This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X24000397

Existence and symmetry breaking results for positive solutions of elliptic Hamiltonian systems *

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Abstract

In this paper we are interested in positive solutions of

$$\begin{cases} -\Delta u = a(x)v^{p-1} & \text{in } \Omega, \\ -\Delta v = b(x)u^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded annular domain (not necessarily an annulus) in $\mathbb{R}^N$ $(N \geq 3)$ and $a(x), b(x)$ are positive continuous functions. We show the existence of a positive solution for a range of supercritical values of p and q when the problem enjoys certain mild symmetry and monotonicity conditions. We shall also address the symmetry breaking phenomena where the system is fully symmetric. Indeed, as a consequence of our results, we shall show that problem (1) has $\left\lfloor \frac{N}{2} \right\rfloor$ (the floor of $\frac{N}{2}$) positive non-radial solutions when $a(x) = b(x) = 1$ and Ω is an annulus with certain assumptions on the radii. In general, for the radial case where the domain is an annulus, we prove the existence of a non-radial solution provided

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right),$$

where λ_H is the best constant for the Hardy inequality on Ω. We remark that the best constant λ_H for the Hardy inequality is just the characteristic of the domain, and is independent of the choices of p and q. For this reason, the aforementioned inequality plays a major role to prove the existence and multiplicity of non-radial solutions when the problem is fully symmetric. Our proofs use a variational formulation on appropriate convex subsets for which the lack of compactness is recovered for the supercritical problem.

Mathematics Subject Classification. 35B06, 35J50, 35J57.
Keywords Symmetry breaking, Hamiltonian Systems, Variational Principles

1 Introduction

The main purpose of the paper is to study the existence and multiplicity of positive solutions for the following system of supercritical nonlinear elliptic equations

$$\begin{cases} -\Delta u = a(x)v^{p-1} & \text{in } \Omega, \\ -\Delta v = b(x)u^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded annular domain (not necessarily an annulus) in $\mathbb{R}^N$, $(N \geq 3)$, $q \geq p > 2$, and $a, b \in C(\bar{\Omega})$ with $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ where a_0 and b_0 are constants. In addition, for the case when $a(x) = b(x) = 1$ and Ω is an annulus defined as

$$\Omega = \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\},$$

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we shall address the symmetry-breaking of the solutions by proving the existence and multiplicity of positive non-radial solutions provided that R_1 and R_2 satisfy certain conditions. Symmetry considerations dominate modern fundamental physics, both in quantum theory and in relativity. Such symmetry breaking is responsible for the existence of magnetism in which rotational invariance is broken.

Introduced independently by Mitidieri [22] and Van der Vorst [32], the Sobolev critical hyperbola

$$\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N} \tag{2}$$

plays a crucial role in the analysis of (1). Our main contribution is to prove existence and multiplicity of positive solutions for the supercritical case by means of the Sobolev critical hyperbola $1/p+1/q = 1-2/N$.

Over the past 30 years, Hamiltonian systems have been widely studied with results including, but not limited to, existence, multiplicity, concentration phenomena, positivity, symmetry, and Liouville theorems. We redirect the interested reader to the surveys [2, 14, 26] for an overview of the topic and to the works [3, 7, 8, 17] for some recent results. One of the first mathematical works studying systems of Hardy-Hénon type equations were done by Calanchi and Ruf in [6]. The system of Hardy-Hénon type equations is given by

$$\begin{cases} -\Delta u = |x|^{\beta} v^{q-1} & \text{in } \Omega, \\ -\Delta v = |x|^{\alpha} u^{p-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $(N \geq 3)$, with $0 \in \Omega$, $p, q > 2$, and $\alpha, \beta > -N$. The authors in [6] presented existence and non-existence of positive solutions along with symmetry breaking results for ground states when Ω is the unit ball in $\mathbb{R}^N$. Calanchi and Ruf remarked that systems of type (3) are closely related to the double weighted Hardy-Littlewood-Sobolev inequality (see [19, 30] for instance). Later on, the authors Bonheure, Moreira dos Santos, and Ramos in [1] presented qualitative properties of ground state solutions corresponding to the following system of equations:

$$\begin{cases} -\Delta u = |x|^{\beta} |v|^{q-2} v & \text{in } B, \\ -\Delta v = |x|^{\alpha} |u|^{p-2} u & \text{in } B, \\ u = v = 0 & \text{on } \partial B, \end{cases} \tag{4}$$

where B denotes the open unit ball in $\mathbb{R}^N, N \geq 1$, $\alpha, \beta \geq 0$, and $p, q > 1$. Here, the authors describe the system (4) as a Lane-Emden system with Hénon-type weights. Consider the following Hénon equation

$$\begin{cases} -\Delta u = |x|^{\alpha} |u|^{p-2} u & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $\alpha > 0$, and $p > 2$. As $|x|^{\alpha}$ increases with respect to $|x|$, we observe that reflection and symmetric arguments are inapplicable to prove radial symmetry of either positive or ground state solutions to the Hénon equation. According to [28], the authors Smets, Su, and Willem proved that the radial symmetry holds for small values of α whereas the symmetry breaks for sufficiently large values of α. However, in [24, 27], the authors showed that the ground state solutions still possess a residual symmetry, namely, the foliated Schwarz symmetry.

We would like to remark that in the Hardy-Hénon system, one gets improved compactness due to the presence of the terms $|x|^{\alpha}$ and $|x|^{\beta}$. In this paper we assume that the functions a and b in (1) are strictly positive and away from zero. As a result, no improved compactness is induced from these functions.

