinite number of times in $i_1, i_2, \ldots$, we have a cluster-tilting subcategory $\mathrm{add}\{\Lambda/I_{s_{i_1} \cdots s_{i_k}} \mid 0 \le k\}$ in $\mathrm{f.\,l.}\,\Lambda$.*

We end this subsection by showing that the subcategories $\mathrm{Sub}\,\Lambda/I_w$ can be characterized by means of torsionfree classes.

Theorem III.2.9. *Let Λ be the completed preprojective algebra of a connected non-Dynkin quiver without loops. Let $\mathcal{C}$ be a torsionfree class in $\mathrm{f.\,l.}\,\Lambda$ with some cluster-tilting object. Then $\mathcal{C} = \mathrm{Sub}\,\Lambda/I_w$ for some element w in the Coxeter group associated with Λ.*

Proof. We first prove that if M is a cluster-tilting object in $\mathcal{C}$, then $\mathcal{C} = \mathrm{Sub}\ M$. We only need to show that $\mathcal{C} \subset \mathrm{Sub}\ M$. For any $X \in \mathcal{C}$, take a projective resolution $\mathrm{Hom}_\Lambda(M, N) \to \mathrm{Ext}^1_\Lambda(M, X) \to 0$ $(*)$ of $\mathrm{End}_\Lambda(M)$-modules with $N \in \mathrm{add}\ M$. Replacing M in $(*)$ by N, we get an exact sequence $0 \to X \to Y \to N \to 0$ as the image of the identity $1_N \in \mathrm{End}_\Lambda(N)$ in $\mathrm{Ext}^1_\Lambda(N, X)$. Since $\mathcal{C}$ is extension-closed, $Y \in \mathcal{C}$. We have an exact sequence $\mathrm{Hom}_\Lambda(M, N) \to \mathrm{Ext}^1_\Lambda(M, X) \to \mathrm{Ext}^1_\Lambda(M, Y) \to \mathrm{Ext}^1_\Lambda(M, N) = 0$. Since $(*)$ is exact, $\mathrm{Ext}^1_\Lambda(M, Y) = 0$. Thus we have $Y \in \mathrm{add}\ M$ and $X \in \mathrm{Sub}\ M$.

Now let I be the annihilator $\mathrm{ann}_{\,\Lambda} M$ of M in Λ. Then I is clearly a cofinite ideal in Λ, and $\mathrm{ann}_{\,\Lambda/I} M = 0$. Further, $\mathrm{Sub}\ M$ is extension-closed also in $\mathrm{mod}\ \Lambda/I$. Hence the direct sum A comprising one copy of each of the non-isomorphic indecomposable Ext-injective Λ/I-modules in $\mathrm{Sub}\ M$ is a cotilting Λ/I-module satisfying $\mathrm{id}_{\,\Lambda/I} A \le 1$ and $\mathrm{Sub}\ M = \mathrm{Sub}\ A$, by [Sma84]. Since $\mathrm{Sub}\ M$ is extension-closed in the derived 2-CY category $\mathrm{f.\,l.}\,\Lambda$, the Ext-injective Λ/I-modules in $\mathrm{Sub}\ M$ coincide with the Ext-projective ones, which are the projective Λ/I-modules. Hence we have that A is a progenerator of Λ/I and that $\mathrm{Sub}\ M = \mathrm{Sub}\,\Lambda/I$. Since $\mathrm{Sub}\,\Lambda/I$ is extension-closed in $\mathrm{f.\,l.}\,\Lambda$, we have $\mathrm{Ext}^1_\Lambda(\Lambda/I, \Lambda/I) = 0$.

By Theorem III.1.6, we only have to show that I is a partial tilting left ideal. By Bongartz completion, we need only show $\mathrm{Ext}^1_\Lambda(I, I) = 0$. The natural surjection $\Lambda \to \Lambda/I$ clearly induces a surjection $\mathrm{Hom}_\Lambda(\Lambda/I, \Lambda/I) \to \mathrm{Hom}_\Lambda(\Lambda, \Lambda/I)$. Since Λ is derived 2-CY, we have injections$\mathrm{Ext}^2_\Lambda(\Lambda/I, \Lambda) \to \mathrm{Ext}^2_\Lambda(\Lambda/I, \Lambda/I)$ and $\mathrm{Ext}^1_\Lambda(I, \Lambda) \to \mathrm{Ext}^1_\Lambda(I, \Lambda/I)$. Using the exact

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sequence $0 \to I \to \Lambda \to \Lambda/I \to 0$, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathrm{Ext}^1_\Lambda(\Lambda/I, \Lambda/I) = 0 & & & & & & \\
 & & \uparrow & & & & & & \\
\mathrm{Hom}_\Lambda(I, \Lambda) & \to & \mathrm{Hom}_\Lambda(I, \Lambda/I) & \to & \mathrm{Ext}^1_\Lambda(I, I) & \to & \mathrm{Ext}^1_\Lambda(I, \Lambda) & \to & \mathrm{Ext}^1_\Lambda(I, \Lambda/I) \\
\uparrow & & \uparrow & & \uparrow & & & & \\
\Lambda & \to & \Lambda/I & \to & 0 & & & &
\end{array}$$

of exact sequences. Thus we have $\mathrm{Ext}^1_\Lambda(I, I) = 0$. □

Note that we have proved that an extension-closed subcategory of f. l. Λ of the form Sub X for some X in f. l. Λ with some cluster-tilting object must be Sub Λ/I_w for some element w in the Coxeter group associated with Λ.

We point out that there are other extension-closed subcategories of f. l. Λ with some cluster-tilting object. Let Q be an extended Dynkin quiver and Q' a Dynkin subquiver, and let Λ and Λ' be the corresponding completed preprojective algebras. Then, clearly, mod $\Lambda' =$ f. l. Λ' is an extension-closed subcategory of f. l. Λ. Any extension-closed subcategory of mod Λ' is therefore extension-closed in f. l. Λ, so Example 1 in § II.3 is an example of an extension-closed stably 2-CY subcategory of f. l. Λ with some cluster-tilting object, but which is not closed under submodules.

III.3 Realization of cluster categories and stable categories for preprojective algebras of Dynkin type

In this section we show that for an appropriate choice of T as a product of tilting ideals $I_j = \Lambda(1 - e_j)\Lambda$, any cluster category is equivalent to some $\underline{\mathrm{Sub}}\, \Lambda/T$. In particular, any cluster category can be realized as the stable category of a Frobenius category with finite-dimensional homomorphism spaces. We also show that the stable categories for preprojective algebras of Dynkin type can be realized in this way.

Let Q be a finite connected quiver without loops, KQ the associated path algebra, and Λ the completion of the preprojective algebra of Q. Choose a complete set of orthogonal primitive idempotents $e_1, \ldots, e_n$ of KQ. We can assume that $e_i(KQ)e_j = 0$ for any $i > j$. We regard $e_1, \ldots, e_n$ as a complete set of orthogonal primitive idempotents of Λ, and we take, as before, $I_i = \Lambda(1 - e_i)\Lambda$.

Assume first that Q is not Dynkin. We consider a stably 2-CY category associated to the square w^2 of a Coxeter element $w = s_1 s_2 \cdots s_n \in W$. Let $\Lambda_i = \Lambda/I_1 I_2 \cdots I_i$ and $\Lambda_{i+n} = \Lambda/I_1 I_2 \cdots I_n I_1 \cdots I_i$ for $1 \le i \le n$. We have seen in § III.2 that Sub Λ_{2n}, and also $\underline{\mathrm{Sub}}\, \Lambda_{2n}$, has a cluster-tilting object $M = \bigoplus_{i=1}^{2n} \Lambda_i$.

We will need the following lemma.

Lemma III.3.1. *Assume that Q is not Dynkin. Then $I_1 \cdots I_n I_1 \cdots I_n I_1 \cdots$ gives rise to a strict descending chain of tilting ideals. In particular, $s_1 \cdots s_n s_1 \cdots s_n s_1 \cdots$ is reduced.*

Proof. Assume to the contrary that the descending chain of ideals is not strict. Let $T_i = I_1 \cdots I_i$ and $U_i = I_1 \cdots I_{i-1} I_{i+1} \cdots I_n$ for $i = 1, \ldots, n$. Then we have $T_n^k T_{i-1} = T_n^k T_i$ for some $i = 1, \ldots, n$ and $k \ge 0$, where $T_0 = \Lambda$. Hence we obtain $T_n^{k+1} = T_n^k U_i$. Then we get $T_n^{k+m} = T_n^k U_i^m$ for any $m > 0$. Since $U_i e_i = \Lambda e_i$ and $J \supseteq T_n$, we have $J^{m+k} e_i \supseteq T_n^{m+k} e_i = T_n^k U_i^m e_i = T_n^k e_i$. Since $(\Lambda/T_n^k)e_i$ has finite length, we have $J^{m+k} e_i = T_n^k e_i$ for m sufficiently large. Thus $T_n^k e_i = 0$, which is a contradiction since Λe_i has infinite length. The latter assertion follows from Proposition III.1.11. □

Using Lemma III.3.1, we obtain the following.

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PROPOSITION III.3.2. *Let Q be a finite connected non-Dynkin quiver without loops and with vertices $1, \dots, n$ ordered as above. Let $\Lambda_{2n} = \Lambda/(I_1 \cdots I_n)^2$. Then $\Lambda_n = \Lambda/I_1 \cdots I_n$ is a cluster-tilting object in $\underline{\mathrm{Sub}}\, \Lambda_{2n}$ with $\mathrm{End}_{\underline{\mathrm{Sub}}\Lambda_{2n}}(\Lambda_n) \simeq KQ$.*

Proof. Since the associated chain of ideals is strict descending by Lemma III.3.1, our general theory applies. We have a cluster-tilting object $\bigoplus_{i=1}^{2n} \Lambda_i$ in $\underline{\mathrm{Sub}}\, \Lambda_{2n}$ by Theorem III.2.6. We also have add $\bigoplus_{i=1}^{2n} \Lambda_i = \Lambda_n \oplus \Lambda_{2n}$ in f. l. Λ by Proposition III.1.11. Thus Λ_n is a cluster-tilting object in $\underline{\mathrm{Sub}}\, \Lambda_{2n}$.

Note that the path algebra KQ is, in a natural way, a factor algebra of Λ; hence KQ is a Λ-module. We want to show that the Λ-modules Λ_n and KQ are isomorphic.

Let P_j be the indecomposable projective Λ-module corresponding to the vertex j. Then $I_{j+1} \cdots I_n P_j = P_j$ and $I_j P_j = JP_j$, the smallest submodule of P_j such that the corresponding factor has only composition factors S_j. Further, $I_{j-1} I_j P_j$ is the smallest submodule of $I_j P_j = JP_i$ such that the factor has only composition factors S_{j-1}, and so on. By our choice of ordering, this means that the paths starting at j, with decreasing indexing on the vertices, give a basis for $P_j / I_1 \cdots I_n P_j$. In other words, we have $P_j/I_1 \cdots I_n P_j \simeq (KQ)e_j$. Hence the Λ-modules $\Lambda_n = \Lambda/I_1 \cdots I_n$ and KQ are isomorphic, and so $\mathrm{End}_\Lambda(\Lambda_n) \simeq KQ$.

It remains to show that $\mathrm{End}_{\Lambda_{2n}}(\Lambda_n) \simeq \mathrm{End}_{\underline{\mathrm{Sub}}\,\Lambda_{2n}}(\Lambda_n)$. By Lemma III.1.14, any morphism from Λ_n to Λ_{2n} is given by a right multiplication of an element in $(I_1 \cdots I_n)/(I_1 \cdots I_n)^2$. This implies that $\mathrm{Hom}_\Lambda(\Lambda_n, \Lambda_{2n})\, \mathrm{Hom}_\Lambda(\Lambda_{2n}, \Lambda_n) = 0$, and the assertion follows. □

We now show that we have the same kind of result for Dynkin quivers.

PROPOSITION III.3.3. *Let Q' be a Dynkin quiver with vertices $1, \dots, m$ contained in a finite connected non-Dynkin quiver Q without oriented cycles and with vertices $1, \dots, n$ ordered as before. Let Λ be the preprojective algebra of Q and let $\Lambda_{n+m} = \Lambda/(I_1 \cdots I_n I_1 \cdots I_m)$. Then $\Lambda_m = \Lambda/I_1 \cdots I_m$ is a cluster tilting object in $\underline{\mathrm{Sub}}\, \Lambda_{n+m}$ with $\mathrm{End}_{\underline{\mathrm{Sub}}\,\Lambda_{n+m}}(\Lambda_m) \simeq KQ'$.*

Proof. Since, as seen in Lemma III.3.1, the product $(I_1 \cdots I_n)^2$ gives rise to a strict descending chain of ideals, the same must hold for $I_1 \cdots I_n I_1 \cdots I_m$. The assertions then follow as in the proof of Proposition III.3.2. □

Recall from [KR08] that if a connected algebraic triangulated 2-CY category has a cluster-tilting object M whose quiver Q has no oriented cycles, then $\mathcal{C}$ is triangle-equivalent to the cluster category $\mathcal{C}_{KQ}$. We then have the following consequence of the previous two results.

THEOREM III.3.4. *Let Q' be a finite connected quiver without oriented cycles. Let $Q = Q'$ if Q' is not Dynkin, and let Q be as in Proposition III.3.3 if Q' is Dynkin. Let Λ be the preprojective algebra of Q. Then there is a tilting ideal I in Λ such that $\underline{\mathrm{Sub}}\, \Lambda/I$ is triangle-equivalent to the cluster category $\mathcal{C}_{KQ'}$ of Q'.*

Finally, we show that the categories $\underline{\mathrm{mod}}\, \Lambda'$, where Λ' is the preprojective algebra of a Dynkin quiver Q', can also be realized in this way.

THEOREM III.3.5. *Let Q' be a Dynkin quiver contained in a finite connected non-Dynkin quiver Q without loops. Let Λ' denote the preprojective algebra of Q', W' the subgroup of W generated by $\{s_i \mid i \in Q'_0\}$, and w_0 the longest element in W'. Then Λ' is isomorphic to Λ/I_{w_0} and $\underline{\mathrm{mod}}\, \Lambda' = \underline{\mathrm{Sub}}\, \Lambda/I_{w_0}$.*

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Proof. Let $I_{Q'} = \Lambda(\sum_{i \in Q_0 \setminus Q'_0} e_i)\Lambda$. Since we have $\Lambda/I_{Q'} \simeq \Lambda'$, we only have to show that $I_{w_0} = I_{Q'}$. We use the fact that $I_{Q'}$ is maximal amongst all two-sided ideals I of Λ such that any composition factor of Λ/I is S_i for some $i \in Q'_0$.

Since w_0 is a product of s_i ($i \in Q'_0$), any composition factor of Λ/I_{w_0} is S_i for some $i \in Q'_0$. Thus we have $I_{w_0} \supseteq I_{Q'}$. On the other hand, since w_0 is the longest element of W', we have $l(s_i w_0) < l(w_0)$ for any $i \in Q'_0$. By Proposition III.1.10, we have $I_i I_{w_0} = I_{w_0}$ for any $i \in Q'_0$. This implies $I_{w_0} = I_{Q'}$. □

Using Theorem III.3.5, we see that our theory applies also to preprojective algebras of Dynkin type. In particular, we can specialize Theorem III.2.8 to recover a result from [GLS06], which is stated as part (a) of the corollary below.

COROLLARY III.3.6.

(a) *For a preprojective algebra Λ' of a Dynkin quiver, the number of non-isomorphic indecomposable summands in a cluster-tilting object is equal to the length $l(w_0)$ of the longest element in the associated Weyl group, which is equal to the number of positive roots.*

(b) *Statements (a) and (b) of Theorem III.2.8 hold also for Dynkin quivers.*

We also obtain a large class of cluster-tilting objects associated with the different reduced expressions of w_0.

Moreover, our results can be viewed as giving an interpretation in terms of tilting theory of some functors $\mathcal{E}_i$ used in [GLS08, Proposition 5.1].

III.4 Quivers of cluster-tilting subcategories

In this subsection, we show that the quivers of standard cluster-tilting subcategories associated with a reduced expression can be described directly from the reduced expression.

Let $s_{i_1} s_{i_2} \cdots s_{i_k} \cdots$ be a (finite or infinite) reduced expression associated with a graph Δ with vertices $1, \dots, n$. We associate with this sequence a quiver $Q(i_1, i_2, \dots)$ as follows, where the vertices correspond to the s_{i_k}.

- For two consecutive i ($i \in \{1, \dots, n\}$), draw an arrow from the second one to the first one.
- For each edge $i \overset{d_{ij}}{-} j$, pick out the expression consisting of the i_k which are i or j, so that we have $\cdots ii \cdots ijj \cdots jii \cdots i \cdots$. Draw d_{ij} arrows from the last i in a connected set of i to the last j in the next set of j, and do the same from j to i. (Note that since, by assumption, both i and j occur an infinite number of times if the expression is infinite, each connected set of i or set of j is finite.)

Note that in the Dynkin case, essentially the same quiver has been used in [BFZ05].

For a finite reduced expression $s_{i_1} \cdots s_{i_k}$, we denote by $\underline{Q}(i_1, \dots i_k)$ the quiver obtained from $Q(i_1, \dots i_k)$ by removing the last i for each i in Q_0.

We denote by $\Lambda = T_0 \supsetneq T_1 \supsetneq \cdots$ the associated strict descending chain of tilting ideals. Then we have a cluster-tilting subcategory $\mathcal{M}(i_1, i_2, \dots) = \operatorname{add}\{\Lambda_k \mid k > 0\}$ for $\Lambda_k = \Lambda/T_k$.

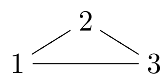
THEOREM III.4.1. *Let the notation be as above.*

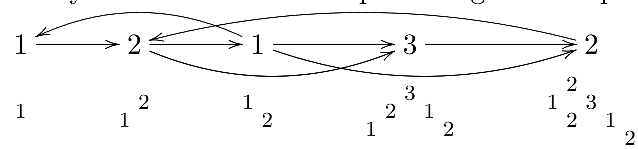
(a) *The quiver of the cluster-tilting subcategory $\mathcal{M}(i_1, i_2, \dots)$ is $Q(i_1, i_2, \dots)$.*

(b) *The quiver of $\underline{\operatorname{End}}_\Lambda(\mathcal{M}(i_1, \dots i_k))$ is $\underline{Q}(i_1, \dots i_k)$.*

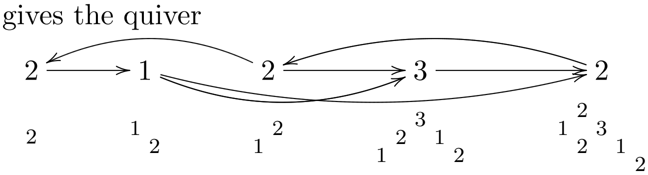
Before presenting the proof, we give some examples and consequences.