As we are dealing with Hamiltonian systems, we highlight some further contributions on problems of type (4) presented in [15, 20]. As for non-existence of solutions, we refer the interested reader to the works of

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[15, 20] and in particular, Theorem 2(a) in [6]. Specifically speaking, Theorem 2(a) states that the problem (4) possesses no positive solutions, u, v in the open unit ball B in $\mathbb{R}^N$ for the case

$$\frac{N+\alpha}{p} + \frac{N+\beta}{q} \leq N-2, \quad \text{provided that } p, q > 1, N \geq 3.$$

As a result, this is a consequence of a suitable Pohožaev type identity. The authors in [1] presented that the hyperbola

$$\frac{N+\alpha}{p} + \frac{N+\beta}{q} = N-2$$

is in fact, the exact threshold for the existence of positive solutions associated to (4).

Prior to introducing the main results of this paper, we conclude with some works pertaining to the Dirichlet problem for the generalized Hénon equation

$$\begin{cases} -\Delta u + \kappa u = |x|^\alpha |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and its corresponding problem for a Hénon-Schrödinger system

$$\begin{cases} -\Delta u + \kappa_1 u = |x|^\alpha \partial_u F(u,v) & \text{in } \Omega, \\ -\Delta v + \kappa_2 v = |x|^\alpha \partial_v F(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where Ω is the unit ball in $\mathbb{R}^N, N \geq 2$, $\kappa, \kappa_1, \kappa_2 \geq 0, p > 2, \alpha > -1$ and where $F : \mathbb{R}^2 \to \mathbb{R}$ is homogenous of degree $p > 2$.

We remark that problem (5) is called the Hénon equation when $\kappa \equiv 0$. In [21], Lou, Weth, and Zhang observed that the Morse index of nontrivial radial solutions corresponding to (6) (positive or sign-changing) tends to infinity as α tends to infinity. Moreover in [10], Clapp and Soares studied a related problem

$$-\Delta u_i + u_i = \sum_{j=1}^{l} \beta_{ij}|u_j|^p |u_i|^{p-2}u_i, \quad u_i \in H^1(\mathbb{R}^N), \quad i = 1, \ldots, l,$$

where $N \geq 4, 1 < p < N/(N-2)$, and (β_{ij}) represents a symmetric matrix admitting a block decomposition with entries either positive or zero within each block and negative for all remaining entries. The authors resulted in the existence of fully nontrivial solutions, that is, nontrivial solutions component-wise, provided certain conditions are satisfied for the symmetric matrix (β_{ij}). Furthermore, the authors derived the existence of solutions with positive and non-radial sign-changing components to the system of singularly perturbed elliptic equations

$$-\epsilon^2 \Delta u_i + u_i = \sum_{j=1}^{l} \beta_{ij}|u_j|^p |u_i|^{p-2}u_i, \quad u_i \in H_0^1(B_1(0)), \quad i = 1, \ldots, l,$$

where $B_1(0)$ is the unit ball exhibiting two different kinds of asymptotic behaviour — the first being solutions whose components decouple as $\epsilon \to 0$, while the second behaviour being solutions whose components remain coupled up to their limit.

In this work we are concerned with domains $\Omega \subset \mathbb{R}^N$ that are invariant by the group action $O(m) \times O(n)$ for $N = m + n$ and $m, n \geq 1$. We refer to Section 2 for the official definitions and further details. Here, we briefly introduce this class of domains in order to be able to state our main results in this paper. Inspired by the work [5], for each $x = (x_1, x_2, ..., x_N) \in \Omega \subset \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, we shall consider the change of variable

$$s := \{x_1^2 + \cdots + x_m^2\}^{\frac{1}{2}}, \quad t := \{x_{m+1}^2 + \cdots + x_N^2\}^{\frac{1}{2}}.$$

Thus the domain Ω can be represented in the (s,t) variable as follows

$$\widehat{\Omega} = \{(s,t) \in U : s > 0, t > 0\},$$

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for some appropriate domain $U \in \mathbb{R}^2$. Using polar coordinates, we can set $s = r\cos(\theta), t = r\sin(\theta)$ where $r = |x| = |(s,t)|$ and θ the usual polar angle in the (s,t)-plane. To describe the domains in terms of the above polar coordinates, we write

$$\widetilde{\Omega} := \{(\theta, r) : (s,t) \in \widehat{\Omega}\}. \tag{7}$$

We say that Ω is an annular domain if its associated domain given by $\widehat{\Omega}$ in the (s,t)-plane in $\mathbb{R}^2$ is of the form

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\}$$

in polar coordinates. Here, $g_i > 0$ is smooth on $\left[0, \frac{\pi}{2}\right]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $\left[0, \frac{\pi}{2}\right]$. Moreover, we say that Ω is an annular domain with monotonicity if g_1 is increasing and g_2 is decreasing on $\left(0, \frac{\pi}{2}\right)$. The class of annular domains with monotonicity is indeed quite rich. For instance, a regular annulus

$$\Omega = \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\},$$

is an annular domain with monotonicity. We can also consider a slightly more general version where the inner and outer boundaries are replaced with ellipsoids instead of balls. Take Ω to have outer boundary given by the ellipsoid

$$\sum_{k=1}^{m} \frac{x_k^2}{A^2} + \sum_{k=m+1}^{N} \frac{x_k^2}{B^2} = 1,$$

and the inner boundary given by

$$\sum_{k=1}^{m} \frac{x_k^2}{C^2} + \sum_{k=m+1}^{N} \frac{x_k^2}{D^2} = 1,$$

where $A, B, C, D > 0$ are chosen such that the resulting domain is an annular region.

We also assume that the function a (resp. b) is a continuous and strictly positive function of (s,t) that is $a(x) = a(s,t)$. Moreover, we say that a (resp. b) satisfies $(\mathcal{A})$ if a (resp. b) is a continuously differentiable function with respect to (s,t) and $sa_t - ta_s \le 0$ (resp. $sb_t - tb_s \le 0$) in $\widehat{\Omega}$.

As observed in [9], for problems having the $O(m) \times O(n)$ symmetry (with $N = m + n$) on an annular domain that is also invariant by $O(m) \times O(n)$, the hyperbola

$$\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N},$$

is no longer the critical hyperbola, as one has the required compactness for the following improved inequality

$$\frac{1}{p} + \frac{1}{q} \ge \max\left\{1 - \frac{2}{n+1}, 1 - \frac{2}{m+1}\right\}.$$

Our main contribution in this paper is to go well beyond the latter inequality for the lower bound of $1/p + 1/q$ and to prove the existence for

$$\frac{1}{p} + \frac{1}{q} \ge \min\left\{1 - \frac{2}{n+1}, 1 - \frac{2}{m+1}\right\}.$$

We begin with the statement of the first main result arising in this paper.

Theorem 1.1. *Suppose Ω is an annular domain with monotonicity in $\mathbb{R}^N$ for $N \ge 3$. Let $N = m + n$ for $1 \le n \le m$. In addition, assume that a and b satisfy $(\mathcal{A})$. Let $q \ge p > 2$. If*

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n+1} = \min\left\{1 - \frac{2}{n+1}, 1 - \frac{2}{m+1}\right\} \quad \text{for } n > \frac{p+1}{p-1},$$

then equation (1) *has a positive weak solution (u,v) that is invariant under the group action $O(m) \times O(n)$.*

We would like to remark that in Theorem 1.1, we are not imposing any lower bound condition on $1/p+1/q$ for the case where $n \leq (p+1)/(p-1)$. We would also like to remind the reader that the functions a and b do not add any compactness to the problem. In addition, we note that the same proof in Theorem 1.1 is valid for the case when $a = b = 1$. Similar results have been proved in an influential paper by Y. Y. Li [18] in the scalar version.

As for our remaining results, we consider a specific problem of (1) given by

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = u^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where the conditions in problem (1) are carried over to problem (8) with the exception that $a = b = 1$ and Ω is an annulus defined as

$$\Omega = \left\{ x \in \mathbb{R}^N : R_1 < |x| < R_2 \right\},$$

where the radii R_1 and R_2 satisfy certain conditions. We shall see in the following theorem that the solution obtained from Theorem 1.1 is non-radial.

Theorem 1.2. *Let $m, n \geq 1$ with $N = m + n$, and $q \geq p > 2$. Suppose (u, v) is the solution of (8) obtained in Theorem 1.1 that is invariant under the group action $O(m) \times O(n)$. Define*

$$\lambda_H := \inf_{0 \neq \eta \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla \eta|^2}{\int_\Omega \frac{|\eta|^2}{|x|^2}} dx.$$

If

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right),$$

then (u, v) is non-radial.

We remark that λ_H is the optimal constant in the classical Hardy inequality on Ω, and is independent of the choices of p and q. Indeed, λ_H is the characteristic of the domain Ω and not the supercritical nonlinearities in the system of equations (1). The following theorem addresses the multiplicity of positive solutions corresponding to problem (8).

Theorem 1.3. *For each $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, where $\lfloor x \rfloor$ is the floor function of x, and $q \geq p > 2$, the equation (8) has k distinct positive non-radial solutions if*

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right)$$

and either of the following two conditions hold:

1. *$k > (p+1)/(p-1)$ and*
$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{k+1}$$
or;
2. *$k \leq (p+1)/(p-1)$ and no lower bound condition imposed for $1/p + 1/q$.*

The following corollary states that under certain conditions on the radii, we conclude that there is a range of p and q for which λ_H becomes sufficiently large. We intend to use Theorem 1.3 to validate this corollary.

Corollary 1.4. *The following assertions hold:*

1. *For $0 < R_1 < R_2 < \infty$ and sufficiently large $(p-1)(q-1)(p/q)$, there are at least $\lfloor \frac{p+1}{p-1} \rfloor$ distinct positive non-radial solutions of (8).*

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2. For fixed

$$\frac{1}{p}+\frac{1}{q}>1-\frac{2}{\lfloor\frac{N}{2}\rfloor+1}$$

and

$$(p-1)(q-1)>\frac{q}{p}$$

with λ_H sufficiently large, there are $\lfloor\frac{N}{2}\rfloor$ distinct positive non-radial solutions of (8). For instance, under either of the following conditions, λ_H can be sufficiently large and therefore there are $\lfloor\frac{N}{2}\rfloor$ distinct positive non-radial solutions of (8):

2.(a): Let $R_1=R$ and $R_2=R+1$. Then λ_H is sufficiently large for large values of R. Note by scaling, we can take $R_1=1$ and $R_2=1+\frac{1}{R}$ and obtain the same result for large R.

2.(b): Let $R<\gamma(R)$ with $\frac{\gamma(R)}{R}\to 1$ as $R\to\infty$. With $\Omega_R=\left\{x\in\mathbb{R}^N : R<|x|<\gamma(R)\right\}$, we have that for R large enough, the λ_H corresponding to Ω_R is sufficiently large.

The structure of the paper is presented as follows. In section 2, we present some fundamental background on domains of double revolution along with some important definitions and results arising from convex analysis and minimax principles for lower semi-continuous functions. Afterwards in section 3, we use a variational formulation on convex closed subsets of an appropriate Sobolev space that plays a detrimental role in proving our main results of the paper. We conclude the paper with section 4 on the proofs of the remaining results which deal with multiplicity results of positive non-radial solutions when Ω is an annulus.