It follows from the definition that we get the same quiver if we interchange two neighbors in the expression of w which are not connected with any edge in Δ. But if we take two reduced expressions in general, we may get different quivers, as the following examples show.

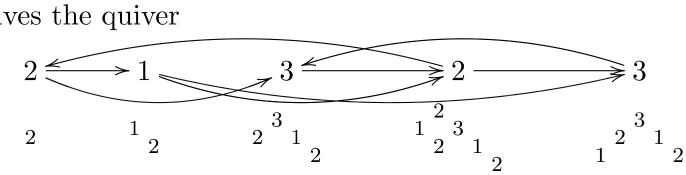
Let Δ be the graph  and let $w = s_1 s_2 s_1 s_3 s_2 = s_2 s_1 s_2 s_3 s_2 = s_2 s_1 s_3 s_2 s_3$ be expressions which are clearly reduced. The first expression gives the quiver



while the second one gives the quiver



and the third one gives the quiver



We now investigate the relationship between the cluster-tilting objects given by different reduced expressions of the same element.

LEMMA III.4.2. *Let $w = s_{i_1} \cdots s_{i_m} = s_{i'_1} \cdots s_{i'_m}$ be reduced expressions and let $\Lambda = T_0 \supsetneq T_1 \supsetneq \cdots$ and $\Lambda = T'_0 \supsetneq T'_1 \supsetneq \cdots$ be corresponding tilting ideals.*

(a) *Assume that for some k we have $i_k = i'_{k+1}$, $i'_k = i_{k+1}$ and $i_j = i'_j$ for any $j \neq k, k+1$. Then the corresponding cluster-tilting objects are isomorphic.*

(b) *Assume that for some k we have $i_{k-1} = i'_k = i_{k+1}$, $i'_{k-1} = i_k = i'_{k+1}$ and $i_j = i'_j$ for any $j \neq k, k \pm 1$. Then the corresponding cluster-tilting objects are in the relationship of exchanges of $T_{k-1} e_{i_{k-1}}$ and $T'_{k-1} e_{i'_{k-1}}$.*

Proof. (a) Obviously, we have $T_j = T'_j$ for any $j < k$. Since $s_{i_k} s_{i_{k+1}} = s_{i'_k} s_{i'_{k+1}}$, we have $I_{i_k} I_{i_{k+1}} = I_{i'_k} I_{i'_{k+1}}$. Thus $T_j = T'_j$ for any $j > k+1$. In particular, $T_j e_{i_j} = T'_j e_{i'_j}$ for any $j \neq k, k+1$. Since $I_{i_k} e_{i_k} = I_{i'_k} I_{i'_{k+1}} e_{i'_{k+1}}$, we have $T_k e_{i_k} = T'_{k+1} e_{i'_{k+1}}$. Similarly, we have $T_{k+1} e_{i_{k+1}} = T'_k e_{i'_k}$. Thus the assertion follows.

(b) Since $I_{i_{k-1}} I_{i_k} I_{i_{k+1}} = I_{i'_{k-1}} I_{i'_k} I_{i'_{k+1}}$, we have $T_j e_{i_j} = T'_j e_{i'_j}$ for any $j \neq k, k \pm 1$. Since $I_{i_{k-1}} I_{i_k} e_{i_k} = I_{i'_{k-1}} I_{i'_k} I_{i'_{k+1}} e_{i_{k+1}}$, we have $T_k e_{i_k} = T'_{k+1} e_{i'_{k+1}}$. Similarly, we have $T_{k+1} e_{i_{k+1}} = T'_k e_{i'_k}$. Thus the assertion is proved. $\square$

As an illustration, note that in the above example we obtain the second quiver from the first by mutation at the left vertex. Immediately, we obtain the following conclusion.

PROPOSITION III.4.3. *All cluster-tilting objects in* $\mathrm{Sub}(\Lambda/I_w)$ *obtained from reduced expressions of w can be obtained from each other under repeated exchanges.*

Proof. This is immediate from Lemma III.4.2 and [BB05, Theorem 3.3.1], since we get from one reduced expression to another by applying the operations described in Lemma III.4.2. □

Using Theorem III.3.5, it can be seen that for preprojective algebras of Dynkin quivers, we get the quivers of the endomorphism algebras of cluster-tilting objects associated with reduced expressions of the longest element w_0.

For a stably 2-CY category or a triangulated 2-CY category $\mathcal{C}$ with cluster-tilting subcategories, we have an associated *cluster tilting graph* that is defined as follows. The vertices correspond to the non-isomorphic basic cluster-tilting objects, and two vertices are connected with an edge if the corresponding cluster-tilting objects differ in exactly one indecomposable summand. For cluster categories this graph is known to be connected [BMRRT06], but this is an open problem in general. For the categories $\operatorname{Sub}\Lambda/I_w$ or $\underline{\operatorname{Sub}}\,\Lambda/I_w$, it follows from Proposition III.4.3 that all standard cluster-tilting objects belong to the same component of the cluster-tilting graph, and we call this the *standard component*.

We now exhibit some classes of examples. Let Q be a connected non-Dynkin quiver without oriented cycles and with vertices $1, \ldots, n$, where there is no arrow $i \to j$ for $j > i$.

(a) Let $w = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n$. The last n vertices correspond to projectives, so the quiver for the cluster-tilting object in the stable category $\underline{\operatorname{Sub}}\,\Lambda/I_w$ is Q, which has no oriented cycles. Thus we get an alternative proof of Proposition III.3.2.

(b) Choose $w = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$. Ordering the indecomposable preprojective modules as $P_1, \ldots, P_n, \tau^{-1}P_1, \ldots, \tau^{-1}P_n, \ldots, \tau^{-i}P_1, \ldots, \tau^{-i}P_n, \ldots$, where P_i is the projective module associated with vertex i, we have a bijection between the indecomposable preprojective modules and the terms in the expression for w. Then the quiver of the corresponding cluster-tilting subcategory is the preprojective component of the AR-quiver of KQ, with an additional arrow from X to τX for each indecomposable preprojective module X. This is a direct consequence of our rule, since we know by Lemma III.3.1 that the expression for w is reduced.

(c) Now take a part $\mathcal{P}$ of the AR-quiver of the preprojective component, closed under predecessors. Consider the expression obtained from $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ by keeping only the terms corresponding to the objects in $\mathcal{P}$ under our given bijection. We show below that this new expression is reduced. It then follows directly from our rule that upon adding arrows $X \to \tau X$ when X is non-projective in $\mathcal{P}$, we get the quiver of the cluster-tilting object given by the above reduced expression. That this quiver is the quiver of a cluster-tilting object was also shown in [GLS07a] for $\mathcal{P}$ being the AR-quiver of a Dynkin quiver, and in [GLS07C] for the general case.

Lemma III.4.4. *The word associated with $\mathcal{P}$ obtained in this way is reduced.*

Proof. The word satisfies the following conditions.

(a) For each pair (i, j) of vertices connected by some edge, i and j occur every other time after removing the other vertices.

(b) $w = A_1 A_2 \cdots A_t$, where each A_s is a strictly increasing sequence of numbers in $\{1, \ldots, n\}$ such that if $j \notin A_s$, then $j \notin A_{s+1}$, and if $i < j$ are connected with an edge and $i \notin A_s$ (respectively, $j \notin A_s$), then $j \notin A_s$ (respectively, $i \notin A_{s+1}$).

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Condition (a) is immediate from the construction. For each pair (i, j) connected by some edge, we have in the AR-quiver the part

$$\cdots \qquad \begin{matrix} & & j & & & & j & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ i & & & & i & & & & i \end{matrix} \qquad \cdots$$

involving i and j; hence i and j must occur every other time to give this quiver. Thus condition (a) is satisfied.

If $A_1 \neq (1, \ldots, n)$, then w is a subsequence of $(1, \ldots, n)$. So the word is clearly reduced in this case. Thus we can assume that $A_1 = (1, \ldots, n)$. We show that for any word satisfying the conditions (a), (b) and $A_1 = (1, \ldots, n)$, the corresponding descending chain $\Lambda = T_0 \supseteq T_1 \supseteq \cdots$ is strict. Then the word is reduced by Proposition III.1.11.

We assume that $T_{k-1} = T_k$ for some k. So $I_{i_1} \cdots I_{i_k} = I_{i_1} \cdots I_{i_{k-1}}$. Take s minimal such that $A_s \neq (1, \ldots, n)$, and take i minimal such that $i \notin A_s$. By condition (b), all terms appearing after the position of i in A_s are not connected with i by an edge in Δ. In particular, the corresponding ideals commute with I_i. Upon multiplying byI_i from the right and using commutative relations, we get an equality where i is inserted in A_s after $1, \ldots, i-1$. By repeating this process, we get an equality

$$I_1 \cdots I_n I_1 \cdots I_n \cdots I_1 \cdots I_{i_k} = I_1 \cdots I_n I_1 \cdots I_n \cdots I_1 \cdots I_{i_k - 1}.$$

This contradicts Lemma III.3.1. □

In the rest of this subsection we give a proof of Theorem III.4.1. Note that statement (b) of the theorem follows directly from statement (a) and Proposition III.1.11(b).