2 Preliminaries

2.1 Domains of double revolution

We dedicate this section to introduce some fundamental background on domains of double revolution. Unless otherwise stated, we assume that our domain is of double revolution. We begin with some notations. Let $\mathbb{R}^N=\mathbb{R}^m\times\mathbb{R}^n$, where $m,n\geq 1$ and $m+n=N$. For each $x=(x_1,x_2,...,x_N)\in\Omega\subset\mathbb{R}^N$, we shall consider the change of variables in terms of s and t as

$$s:=\{x_1^2+\cdots+x_m^2\}^{\frac{1}{2}},\quad t:=\{x_{m+1}^2+\cdots+x_N^2\}^{\frac{1}{2}}.$$

Definition 2.1. *We say that $\Omega\subset\mathbb{R}^N$ is a domain of double revolution if it is invariant under rotations of the first m variables and invariant under rotations of the last n variables. Equivalently, Ω is of the form $\Omega=\{x\in\mathbb{R}^N:(s,t)\in U\}$ where U is a domain in $\mathbb{R}^2$ which is symmetric with respect to the two coordinate axes. In fact,*

$$U=\{(s,t)\in\mathbb{R}^2: x=(x_1=s,x_2=0,\ldots,x_m=0,x_{m+1}=t,\ldots,x_N=0)\in\Omega\},$$

is the intersection of Ω with the (x_1,x_{m+1})-plane.

We remark that U is smooth if and only if Ω is smooth. Next, we denote $\widehat{\Omega}$ to be the intersection of U with the first quadrant of $\mathbb{R}^2$, in other words,

$$\widehat{\Omega}=\{(s,t)\in U: s>0,t>0\}. \tag{9}$$

Using polar coordinates, we can set $s=r\cos(\theta),t=r\sin(\theta)$ where $r=|x|=|(s,t)|$ and θ the usual polar angle in the (s,t)-plane.

In this paper, we consider domains to be annular with a certain monotonicity (or convexity) assumption with respect to the polar angle. In addition, all domains under consideration will be bounded in $\mathbb{R}^N$ with smooth boundary unless explicitly stated. We describe the domains in terms of the above polar coordinates by

$$\widetilde{\Omega}:=\{(\theta,r):(s,t)\in\widehat{\Omega}\}. \tag{10}$$

Now we can formally define an annular domain stated as follows.

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Definition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a domain of double revolution in $\mathbb{R}^N$ with $N = m + n$ for $m, n \geq 1$. We say that Ω is an annular domain if its associated domain given by $\widehat{\Omega}$ in the (s,t)-plane in $\mathbb{R}^2$ is of the form*

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\}$$

in polar coordinates. Here, $g_i > 0$ is smooth on $\left[0, \frac{\pi}{2}\right]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $\left[0, \frac{\pi}{2}\right]$. Moreover, we say that Ω is an annular domain with monotonicity if g_1 is increasing and g_2 is decreasing on $\left(0, \frac{\pi}{2}\right)$.

We refer the interested reader to the paper [12] for further explicit examples of annular domains. Now, we provide some assumptions on the functions a and b in which we encounter later in the paper.

Definition 2.3. *We assume that a and b are continuous and strictly positive functions of (s,t) that is $a(x) = a(s,t)$ (resp. $b(x) = b(s,t)$). Moreover, we say that a (resp. b) satisfies $(\mathcal{A})$ if a (resp. b) is a continuously differentiable function with respect to (s,t) and $sa_t - ta_s \leq 0$ (resp. $sb_t - tb_s \leq 0$) in $\widehat{\Omega}$.*

2.2 Convex analysis and minimax principles for lower semi-continuous functions

In this section, we lay out some important definitions and fundamental results from convex analysis and minimax principles for lower semi-continuous functions. Consider V to be a real Banach space, V^* to be its topological dual, and we denote the pairing of V and V^* by $\langle \cdot, \cdot \rangle$. We denote the weak topology on V induced by the pairing $\langle \cdot, \cdot \rangle$ to be $\sigma(V, V^*)$. We say a function $\Psi : V \to \mathbb{R}$ is weakly lower semi-continuous if for each $u \in V$ and for any sequence $\{u_n\}_{n=1}^{\infty}$ approaching u in the weak topology $\sigma(V, V^*)$,

$$\Psi(u) \leq \liminf_{n\to\infty} \Psi(u_n).$$

Consider $\Phi : V \to \mathbb{R} \cup \{\infty\}$ to be a proper convex function. We define the subdifferential $\partial\Psi$ of Ψ to be the following set-valued operator: if $u \in Dom(\Psi) = \{v \in V : \Psi(v) < \infty\}$, then we set

$$\partial\Psi(u) = \{u^* \in V^*; \langle u^*, v - u\rangle + \Psi(u) \leq \Psi(v), \forall v \in V\}$$

and if $u \notin Dom(\Psi)$, we set $\partial\Psi(u) = \emptyset$. If Ψ is Gâteaux differentiable at u, then we denote the derivative of Ψ at u by $D\Psi(u)$. In this case $\partial\Psi(u) = \{D\Psi(u)\}$.

Now, we arrive to the topic on minimax principles for lower semi-continuous functions. We begin with the definition of a critical point arising in Szulkin [31].

Definition 2.4. *Let V be a real Banach space, $\Phi \in C^1(V, \mathbb{R})$, and $\Psi : V \to (-\infty, \infty]$ be a proper (i.e., $Dom(\Psi) \neq \emptyset$), convex and lower semi-continuous function. A point $u \in V$ is said to be a critical point of*

$$I := \Psi - \Phi$$

if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in V.$$

We utilize the following important property of uniformly convex spaces.