To prove (a), let J be the Jacobson radical of Λ, let $\mathcal{M} = \mathcal{M}(i_1, i_2, \ldots)$, and let $T_{l,k} = I_{i_l}\ I_{i_{l+1}} \cdots I_{i_k}$. For $l > k$, this means that $T_{l,k} = \Lambda$. In what follows we will often use the equalities

$$e_i\ J\ e_{i'} = \begin{cases} e_i\ I_i\ e_{i'} & (i = i') \\ e_i\ \Lambda\ e_{i'} & (i \neq i'), \end{cases} \quad I_i\ e_{i'} = \begin{cases} J\ e_{i'} & (i = i') \\ \Lambda\ e_{i'} & (i \neq i'), \end{cases} \quad \text{and} \quad e_{i'}\ I_i = \begin{cases} e_{i'}\ J & (i = i') \\ e_{i'}\ \Lambda & (i \neq i'). \end{cases}$$

We have $\operatorname{Hom}_\Lambda(\Lambda_l\ e_{i_l},\ \Lambda_k\ e_{i_k}) = e_{i_l}\ (T_{l+1,k}/T_k)\ e_{i_k}$, by Lemma III.1.14, and $\operatorname{rad}_{\mathcal{M}}(\Lambda_l\ e_{i_l},\ \Lambda_k\ e_{i_k}) = e_{i_l}\ (T_{(l+1-\delta_{l,k}),k}/T_k)\ e_{i_k}$. Moreover,

$$\operatorname{rad}^2_{\mathcal{M}}(\Lambda_l\ e_{i_l},\ \Lambda_k\ e_{i_k}) = e_{i_l}\ \left(\left(T_k + \sum_{j>0} T_{(l+1-\delta_{l,j}),j}\ e_{i_j}\ T_{(j+1-\delta_{j,k}),k}\right)/T_k\right)\ e_{i_k}.$$

To get the quiver of $\mathcal{M}$, we have to compute $(\operatorname{rad}_{\mathcal{M}} / \operatorname{rad}^2_{\mathcal{M}})(\Lambda_l\ e_{i_l},\ \Lambda_k\ e_{i_k}) = E_{l,k}/D_{l,k}$ for

$$E_{l,k} = e_{i_l}\ T_{(l+1-\delta_{l,k}),k}\ e_{i_k} \supseteq D_{l,k} = e_{i_l}\ T_k\ e_{i_k} + \sum_{j>0} e_{i_l}\ T_{(l+1-\delta_{l,j}),j}\ e_{i_j}\ T_{(j+1-\delta_{j,k}),k}\ e_{i_k}.$$

We denote by k^+ the minimal number that satisfies $k < k^+$ and $i_k = i_{k^+}$, if it exists.

(i) Consider the case where there are no arrows in Q from l to k. We shall show that $E_{l,k} = D_{l,k}$.

If $l > k$ and $i_l = i_k$, then we have $l > k^+ > k$ and thus

$$E_{l,k} = e_{i_l}\ \Lambda\ e_{i_k} = e_{i_l}\ \Lambda\ e_{i_{k^+}}\ \Lambda\ e_{i_k} \subseteq D_{l,k}.$$

For the rest we shall assume that either $l \leq k$ or $i_l \neq i_k$ holds. First, let us show that

$$E_{l,k} = e_{i_l}\ T_{l+1,k-1}\ (1 - e_{i_k})\ \Lambda\ e_{i_k} = \sum_{a \neq i_k} e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k}$$

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by the following case by case study.

- If $l < k$, then $E_{l,k} = e_{i_l}\ T_{l+1,k-1} I_{i_k}\ e_{i_k} = e_{i_l}\ T_{l+1,k-1}\ (1 - e_{i_k})\ \Lambda\ e_{i_k}$.
- If $l = k$, then $E_{l,k} = e_{i_l}\ I_{i_k}\ e_{i_k} = e_{i_l}\ \Lambda\ (1 - e_{i_k})\ \Lambda\ e_{i_k}$.
- If $l > k$ and $i_l \neq i_k$, then $E_{l,k} = e_{i_l}\ \Lambda\ e_{i_k} = e_{i_l}\ I_{i_k}\ e_{i_k} = e_{i_l}\ \Lambda\ (1 - e_{i_k})\ \Lambda\ e_{i_k}$.

Thus we only have to show that $e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k} \subseteq D_{l,k}$ for any $a \neq i_k$. The following three possibilities exist.

- If $a \notin \{i_1, i_2, \dots, i_{k-1}\}$, then $T_{l+1,k-1}\ e_a = I_{i_{l+1}} \cdots I_{i_{k-1}}\ e_a = \Lambda e_a = I_{i_1} \cdots I_{i_{k-1}}\ e_a = T_{k-1} e_a$ and thus
$$e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{k-1}\ e_a\ \Lambda\ e_{i_k}\ \subseteq e_{i_l}\ T_k\ e_{i_k} \subseteq D_{l,k}.$$
- If $a \notin \{i_{k+1}, i_{k+2}, \dots, i_{k^+-1}\}$, then $e_a\ \Lambda = e_a\ I_{i_k} \cdots I_{i_{k^+}} = e_a\ T_{k,k^+}$ and thus
$$e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{l+1,k-1}\ e_a\ T_{k,k^+}\ e_{i_k} \subseteq e_{i_l}\ T_{l+1,k^+}\ e_{i_k} = e_{i_l}\ T_{l+1,k^+}\ e_{i_{k^+}}\ \Lambda\ e_{i_k} \subseteq D_{l,k}$$
since $l \neq k^+ \neq k$.
- Otherwise, there is an arrow $j \to k$ with $i_j = a$. Since $a \notin \{i_{j+1}, i_{j+2}, \dots, i_{k-1}\}$, we have $T_{j+1,k-1}\ e_a\ \Lambda = I_{i_{j+1}} \cdots I_{i_{k-1}}\ e_a\ \Lambda = \Lambda\ e_a\ \Lambda = \Lambda\ e_a\ I_{i_{j+1}} \cdots I_{i_{k-1}} = \Lambda\ e_a T_{j+1,k-1}$ and thus
$$e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{l+1,j}\ T_{j+1,k-1}\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{l+1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq D_{l,k}$$
since $l \neq j \neq k$.

In each case, we have $e_{i_l}\ T_{l+1,k-1}\ e_a\ \Lambda\ e_{i_k} \subseteq D_{l,k}$ for any $a \neq i_k$.

(ii) Consider the case $l = k^+$. We have $E_{k^+,k} = e_{i_{k^+}}\ \Lambda\ e_{i_k}$, and we shall show that $D_{k^+,k} = e_{i_l}\ J\ e_{i_k}$.

Clearly, $D_{k^+,k} \subseteq e_{i_{k^+}}\ J\ e_{i_k}$. Conversely, we have
$$e_{i_{k^+}}\ J\ e_{i_k} = e_{i_{k^+}}\ I_{i_k}\ e_{i_k} = e_{i_{k^+}}\ \Lambda\ (1 - e_{i_k})\ \Lambda\ e_{i_k} = \sum_{a \neq i_k} e_{i_{k^+}}\ \Lambda\ e_a\ \Lambda\ e_{i_k}.$$

Thus we need only show that $e_{i_{k^+}}\ \Lambda\ e_a\ \Lambda\ e_{i_k} \subseteq D_{k^+,k}$ for any $a \neq i_k$. The following two possibilities exist.

- If $a \notin \{i_1, i_2, \dots, i_{k-1}\}$, then $\Lambda e_a = I_{i_1} \cdots I_{i_{k-1}} e_a = T_{k-1}\ e_a$ and thus
$$e_{i_{k^+}}\ \Lambda\ e_a\ \Lambda\ e_{i_k} = e_{i_{k^+}}\ T_{k-1}\ e_a\ I_{i_k}\ e_{i_k} \subseteq e_{i_{k^+}}\ T_k\ e_{i_k} \subseteq D_{k^+,k}.$$
- If $a \in \{i_1, i_2, \dots, i_{k-1}\}$, then take the largest j such that $i_j = a$. We then have $\Lambda = T_{k^++1,j}$ and $e_a\ \Lambda = e_a\ I_{i_{j+1}} \cdots I_{i_k} = e_a\ T_{j+1,k}$. Thus
$$e_{i_{k^+}}\ \Lambda\ e_a\ \Lambda\ e_{i_k} = e_{i_{k^+}}\ T_{k^++1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq D_{k^+,k}$$
since $k^+ \neq j \neq k$.

In each case, we have $e_{i_{k^+}}\ \Lambda\ e_a\ \Lambda\ e_{i_k} \subseteq D_{k^+,k}$ for any $a \neq i_k$.

(iii) Finally, consider the case where $l \neq k^+$ and there is an arrow in Q from l to k. Then $l < k$, and we have $E_{l,k} = e_{i_l}\ J\ e_{i_k}$. We shall show that $D_{l,k} = e_{i_l}\ J^2\ e_{i_k}$.

First, we show that $D_{l,k} \subseteq e_{i_l}\ J^2\ e_{i_k}$. We have $e_{i_l}\ T_k\ e_{i_k} \subseteq e_{i_l}\ I_{i_l}\ I_{i_k}\ e_{i_k} = e_{i_l}\ J^2\ e_{i_k}$, and the following three possibilities exist.

- Suppose $l \leq j \leq k$; then $e_{i_l}\ T_{(l+1-\delta_{l,j}),j}\ e_{i_j}\ T_{(j+1-\delta_{j,k}),k}\ e_{i_k} \subseteq e_{i_l}\ J\ e_{i_j}\ J\ e_{i_k} \subseteq e_{i_l}\ J^2\ e_{i_k}$.

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- Suppose $k < j$. If $i_j \neq i_k$, then $e_{i_l}\ T_{l+1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq e_{i_l}\ I_{i_j}\ e_{i_j}\ J\ e_{i_k} \subseteq e_{i_l}\ J^2\ e_{i_k}$. If $i_j = i_k$, then $j \geq k^+$. Since there is an arrow $l \to k$, we have $i_l \in \{i_{k+1}, i_{k+2}, \dots, i_{k^+-1}\} \subseteq \{i_{l+1}, i_{l+2}, \dots, i_{j-1}\}$ and thus $e_{i_l}\ T_{l+1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq e_{i_l}\ I_{i_l}\ I_{i_j}\ e_{i_j}\ \Lambda\ e_{i_k} \subseteq e_{i_l}\ J^2\ e_{i_k}$.
- Suppose $l > j$. If $i_j \neq i_l$, then $e_{i_l}\ T_{l+1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq e_{i_l}\ J\ e_{i_j}\ I_k\ e_{i_k} \subseteq e_{i_l}\ J^2\ e_{i_k}$. If $i_j = i_l$, then $e_{i_l}\ T_{l+1,j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq e_{i_l}\ \Lambda\ e_{i_j}\ I_{i_l}\ I_{i_k}\ e_{i_k} \subseteq e_{i_l}\ J^2\ e_{i_k}$.

Next, we show that $e_{i_l}\ J^2\ e_{i_k} \subseteq D_{l,k}$. We have

$$e_{i_l}\ J^2\ e_{i_k} = e_{i_l}\ J\ (1 - e_{i_k})\ \Lambda\ e_{i_k} = \sum_{a \neq i_k} e_{i_l}\ J\ e_a\ \Lambda\ e_{i_k}.$$

Thus we need only show that $e_{i_l}\ J\ e_a\ \Lambda\ e_{i_k} \subseteq D_{l,k}$ for any $a \neq i_k$. The following two possibilities exist.

- If $a \notin \{i_1, i_2, \dots, i_{k-1}\}$, then $e_{i_l}\ J\ e_a = e_{i_l}\ \Lambda\ e_a = e_{i_l}\ I_{i_1} \cdots I_{i_{k-1}}\ e_a = e_{i_l}\ T_{k-1}\ e_a$ and thus
$$e_{i_l}\ J\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{k-1}\ e_a\ I_{i_k}\ e_{i_k} \subseteq e_{i_l}\ T_k\ e_{i_k} \subseteq D_{l,k}.$$
- If $a \in \{i_1, i_2, \dots, i_{k-1}\}$, then take the largest j such that $i_j = a$. We therefore have $e_a\ \Lambda = e_a\ I_{i_{j+1}} \cdots I_{i_k} = e_a\ T_{j+1,k}$. Moreover, if $j = l$, then $e_{i_l}\ J\ e_a = e_{i_l}\ T_{(l+1-\delta_{l,j}),j}\ e_a$; if $j \neq l$, then $e_{i_l}\ J\ e_a \subseteq e_{i_l}\ I_{i_{l+1}} \cdots I_{i_j}\ e_a = e_{i_l}\ T_{l+1,j}\ e_a$. Thus
$$e_{i_l}\ J\ e_a\ \Lambda\ e_{i_k} = e_{i_l}\ T_{(l+1-\delta_{l,j}),j}\ e_{i_j}\ T_{j+1,k}\ e_{i_k} \subseteq D_{l,k}$$
since $j \neq k$.

In each case, we have $e_{i_l}\ J\ e_a\ \Lambda\ e_{i_k} \subseteq D_{l,k}$ for any $a \neq i_k$. □

III.5 Substructure

Here we point out that the work in § III provides several illustrations of substructures of cluster structures. We also give some concrete examples of 2-CY categories and their cluster tilting objects, to be discussed in § IV.

Let $s_{i_1} s_{i_2} \cdots s_{i_t} \cdots$ be an infinite reduced expression which contains each $i \in \{1, \dots, n\}$ an infinite number of times. Let $T_t = I_{i_1} \cdots I_{i_t}$ and $\Lambda_t = \Lambda / T_t$. Recall that for $t < m$ we have $\operatorname{Sub} \Lambda_t \subseteq \operatorname{Sub} \Lambda_m \subseteq \mathrm{f.l.}\,\Lambda$. We then have the following result.

Theorem III.5.1. *Let the notation be as above.*

(a) $\operatorname{Sub} \Lambda_m$, $\underline{\operatorname{Sub}}\, \Lambda_m$ *and* $\mathrm{f.l.}\,\Lambda$ *have a cluster structure using the cluster-tilting subcategories with the indecomposable projectives as coefficients.*

(b) *For $t < m$, the cluster-tilting object $\Lambda_1 \oplus \cdots \oplus \Lambda_t$ in $\operatorname{Sub}\ \Lambda_t$ can be extended to a cluster-tilting object $\Lambda_1 \oplus \cdots \oplus \Lambda_t \oplus \cdots \oplus \Lambda_m$ in $\operatorname{Sub}\ \Lambda_m$, and it determines a substructure of* $\operatorname{Sub} \Lambda_m$.

(c) *The cluster-tilting object $\Lambda_1 \oplus \cdots \oplus \Lambda_t$ in $\operatorname{Sub}\ \Lambda_t$ can be extended to the cluster-tilting subcategory $\{\Lambda_i \mid i \geq 0\}$ in $\mathrm{f.l.}\,\Lambda$, and it determines a substructure of* $\mathrm{f.l.}\,\Lambda$.

Proof. (a) Since $\operatorname{Sub} \Lambda_m$ and $\underline{\operatorname{Sub}}\, \Lambda_m$ are stably 2-CY and triangulated 2-CY, respectively, they have a weak cluster structure. It follows from Proposition II.1.11 that we have no loops or 2-cycles, using the cluster-tilting objects. Then it follows from Theorem II.1.6 that we have a cluster structure for $\operatorname{Sub} \Lambda_m$ and $\underline{\operatorname{Sub}}\, \Lambda_m$.

That $\mathrm{f.l.}\,\Lambda$ also has a cluster structure comes from the fact that this is the case for all the $\operatorname{Sub} \Lambda_m$.

Parts (b) and (c) follow directly from the definition of substructure and previous results. □

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We now consider the Kronecker quiver $1 \rightrightarrows 0$ and let Λ be the associated preprojective algebra over the field K. The only strict descending chains are

$$\begin{aligned} &I_0 \supsetneq I_0 I_1 \supsetneq I_0 I_1 I_0 \supsetneq \cdots (I_0 I_1)^j \supsetneq (I_0 I_1)^j I_0 \supsetneq \cdots \quad \text{and} \\ &\qquad I_1 \supsetneq I_1 I_0 \supsetneq I_1 I_0 I_1 \supsetneq \cdots (I_1 I_0)^j \supsetneq (I_1 I_0)^j I_1 \supsetneq \cdots . \end{aligned}$$

We let T_t be the product of the first t ideals, and let $\Lambda_t = \Lambda/T_t$. Both I_0 and I_1 occur an infinite number of times in each chain. The indecomposable projective Λ-modules P_0 and P_1 have the structure

$$P_0 = \begin{smallmatrix} & & 0 & & \\ & 1 & & 1 & \\ 0 & & 0 & & 0 \\ & & \vdots & & \end{smallmatrix} \qquad P_1 = \begin{smallmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 1 & & 1 \\ & & \vdots & & \end{smallmatrix}$$

where radical layer number $2i$ has $2i$ copies of 1 for P_0 and $2i$ copies of 0 for P_1, while radical layer number $2i+1$ has $2i+1$ copies of 0 for P_0 and $2i+1$ copies of 1 for P_1. We write $P_{0,t} = P_0/J^t P_0$ and $P_{1,t} = P_1/J^t P_1$. It is then easy to see that for the chain $I_0 \supsetneq I_0 I_1 \supsetneq \cdots$, we have $\Lambda_1 = \Lambda/I_0 = P_{0,1} = (0)$, $\Lambda_2 = \Lambda/I_0 I_1 = P_{0,1} \oplus P_{1,2} = (0) \oplus \left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \end{smallmatrix}\right)$, $\Lambda_3 = \Lambda/I_0 I_1 I_0 = P_{0,3} \oplus P_{1,2}, \ldots,$ $\Lambda_{2t} = P_{0,2t+1} \oplus P_{1,2t}$, $\Lambda_{2t+1} = P_{0,2t-1} \oplus P_{1,2t}$, and so on.