Proposition 2.1. *Suppose that V is a uniformly convex Banach space. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in V such that $u_n \rightharpoonup u$ weakly $\sigma(V, V^*)$ and*

$$\limsup_{n\to\infty} \|u_n\| \leq \|u\|.$$

Then $u_n \to u$ strongly.

The following definition leads to the mountain pass theorem in which we primarily use to prove our first main result.

Definition 2.5. *We say that I satisfies the Palais-Smale compactness condition (PS) if for every sequence $\{u_n\}_{n=1}^{\infty}$ such that*

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(i) $I(u_n) \to c \in \mathbb{R}$,

(ii) $\langle D\Phi(u_n), u_n - v\rangle + \Psi(v) - \Psi(u_n) \geq -\epsilon_n\|v - u_n\|, \quad \forall v \in V$,

where $\epsilon_n \to 0$, *we have* $\{u_n\}_{n=1}^{\infty}$ *possessing a convergent subsequence.*

Now, we present the mountain pass theorem provided by Szulkin [31].

Theorem 2.6 (Mountain Pass Theorem)**.** *Let* $I : V \to (-\infty, \infty]$ *be of the form*

$$I := \Psi - \Phi$$

where $\Psi : V \to (-\infty, \infty]$ *is a proper convex and lower semi-continuous function and* $\Phi \in C^1(V, \mathbb{R})$*. Suppose that* I *satisfies the Palais-Smale condition and the mountain pass geometry (MPG):*

(i) $I(0) = 0$,

(ii) there exists $e \in V$ *such that* $I(e) \leq 0$,

(iii) there exists some ρ *such that* $0 < \rho < \|e\|$ *and for every* $u \in V$ *with* $\|u\| = \rho$ *one has* $I(u) > 0$.

Then I *has a critical value* $c > 0$ *which is characterized by*

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I[\gamma(t)],$$

where $\Gamma = \{\gamma \in C([0, 1], V) : \gamma(0) = 0, \gamma(1) = e\}$.

3 A variational formulation and the proof of Theorem 1.1

Our interest in this paper lies within solving the following system

$$\begin{cases} -\Delta u = a(x)v^{p-1} & \text{in } \Omega, \\ -\Delta v = b(x)u^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where Ω is a bounded annular domain (not necessarily an annulus) in $\mathbb{R}^N$, ($N \geq 3$), $q \geq p > 2$, and $a, b \in C(\bar{\Omega})$ with $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ where a_0 and b_0 are constants. Let $p' = p/(p-1)$ and consider the Banach space $V = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \cap L^q(\Omega)$ equipped with the following norm

$$\|u\|_V = \|u\|_{W^{2,p'}(\Omega)} + \|u\|_{W_0^{1,p'}(\Omega)} + \|u\|_{L^q(\Omega)}.$$

Recall the duality pairing between V and its dual space V^* is defined by

$$\langle u, u^*\rangle = \int_\Omega u(x)u^*(x)dx, \quad \forall u \in V, u^* \in V^*.$$

Following for instance the work by Wang [33], one can get from (11) that

$$v = (-\Delta u)^{\frac{1}{p-1}} a(x)^{-\frac{1}{p-1}}.$$

Inserting this equation into the second equation of (11) results in the following scalar equation corresponding to the u-component:

$$-\Delta \left((-\Delta u)^{\frac{1}{p-1}} a(x)^{-\frac{1}{p-1}} \right) = b(x)u^{q-1}.$$

Considering the fact that $p' - 1 = 1/(p-1)$ we arrive at

$$-\Delta \left((-\Delta u)^{p'-1} a(x)^{-(p'-1)} \right) = b(x)u^{q-1}. \tag{12}$$

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Formally, the Euler-Lagrange functional associated to problem (12) is given by

$$I(u) := \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{q}\int_\Omega b(x)|u|^q dx.$$

We define $\Psi : V \to \mathbb{R}$ and $\Phi : V \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx$$

and

$$\Phi(u) = \frac{1}{q}\int_\Omega b(x)|u|^q dx,$$

respectively. Let K be a convex subset of V. Finally, we introduce the functional $I_K : V \to (-\infty, \infty]$ to be defined by

$$I_K(u) := \Psi_K(u) - \Phi(u) \tag{13}$$

where the restriction of Ψ on K at u, denoted by $\Psi_K(u)$ is defined by

$$\Psi_K(u) = \begin{cases} \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx, & u \in K, \\ +\infty, & u \notin K. \end{cases}$$

We denote the functional I_K the Euler-Lagrange functional corresponding to (12) restricted on K.

The following proposition states the existence of a critical point for the functional I_K and we use Theorem 2.6 to prove the proposition.

Proposition 3.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $q \geq p > 2$. Let $a, b \in C(\bar{\Omega})$ with $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ where a_0 and b_0 are constants. Consider the Euler-Lagrange functional $I : V \to \mathbb{R}$ associated to problem (12)*

$$I(u) := \frac{1}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{q}\int_\Omega b(x)|u|^q dx.$$

Let K be a weakly closed convex subset of $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ which is compactly embedded in $L^q(\Omega)$. Then the functional I has a critical point $\bar{u}$ on K by means of Definition 2.4.