Note that this calculation also shows that both our infinite chains are strict descending.

It follows from § III.4 (and is also easily seen directly) that the quiver of the cluster-tilting subcategory $\{\Lambda_i \mid i \geq 1\}$ is the following.

$$\begin{tikzcd}[column sep=small] & P_{1,2} \arrow[dr, shift left] \arrow[dr, shift right] & & P_{1,4} \arrow[ll] \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots & & P_{1,2t+2} \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \\ P_{0,1} \arrow[ur, shift left] \arrow[ur, shift right] & & P_{0,3} \arrow[ll] \arrow[ur, shift left] \arrow[ur, shift right] & & P_{0,5} \arrow[ll] & \cdots & P_{0,2t+1} \arrow[ur, shift left] \arrow[ur, shift right] & & P_{0,2t+3} \arrow[ll] & \cdots \end{tikzcd}$$

In particular, we have the cluster-tilting object $P_{0,1} \oplus P_{1,2} \oplus P_{0,3}$ for $\operatorname{Sub} \Lambda_3$, where the last two summands are projective. Hence $P_{0,1}$ is a cluster-tilting object in $\underline{\operatorname{Sub}}\, \Lambda_3$. The quiver of the endomorphism algebra consists of one vertex and no arrows. Hence $\underline{\operatorname{Sub}}\, \Lambda_3$ is equivalent to the cluster category $\mathcal{C}_K$, which has exactly two indecomposable objects. The other one is $JP_{0,3}$, obtained from the exchange sequence $0 \to JP_{0,3} \to P_{0,3} \to P_{0,1} \to 0$. Note that it is also easy to see directly that there are no other indecomposable rigid non-projective objects in $\operatorname{Sub} \Lambda_3$.

For Λ_4, we have the cluster-tilting object $P_{0,1} \oplus P_{1,2} \oplus P_{0,3} \oplus P_{1,4}$ for $\operatorname{Sub} \Lambda_4$. Again, the last two Λ_4-modules are projective, so $P_{0,1} \oplus P_{1,2}$ is a cluster-tilting object in $\underline{\operatorname{Sub}}\, \Lambda_4$. The quiver of the endomorphism algebra is $\cdot \rightrightarrows \cdot$, which has no oriented cycles; hence $\underline{\operatorname{Sub}}\, \Lambda_4$ is triangle-equivalent to the cluster category $\mathcal{C}_{K(\cdot \rightrightarrows \cdot)}$. In particular, the cluster-tilting graph is connected. We can use this fact to get a description of the rigid objects in $\operatorname{Sub} \Lambda_4$.

PROPOSITION III.5.2. *Let $\Lambda_4 = \Lambda/I_0 I_1 I_0 I_1$ be the algebra defined above. Then the indecomposable rigid Λ_4-modules in $\operatorname{Sub} \Lambda_4$ are exactly the ones of the form $\Omega^i_{\Lambda_4}(P_{0,1})$ and $\Omega^i_{\Lambda_4}(P_{1,2})$ for $i \in \mathbb{Z}$.*

Proof. For $\mathcal{C}_{K(\cdot \rightrightarrows \cdot)}$, the indecomposable rigid objects are the τ-orbits of the objects induced by the indecomposable projective $K(\cdot \rightrightarrows \cdot)$-modules. Here $\tau = [1]$, and for $\underline{\operatorname{Sub}}\, \Lambda_4$, $\Omega^{-1} = [1]$. This proves the claim. $\square$

The cluster-tilting graphs for $\operatorname{Sub} \Lambda_3$ and $\operatorname{Sub} \Lambda_4$ are $\cdot - \cdot$ and $\cdots \; \cdot - \cdot - \cdot - \cdot - \cdot \; \cdots$.

We end with the following problem.

CONJECTURE III.5.3. For any $w \in W$, the cluster-tilting graph for $\operatorname{Sub} \Lambda/I_w$ is connected.

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IV. Connections to cluster algebras

While the theory of 2-CY categories is interesting in itself, one of the motivations for investigating 2-CY categories comes from the theory of cluster algebras initiated by Fomin and Zelevinsky [FZ02]. In many situations, the 2-CY categories can be used to construct new examples of cluster algebras and also to give a new categorical model for already known examples. This has been done in, for example, [CK08, CK06] and [GLS06]. In this section, we discuss some connections with the theory developed in §§ II–III. We make explicit the notion of subcluster algebra and observe that a substructure of a (stably) 2-CY category together with a cluster map gives rise to a subcluster algebra. This provides a way of constructing new cluster algebras in the Dynkin case, or of modelling old ones, as we illustrate in § IV.2 with examples. In § IV.3 we discuss a conjecture on modelling cluster algebras by the 2-CY categories investigated in § III.

IV.1 Cluster algebras, subcluster algebras and cluster maps

We recall the notion of cluster algebras from [FZ02] and make explicit a notion of *subcluster algebras*. Actually, we shall extend the definition of cluster algebras to include the possibility of clusters with countably many elements. The coordinate rings of unipotent groups of non-Dynkin diagrams are candidates for containing such cluster algebras. We also introduce certain maps, called (strong) *cluster maps*, defined for categories with a cluster structure. The image of a cluster map gives rise to a cluster algebra. Cluster substructures on the category side give rise to subcluster algebras.

First we recall the definition of a cluster algebra, allowing countable clusters. Note that the setting we use here is not the most general one. Let $m \geq n$ be positive integers or countable numbers. Let $\mathbf{F} = \mathbb{Q}(u_1, \ldots, u_m)$ be the field of rational functions over $\mathbb{Q}$ in m independent variables. A cluster algebra is a subring of $\mathbf{F}$ that is constructed in the following way. A *seed* in $\mathbf{F}$ is a triple $(\underline{x}, \underline{c}, \widetilde{B})$, with $\underline{x}$ and $\underline{c}$ being non-overlapping sets of elements in $\mathbf{F}$, where we let $\underline{\widetilde{x}} = \underline{x} \cup \underline{c}$; we sometimes denote the seed by the pair $(\underline{\widetilde{x}}, \widetilde{B})$. Here $\underline{\widetilde{x}} = \{x_1, \ldots, x_m\}$ should be a transcendence basis for $\mathbf{F}$ and $\widetilde{B} = (b_{ij})$ is a locally finite $m \times n$ matrix with integer elements such that the submatrix B of $\widetilde{B}$ consisting of the first n rows is skew-symmetric.

The set $\underline{x} = \{x_1, \ldots, x_n\}$ is called the *cluster* of the seed, and the set $\underline{c} = \{x_{n+1}, \ldots, x_m\}$ is the *coefficient set* of the cluster algebra. The set $\underline{\tilde{x}} = \underline{x} \cup \underline{c}$ is called an *extended cluster*.

For a seed $(\underline{\tilde{x}}, \widetilde{B})$ with $\widetilde{B} = (b_{ij})$ and for $k \in \{1, \ldots, n\}$, a *seed mutation* in direction k produces a new seed $(\underline{\tilde{x}}', \widetilde{B}')$ with $\underline{\tilde{x}}' = (\underline{\tilde{x}} \setminus \{x_k\}) \cup \{x'_k\}$, where

$$x'_k = x_k^{-1}\left(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}\right).$$

This is called an *exchange relation* and $\{x_k, x'_k\}$ is called an *exchange pair*. Furthermore,

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k,\\ b_{ij} + \dfrac{|b_{ik}|\, b_{kj} + b_{ik}\, |b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Fix an (initial) seed $(\underline{\tilde{x}}, \widetilde{B})$, and consider the set $\mathbf{S}$ of all seeds obtained from $(\underline{\tilde{x}}, \widetilde{B})$ by a sequence of seed mutations. The union $\mathbf{X}$ of all elements in the clusters in $\mathbf{S}$ is called the set of *cluster variables*, and for a fixed subset of coefficients $\underline{c}_0 \subseteq \underline{c}$, the *cluster algebra* $\mathbf{A}(\mathbf{S})$ with the coefficients $\underline{c}_0$ inverted is the $\mathbb{Z}[\underline{c}, {\underline{c}_0}^{-1}]$-subalgebra of $\mathbf{F}$ generated by $\mathbf{X}$. Note that, unlike in the original definition, we do not necessarily invert all coefficients. This is to enable us to

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catch examples such as the coordinate ring of a maximal unipotent group in the Dynkin case and the homogeneous coordinate ring of a Grassmannian. Note that we often extend the scalars for cluster algebras to $\mathbb{C}$.

We now make explicit the notion of subcluster algebras. Let $\mathbf{A}$ be a cluster algebra with cluster variables $\mathbf{X}$, coefficients $\underline{c}$ and ambient field $\mathbf{F}=\mathbb{Q}(u_1,\dots,u_m)$. A *subcluster algebra* $\mathbf{A}'$ of $\mathbf{A}$ is a cluster algebra such that there exists a seed $(\underline{x},\underline{c},Q)$ for $\mathbf{A}$ and a seed $(\underline{x}',\underline{c}',Q')$ for $\mathbf{A}'$ such that:

(S1) $\underline{x}'\subseteq\underline{x}$ and $\underline{c}'\subseteq\underline{x}\cup\underline{c}$;

(S2) for each cluster variable $x_i\in\underline{x}'$, the set of arrows entering and leaving i in Q lie in Q';

(S3) the invertible coefficients $\underline{c}'_0\subseteq\underline{c}'$ satisfy $\underline{c}_0\cap\underline{c}'\subseteq\underline{c}'_0$.