Proof. Note that the function a is bounded from above, and is also away from zero. Thus, an equivalent norm on $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ can be defined by

$$\|u\|^{p'}_{W^{2,p'}(\Omega)} = \int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx, \qquad \forall u \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega).$$

By assumption, K is compactly embedded in $L^q(\Omega)$. So there exists a constant $C > 0$ such that

$$\|u\|_{W^{2,p'}(\Omega)} \leq \|u\|_V \leq C\|u\|_{W^{2,p'}(\Omega)} \quad \forall u \in K. \tag{14}$$

In order to satisfy the mountain pass theorem, we must satisfy the (PS)-compactness condition and the mountain pass geometry. We begin by verifying the (PS)-compactness condition. Suppose that $\{u_n\}_{n=1}^\infty$ is a sequence in K such that $I(u_n) \to c \in \mathbb{R}, \epsilon_n \to 0$, and

$$\Psi_K(v) - \Psi_K(u_n) + \langle D\Phi(u_n), u_n - v\rangle \geq -\epsilon_n\|v - u_n\|_V \quad \forall v \in V. \tag{15}$$

We want to prove that $\{u_n\}_{n=1}^\infty$ has a converging subsequence in V. First, we prove that $\{u_n\}_{n=1}^\infty$ is bounded in $W^{2,p'}(\Omega)$. Since $I(u_n) \to c$, it follows that for large values of n, we obtain

$$\begin{aligned} I(u_n) &= \frac{1}{p'}\int_\Omega \frac{|-\Delta u_n|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{q}\int_\Omega b(x)|u_n|^q dx \\ &= \frac{1}{p'}\|u_n\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{q}\int_\Omega b(x)|u_n|^q dx \\ &\leq c + 1. \end{aligned} \tag{16}$$

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Note that

$$\langle D\Phi(u_n), u_n\rangle = \int_\Omega b(x)|u_n|^{q-1}u_n \cdot u_n dx = \int_\Omega b(x)|u_n|^q dx.$$

Since $q > 2 > p'$, there exists $\delta > 0$ such that

$$\delta + 1 > \big(1 + \frac{\delta}{q}\big)^{p'}.$$

Setting $v = ru_n$ in (15) with $r = 1 + \delta/q$, we get

$$\begin{aligned}
&\frac{1}{p'}\int_\Omega \frac{|-\Delta ru_n|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{p'}\int_\Omega \frac{|-\Delta u_n|^{p'}}{a(x)^{p'-1}}dx + \int_\Omega b(x)|u_n|^{q-2}u_n \cdot (u_n - ru_n)dx \geq -\epsilon_n\|ru_n - u_n\|_V \\
\Longrightarrow\ &\frac{r^{p'}}{p'}\int_\Omega \frac{|-\Delta u_n|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{p'}\int_\Omega \frac{|-\Delta u_n|^{p'}}{a(x)^{p'-1}}dx + \int_\Omega b(x)|u_n|^{q-2}u_n \cdot (u_n - ru_n)dx \geq -(r-1)\epsilon_n\|u_n\|_V \\
\Longrightarrow\ &\frac{r^{p'}-1}{p'}\|u_n\|^{p'}_{W^{2,p'}(\Omega)} + (1-r)\int_\Omega b(x)|u_n|^q dx \geq -(r-1)\epsilon_n\|u_n\|_V \\
\Longrightarrow\ &\frac{1-r^{p'}}{p'}\|u_n\|^{p'}_{W^{2,p'}(\Omega)} + (r-1)\int_\Omega b(x)|u_n|^q dx \leq (r-1)\epsilon_n\|u_n\|_V. \qquad (17)
\end{aligned}$$

Multiplying (16) by δ and adding the result by (17) yield that

$$\Big(\frac{\delta}{p'} + \frac{1-r^{p'}}{p'}\Big)\|u_n\|^{p'}_{W^{2,p'}(\Omega)} \leq \delta c + \delta + \frac{\epsilon_n\delta}{q}\|u_n\|_V$$

Note that for n large enough, by applying (14), we obtain

$$\begin{aligned}
\|u_n\|^{p'}_{W^{2,p'}(\Omega)} &< C_0(1 + \|u_n\|_V) \\
&\leq C_0\Big(1 + C\|u_n\|_{W^{2,p'}(\Omega)}\Big).
\end{aligned}$$

for a constant C_0. Thus, we conclude that $\{u_n\}_{n=1}^\infty$ is bounded in $W^{2,p'}(\Omega)$. Since $\{u_n\}_{n=1}^\infty$ is bounded in $W^{2,p'}(\Omega)$, it follows that, up to a subsequence, there exists $\bar{u} \in W^{2,p'}(\Omega)$ such that $u_n \rightharpoonup \bar{u}$ weakly in $W^{2,p'}(\Omega)$ and $u_n \to \bar{u}$ a.e.. By assumption that K is compactly embedded in $L^q(\Omega)$, we can deduce from boundedness of $\{u_n\}_{n=1}^\infty \subset K$ in $W^{2,p'}(\Omega)$ strong convergence of u_n to $\bar{u}$ in $L^q(\Omega)$. Setting $v = \bar{u}$ in (15), we get

$$\frac{1}{p'}\Big(\|\bar{u}\|^{p'}_{W^{2,p'}(\Omega)} - \|u_n\|^{p'}_{W^{2,p'}(\Omega)}\Big) + \int_\Omega b(x)|u_n|^{q-2}u_n \cdot (u_n - \bar{u})dx \geq -\epsilon_n\|u_n - \bar{u}\|_V. \qquad (18)$$

Taking $\limsup_{n\to\infty}$ on both sides of (18), we obtain

$$\frac{1}{p'}\Big(\limsup_{n\to\infty}\|u_n\|^{p'}_{W^{2,p'}(\Omega)} - \|\bar{u}\|^{p'}_{W^{2,p'}(\Omega)}\Big) \leq 0.$$

By Proposition 2.1 we have

$$u_n \to \bar{u} \quad \text{strongly in } W^{2,p'}(\Omega)$$

and therefore, we conclude that $u_n \to \bar{u}$ strongly in V, as desired. Now, we verify the mountain pass geometry for the functional I_K. Clearly, $I_K(0) = 0$ which satisfies condition (i). For condition (ii), let $u \in K$. Then for $t \geq 0$,

$$\begin{aligned}
I_K(tu) &= \frac{1}{p'}\int_\Omega \frac{|-\Delta tu|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{q}\int_\Omega b(x)|tu|^q dx \\
&= \frac{t^{p'}}{p'}\int_\Omega \frac{|-\Delta u|^{p'}}{a(x)^{p'-1}}dx - \frac{t^q}{q}\int_\Omega b(x)|u|^q dx.
\end{aligned}$$

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Since $q > 2 > p'$, it follows that for t large enough, we obtain $I_K(tu) < 0$ and setting $e := tu$, condition (ii) holds. To satisfy condition (iii), take $u \in K$ with $\|u\|_V = \rho > 0$. Then

$$I_K(u) = \frac{1}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{q}\int_\Omega b(x)|u|^q dx.$$

By (14), there exists a constant $C > 0$ such that for all $u \in K$, we have

$$\|u\|_{W^{2,p'}(\Omega)} \le \|u\|_V \le C\|u\|_{W^{2,p'}(\Omega)}. \tag{19}$$

In addition, we have

$$\int_\Omega b(x)|u|^q dx \le C_1\|u\|^q_V,$$

for some constant $C_1 > 0$. So,

$$\begin{aligned} I_K(u) &= \frac{1}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)} - \frac{1}{q}\int_\Omega b(x)|u|^q dx \\ &\ge \frac{1}{p'}\|u\|^{p'}_{W^{2,p'}(\Omega)} - \frac{C_1}{q}\|u\|^q_V \\ &\ge \frac{1}{p'C^{p'}}\|u\|^{p'}_V - \frac{C_1}{q}\|u\|^q_V \\ &= \frac{1}{p'C^{p'}}\rho^{p'} - \frac{C_1}{q}\rho^q > 0, \end{aligned}$$

provided ρ is small enough as $q > 2 > p'$. Note that if $u \notin K$, then $I_K(u) > 0$ by definition of $\Psi_K(u)$. Thus, the mountain pass geometry holds for the functional I_K. By the mountain pass theorem, I_K has a critical point $\bar{u} \in K$ with $I_K(\bar{u}) = c$ where $c > 0$ is the critical value characterized by

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0, \gamma(1) = e, I_K(\gamma(1)) \le 0\}$. □

Lemma 3.1. *Let V be a reflexive Banach space and let $f : V \to \mathbb{R}$ be a convex and differentiable functional. If*

$$f(u) - f(\bar{u}) \ge \langle Df(u), u - \bar{u}\rangle, \tag{20}$$

then $Df(u) = Df(\bar{u})$, where $\langle .,.\rangle$ is the duality pairing between V and V^*. In particular, if f is strictly convex, then $u = \bar{u}$.

Proof. By the convexity of f,

$$f(\bar{u}) - f(u) \ge \langle Df(u), \bar{u} - u\rangle \implies f(u) - f(\bar{u}) \le \langle Df(u), u - \bar{u}\rangle. \tag{21}$$

So, (20) and (21) implies that

$$f(u) - f(\bar{u}) = \langle Df(u), u - \bar{u}\rangle.$$

Note that for all $v \in V$,

$$f(v) - f(u) \ge \langle Df(u), v - u\rangle.$$

Equivalently,

$$\begin{aligned} f(v) \ge f(u) + \langle Df(u), v - u\rangle &\implies f(v) \ge f(u) + \langle Df(u), v\rangle - \langle Df(u), u\rangle \\ &\implies f(v) - \langle Df(u), v\rangle \ge f(u) - \langle Df(u), u\rangle. \end{aligned}$$

Let $G(v) = f(v) - \langle Df(u), v\rangle$. Then for all $v \in V$,

$$G(v) = f(v) - \langle Df(u), v\rangle \ge f(u) - \langle Df(u), u\rangle = G(u),$$

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and when $v = \bar{u}$,

$$G(\bar{u}) = f(\bar{u}) - \langle Df(u), \bar{u}\rangle = f(u) - \langle Df(u), u\rangle = G(u).$$

So G attains its minimum at $v = \bar{u}$, i.e., $DG(\bar{u}) = 0$. Thus,

$$Df(\bar{u}) - Df(u) = 0.$$

Now, we show that $u = \bar{u}$ provided that f is strictly convex. Indeed, it follows that

$$\langle Df(u) - Df(\bar{u}), u - \bar{u}\rangle = 0,$$

from which we obtain the desired result. $\square$

Inspired by an argument in [23], the following proposition links the critical points of I_K to the solutions of the system (11).