Note that a subcluster algebra is not necessarily a subalgebra, since some coefficients may be inverted. Note also that $\mathbf{A}'$ is determined by the seed $(\underline{x}',\underline{c}',Q')$ and the set $\underline{c}'_0$ of invertible coefficients.

The definition implies that clusters in the subcluster algebra can be uniformly extended. The following useful fact follows from the definition.

Proposition IV.1.1.

(a) *Seed mutation in $\mathbf{A}'$ is compatible with seed mutation in $\mathbf{A}$.*

(b) *There is a set $\underline{v}$ consisting of cluster variables and coefficients in $\mathbf{A}$ such that for any extended cluster $\underline{x}'$ in $\mathbf{A}'$, $\underline{x}'\cup\underline{v}$ is an extended cluster in $\mathbf{A}$.*

Inspired by [GLS06, GLS07b] and [CC06, CK08, CK06], we introduce certain maps, which we call (strong) cluster maps, that are defined for a 2-CY category with a (weak) cluster structure in such a way that the image gives rise to a cluster algebra. We show that such maps preserve substructures as defined above and in § II.2.

Recall that a category $\mathcal{C}$ is stably 2-CY if it is either a Frobenius category where $\underline{\mathcal{C}}$ is triangulated 2-CY or a functorially finite extension-closed subcategory $\mathcal{B}$ of a triangulated 2-CY category $\mathcal{C}$.

Let $\mathcal{C}$ be a stably 2-CY category with a cluster structure defined by cluster-tilting objects, where projectives are coefficients. We assume that the cluster-tilting objects have n cluster variables and c coefficients, where $1\le n\le\infty$ and $0\le c\le\infty$. For a cluster-tilting object T, we denote by $B_{\mathrm{End}_{\mathcal{C}}(T)}$ the $m\times n$ matrix obtained by removing the last $m-n$ columns of the skew-symmetric $m\times m$ matrix corresponding to the quiver of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$, where the columns are ordered such that those corresponding to projective summands of T come last. We can also think of this as dropping the arrows between vertices in $\mathrm{End}_{\mathcal{C}}(T)$ corresponding to indecomposable projective summands of T from the quiver of $\mathrm{End}_{\mathcal{C}}(T)$.

Let $\mathbf{F}=\mathbb{Q}(u_1,\dots,u_m)$. Given a connected component Δ of the cluster-tilting graph of $\mathcal{C}$, a *cluster map* (respectively, *strong cluster map*) for Δ is a map $\varphi\colon\ \mathcal{E}=\mathrm{add}\{T\mid T\in\Delta\}\to\mathbf{F}$ (respectively, $\varphi\colon\ \mathcal{C}\to\mathbf{F}$) such that isomorphic objects have the same image, and which satisfies the following three conditions.

(M1) For a cluster-tilting object T in Δ, $\varphi(T)$ is a transcendence basis for $\mathbf{F}$.

(M2) (respectively, (M2$'$)) For all indecomposable objects M and N in $\mathcal{E}$ (respectively, $\mathcal{C}$) with $\dim_k\mathrm{Ext}^1(M,N)=1$, we have $\varphi(M)\varphi(N)=\varphi(V)+\varphi(V')$ where V and V' are in the middle of the non-split triangles or short exact sequences $N\to V\to M$ and $M\to V'\to N$.

(M3) (respectively, (M3′)) $\varphi(A \oplus A') = \varphi(A)\varphi(A')$ for all A, A' in $\mathcal{E}$ (respectively, $\mathcal{C}$).

Note that a pair (M, N) of indecomposable objects in $\mathcal{E}$ is an exchange pair if and only if $\mathrm{Ext}^1(M, N) \simeq K$ (see [BMRRT06]). A map $\varphi\colon \mathcal{C} \to \mathbf{F}$ satisfying (M2′) and (M3′) was called a *cluster character* in [Pal08]. Important examples of (strong) cluster maps have appeared in [CK08, CK06, GLS06] and, more recently, in [GLS07C, Pal08, FK07]. The following proposition is easily seen to hold.

Proposition IV.1.2. *With $\mathcal{C}$ and $\mathcal{E}$ as above, let $\varphi\colon \mathcal{E} \to \mathbf{F}$ be a cluster map. Then the following hold.*

(a) *Let $\mathbf{A}$ be the subalgebra of $\mathbf{F}$ generated by $\varphi(X)$ for $X \in \mathcal{E}$. Then $\mathbf{A}$ is a cluster algebra and $(\varphi(T), B_{\mathrm{End}_{\mathcal{C}}(T)})$ is a seed for $\mathbf{A}$ for any cluster-tilting object T in Δ.*

(b) *Let $\mathcal{B}$ be a subcategory of $\mathcal{C}$ with a substructure, and let $\mathcal{E}'$ be a subcategory of $\mathcal{B}$ defined by a connected component of the cluster-tilting graph of $\mathcal{B}$. Then $\varphi(X)$ for $X \in \mathcal{E}'$ generates a subcluster algebra of $\mathbf{A}$.*

For any subset $\underline{c}_0$ of coefficients of $\mathbf{A}$, we have a cluster algebra $\mathbf{A}[\underline{c}_0^{-1}]$. We say that *the cluster algebra $\mathbf{A}[\underline{c}_0^{-1}]$ is modelled by the cluster map $\varphi : \mathcal{E} \to \mathbf{F}$.*

IV.2 Applications to the Dynkin case

In this section we discuss the GLS-map φ, which is an important example of a strong cluster map. Using examples from § II.3, we illustrate how, via this map, the image of a substructure gives rise to a subcluster algebra.

For a Dynkin quiver Q, let U be a maximal unipotent subgroup of the complex semisimple Lie group G associated with Q, and let U^w be the unipotent cell associated with an element w of the associated Coxeter group W. In [GLS06], a map $\varphi\colon \operatorname{mod} \Lambda \to \mathbb{C}[U]$ was constructed for this case, where Λ is the preprojective algebra of Q; we call φ the GLS-map. We let $\mathbb{C}(U)$ denote the function field of U.

Part (a) of the following result is proved in [GLS07b], and (b) in [GLS06, GLS05].

Theorem IV.2.1.

(a) *The GLS-map $\varphi\colon \operatorname{mod} \Lambda \to \mathbb{C}[U] \subset \mathbb{C}(U)$ satisfies conditions (M2′) and (M3′).*

(b) *The coordinate ring $\mathbb{C}[U]$ (respectively, $\mathbb{C}[U^{w_0}]$ for the longest element w_0) is a cluster algebra modelled by a strong cluster map $\varphi : \operatorname{mod} \Lambda \to \mathbb{C}(U)$ for the standard component of the cluster-tilting graph of* mod Λ *with no (respectively, all) coefficients inverted.*

The image of a substructure $\mathcal{B}$ of mod Λ gives a subcluster algebra of $\mathbb{C}[U]$, and we illustrate this with the examples from § II.3. We omit the calculation involved in proving the isomorphisms between the subcluster algebras arising from $\mathcal{B}$ and the coordinate rings of the varieties under consideration. See [BL00] for general background on Schubert varieties and (isotropic) Grassmannians. For a subset J of size k in $[1 \dots n]$, the symbol $[J]$ will denote the $k \times k$ matrix minor of an $n \times n$ matrix with row set $[1 \dots k]$ and column set J. In the first example, G is $\mathrm{SL}_n(\mathbb{C})$ and U is the subgroup of all upper triangular $n \times n$ unipotent matrices.

Example 1 ($\mathrm{Gr}_{2,5}$-Schubert variety). Let Λ be of type A_4, and let $\mathcal{B}$ be the full additive subcategory of mod Λ from Example 1 in II.3. The associated algebraic group is then $\mathrm{SL}_5(\mathbb{C})$. Consider the Grassmannian $\mathrm{Gr}_{2,5}$ and the Schubert variety $X_{3,5}$ associated with the subset $\{3, 5\}$ of $\{1, 2, 3, 4, 5\}$. Let $w_{3,5} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{smallmatrix}\right)$ be the associated Grassmann permutation in S_5, and let

$U^{w_{3,5}}$ be the unipotent cell in U associated to $w_{3,5}$. Note that the Schubert variety $X_{3,5}$ is birationally isomorphic to the unipotent cell $U^{w_{3,5}}$ (see [BZ97]). Then $\mathbb{C}[U^{w_{3,5}}]$ is known to be a subcluster algebra of $\mathbb{C}[U]$ (see [BFZ05]).

Under the GLS-map φ from mod Λ to $\mathbb{C}[U]$, one can check that $\varphi(M_x) = [x]$, with M_x as defined in Example 1 of II.3. Since $\mathcal{B}$ has a cluster substructure of mod Λ, we know that the image gives rise to a subcluster algebra of $\mathbb{C}[U]$. Then the image of $\mathcal{B}$ under the strong cluster map φ is precisely $\mathbb{C}\big[U^{w_{3,5}}\big]$. To see this, we mutate a seed from [BFZ05] that generates the cluster algebra structure for $\mathbb{C}[U]$ to get a new seed which contains $\varphi(T)$ for the cluster-tilting object T in $\mathcal{B}$ in Example 1 of II.3. Then one proves that the image is $\mathbb{C}[U^{w_{3,5}}]$ after a proper choice of coefficients to invert.

Example 2 (The $\mathrm{SO}_8(\mathbb{C})$-isotropic Grassmannians; see also [GLS08, § 10.4.3]). Let Λ be the preprojective algebra of the Dynkin quiver D_4, let ϱ be the 4×4 anti-diagonal matrix whose (i, j) entry is $(-1)^i \delta_{i,5-j}$, and let J be the 8×8 anti-diagonal matrix, written in block form as $\left(\begin{smallmatrix} 0 & \varrho \\ \varrho^T & 0 \end{smallmatrix}\right)$.

The *even special orthogonal group* $\mathrm{SO}_8(\mathbb{C})$ is the group of 8×8 matrices $\big\{g \in \mathrm{SL}_8(\mathbb{C}) \mid g^T J g = J\big\}$. The *maximal unipotent subgroup* U of $\mathrm{SO}_8(\mathbb{C})$ consists of all 8×8 matrices in $\mathrm{SO}_8(\mathbb{C})$ which are upper triangular and unipotent, i.e. having all diagonal entries equal to 1. A more explicit description in terms of matrices in block form is

$$U = \left\{ \begin{pmatrix} u & u\varrho v \\ 0 & \varrho^T \big(u^{-1}\big)^T \varrho \end{pmatrix} \;\middle|\; \begin{array}{l} u \text{ is upper triangular and unipotent in } \mathrm{SL}_4(\mathbb{C}), \\ v \text{ is skew-symmetric in } \mathrm{M}_4(\mathbb{C}) \end{array} \right\}. \tag{3}$$

The isotropic Grassmannian $\mathrm{Gr}_{2,8}^{\mathrm{iso}}$ is the closed subvariety of the classical Grassmannian $\mathrm{Gr}_{2,8}$ that consists of all isotropic two-dimensional subspaces of $\mathbb{C}^8$. Let $\widehat{\mathrm{Gr}}_{2,8}^{\mathrm{iso}}$ be the corresponding affine cone. Let $q\colon U \to \widehat{\mathrm{Gr}}_{2,8}^{\mathrm{iso}}$ denote the map given by $q(u) = u_1 \wedge u_2$, where u_1 and u_2 are the first two rows of u in U, and let $q^*\colon \mathbb{C}\big[\widehat{\mathrm{Gr}}_{2,8}^{\mathrm{iso}}\big] \longrightarrow \mathbb{C}\big[U\big]$ be the associated homomorphism of coordinate rings.

Let $\varphi\colon$ mod $\Lambda \to \mathbb{C}[U]$ be the GLS φ-map. Then one can show that

$$\begin{array}{llll} \varphi\big(M_{16}\big) = [16], & \varphi\big(M_{24}\big) = [24], & \varphi\big(M_{25}\big) = [25], & \varphi\big(M_{26}\big) = [26], \\ \varphi\big(M_{68}\big) = [68], & \varphi\big(M_{18}\big) = [18], & \varphi\big(M_{-}\big) = \psi_{-}, & \varphi\big(M_{+}\big) = \psi_{+}, \\ \varphi\big(P_1\big) = [8], & \varphi\big(P_2\big) = [78], & \varphi\big(P_3\big) = \dfrac{[678]}{\mathrm{Pfaff}_{[1234]}}, & \varphi\big(P_4\big) = \mathrm{Pfaff}_{[1234]}\,. \end{array}$$

Here $\mathrm{Pfaff}_{[1234]}$ denotes the Pfaffian of the 4×4 skew-symmetric part v, appearing in (3), of the unipotent element, and $\psi_{\pm} = \big([18] - [27] + [36] \pm [45]\big)/2$. The functions $[678]/\,\mathrm{Pfaff}_{[1234]}$ and $\mathrm{Pfaff}_{[1234]}$ are examples of generalized minors of type D (see [FZ99]).

In the notation of Example 2 in § II.3, we have seen that

$$T = M_{16} \oplus M_{24} \oplus M_{25} \oplus M_{26} \oplus M_{68} \oplus M_{18} \oplus M_{-} \oplus M_{+} \oplus P_2$$

is a cluster-tilting object in $\mathcal{B} = \mathrm{Sub}\ P_2$, which can be extended to a cluster-tilting object $\widetilde{T} = T \oplus P_1 \oplus P_3 \oplus P_4$ for mod Λ. One shows that the initial seed used in [BFZ05], which determines a cluster algebra structure for $\mathbb{C}[U]$, is mutation-equivalent to the initial seed determined by $\widetilde{T}$, and hence generates the same cluster algebra. Since the subcategory $\mathcal{B}$ of mod Λ in Example 2 of § II.3 has a substructure as defined in § II, the connected component of the cluster-tilting graph of $\mathcal{B}$ containing T determines a subcluster algebra $\mathbf{A}'$ of $\mathbb{C}[U]$ (where [18], $\psi_\pm$ and [78] are taken as noninverted coefficients). Then we can prove that $\mathbf{A}'$ coincides with $\mathrm{Im}\, q^*$ (which we conjecture to be true more generally).

Notice that we have the cluster algebra structure for $\widehat{\mathrm{Gr}}^{\mathrm{iso}}_{2,8}$ by adjoining the coefficient [12] to $\mathrm{Im}\, q^*$.

IV.3 The non-Dynkin case

Let Q be a non-Dynkin quiver without loops, Λ the associated completed preprojective algebra and W the associated Coxeter group. Let G be the associated Kac–Moody group with a maximal unipotent subgroup U, and let U^w be the unipotent cell associated with an element w in W. In this final subsection, we pose some problems concerning the relationship with the stably 2-CY categories that correspond to the same w.

The GLS-map $\varphi\colon\ \mathrm{f.\,l.}\,\Lambda \to \mathbb{C}[U]$ is defined also in the non-Dynkin case [GLS07C], and the restriction map $\mathbb{C}[U] \to \mathbb{C}[U^w]$ defines a map

$$\varphi_w : \mathrm{Sub}\ \Lambda/I_w \subset \mathrm{f.\,l.}\,\Lambda \xrightarrow{\varphi} \mathbb{C}[U] \to \mathbb{C}[U^w].$$

Using our results from § III, we know that the transcendence degree $l(w)$ of $\mathbb{C}(U^w)$ is equal to the number of non-isomorphic summands of a cluster-tilting object in $\mathrm{Sub}\ \Lambda/I_w$. It is then natural to pose the following conjecture.

Conjecture IV.3.1. For any $w \in W$, the coordinate ring $\mathbb{C}[U^w]$ is a cluster algebra modelled by a strong cluster map $\varphi_w : \mathrm{Sub}\ \Lambda/I_w \to \mathbb{C}[U^w]$ for the standard component of the cluster-tilting graph of $\mathrm{Sub}\ \Lambda/I_w$ with all coefficients inverted.

Recall that any infinite reduced expression where all generators occur an infinite number of times gives rise to a cluster-tilting subcategory with an infinite number of non-isomorphic indecomposable objects. Since the GLS-map $\varphi : \mathrm{f.\,l.}\,\Lambda \to \mathbb{C}[U]$ satisfies (M2′) and (M3′), it is natural to ask the following.

Question IV.3.2. Does the coordinate ring $\mathbb{C}[U]$ contain a cluster algebra modelled by $\varphi\colon\ \mathrm{f.\,l.}\,\Lambda \to \mathbb{C}[U]$ for any connected component of the cluster-tilting graph of $\mathrm{f.\,l.}\,\Lambda$?

As support for Conjecture IV.3.1, we checked that this is true when Q is the Kronecker quiver $1 \rightrightarrows 0$ and the length of w is at most 4. For the case $w_3 = s_0 s_1 s_0$ and $w_4 = s_0 s_1 s_0 s_1$, we have cluster-tilting objects $T_3 = P_{0,1} \oplus P_{1,2} \oplus P_{0,3}$ in $\mathrm{Sub}\ \Lambda/I_{w_3}$ and $T_4 = P_{0,1} \oplus P_{1,2} \oplus P_{0,3} \oplus P_{1,4}$ in $\mathrm{Sub}\ \Lambda/I_{w_4}$, where $P_{i,k} = P_i/J^k P_i$ for $i = 0, 1$ and $k > 0$.

One can check that $\mathbb{C}[U^{w_3}]$ and $\mathbb{C}[U^{w_4}]$ have seeds whose quiver is the same as the quiver of $\mathrm{End}(T_3)$ and $\mathrm{End}(T_4)$ (see § III.5), obtained by dropping the arrows between projective vertices, and that the cluster graphs coincide with the cluster-tilting graphs in this case.

Note that this gives an example of a substructure of a cluster structure coming from the inclusion $\mathrm{Sub}\,\Lambda/I_{w_3} \subset \mathrm{Sub}\,\Lambda/I_{w_4}$ and a cluster map such that we get a subcluster algebra of a cluster algebra, namely $\mathbb{C}[U^{w_3}]$ as a subcluster algebra of $\mathbb{C}[U^{w_4}]$.

We remark that, recently, our conjecture has been verified more generally in [GLS07C] for adaptable elements in the Coxeter group, thus providing stronger evidence for the truth of the conjecture.

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