Proposition 3.2. *Let $\bar{u}$ be a critical point of the functional I_K. If there exists $\tilde{u} \in K$ and $\tilde{v} \in W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega)$ where $1/q + 1/q' = 1$ such that*

$$\begin{cases} -\Delta\tilde{u} = a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta\tilde{v} = b(x)|\bar{u}|^{q-2}\bar{u}, \end{cases} \tag{22}$$

then $\bar{u} = \tilde{u}$, and $(\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} -\Delta u = a(x)|v|^{p-2}v \\ -\Delta v = b(x)|u|^{q-2}u. \end{cases}$$

Proof. Define the functional $F : W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) \to \mathbb{R}$ by

$$F(w) = \frac{1}{p'}\int_\Omega \frac{|-\Delta w|^{p'}}{a(x)^{p'-1}}dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}w dx.$$

We first show that $\tilde{u}$ is a critical point of F. By (22) we have that

$$\begin{cases} -\Delta\tilde{u} = a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta\tilde{v} = b(x)|\bar{u}|^{q-2}\bar{u}. \end{cases}$$

Therefore,

$$\begin{cases} \tilde{v} = \frac{1}{a(x)^{p'-1}}|-\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u}) \\ \bar{u} = \frac{1}{b(x)^{q'-1}}|-\Delta\tilde{v}|^{q'-2}(-\Delta\tilde{v}). \end{cases} \tag{23}$$

Now, take $\eta \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$. It follows that

$$\begin{aligned} \langle F'(\tilde{u}), \eta\rangle &= \int_\Omega \frac{1}{a(x)^{p'-1}}|-\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta\eta)dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}\eta dx \\ &= \int_\Omega \tilde{v}(-\Delta\eta)dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}\eta dx, \qquad \text{(As a result of (23))} \\ &= \int_\Omega (-\Delta\tilde{v})\eta dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}\eta dx \\ &= \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}\eta dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}\eta dx, \qquad \text{(As a result of (22))} \\ &= 0. \end{aligned}$$

Thus, $\tilde{u}$ is a critical point of F. It then follows that

$$0 = \langle F'(\tilde{u}), \tilde{u} - \bar{u}\rangle = \int_\Omega \frac{1}{a(x)^{p'-1}}|-\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u} - \bar{u}))dx - \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}(\tilde{u} - \bar{u})dx,$$

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from which we obtain

$$\int_\Omega \frac{1}{a(x)^{p'-1}}|-\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u}-\bar{u}))dx = \int_\Omega b(x)|\bar{u}|^{q-2}\bar{u}(\tilde{u}-\bar{u})dx. \tag{24}$$

Since $\bar{u}$ is a critical point on I_K, by definition of a critical point, we have

$$\frac{1}{p'}\int_\Omega \frac{|-\Delta w|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{p'}\int_\Omega \frac{|-\Delta\bar{u}|^{p'}}{a(x)^{p'-1}}dx \geq \langle b(x)|\bar{u}|^{q-2}\bar{u}, w-\bar{u}\rangle, \quad \forall w \in K. \tag{25}$$

Plugging (24) into (25) for $w = \tilde{u}$, we get

$$\frac{1}{p'}\int_\Omega \frac{|-\Delta\tilde{u}|^{p'}}{a(x)^{p'-1}}dx - \frac{1}{p'}\int_\Omega \frac{|-\Delta\bar{u}|^{p'}}{a(x)^{p'-1}}dx \geq \int_\Omega \frac{1}{a(x)^{p'-1}}|-\Delta\tilde{u}|^{p'-2}(-\Delta\tilde{u})(-\Delta(\tilde{u}-\bar{u}))dx.$$

Thus, by Lemma 3.1, we obtain

$$\tilde{u} = \bar{u}.$$

The result now follows from (22) considering $\tilde{u} = \bar{u}$. $\square$

So far, we have considered K to be a weakly closed convex subset of $W^{2,p'}(\Omega)$ which is compactly embedded in $L^q(\Omega)$. Now, we explicitly define our convex set K to be given by

$$K = K(m,n) := \left\{0 \leq u = u(s,t) \in W^{2,p'}_G(\Omega) \cap W^{1,p'}_0(\Omega) : su_t - tu_s \leq 0 \text{ a.e. in } \widehat{\Omega}\right\}, \tag{26}$$

where $W^{2,p'}_G(\Omega) := \{u \in W^{2,p'}(\Omega) : gu = u, \quad \forall g \in G\}$ where $G := O(m) \times O(n)$. Here, $O(k)$ is the orthogonal group in $\mathbb{R}^k$ with $gu(x) := u(g^{-1}x)$. We remind the reader that we can express K as functions u such that if we write (s,t) in terms of polar coordinates, we have $u_\theta \leq 0$ on $\widetilde{\Omega}$ defined in (10). Before we introduce the embedding theorem for annular domains, for the convenience of the reader, we recall the following standard embedding theorem for which we make frequent use in this paper.

Theorem 3.2. *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^k$. Let $j \geq 1$ be an integer and let $1 \leq \mathcal{P} < \infty$. Suppose $\mathcal{O}$ satisfies the cone condition. Then the following embeddings are compact:*

(i) If $j\mathcal{P} < k$, then

$$W^{j,\mathcal{P}}(\mathcal{O}) \hookrightarrow L^d(\mathcal{O}), \quad \text{for } 1 \leq d < \mathcal{P}^* = k\mathcal{P}/(k - j\mathcal{P});$$

(ii) If $j\mathcal{P} \geq k$, then

$$W^{j,\mathcal{P}}(\mathcal{O}) \hookrightarrow L^d(\mathcal{O}), \quad \text{for } 1 \leq d < \infty.$$

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^{m+n}$ be an annular domain of double revolution.*

(i) (Embedding without monotonicity). Let $\mathcal{P} > 1$. Suppose Ω has no monotonicity and

$$1 \leq d < \min\left\{\frac{(m+1)\mathcal{P}}{(m+1)-2\mathcal{P}}, \frac{(n+1)\mathcal{P}}{(n+1)-2\mathcal{P}}\right\}.$$

Then the embedding $W^{2,\mathcal{P}}_G(\Omega) \hookrightarrow L^d(\Omega)$ is compact with the obvious interpretation if $(m+1) - 2\mathcal{P} \leq 0$ and $(n+1) - 2\mathcal{P} \leq 0$.

(ii) (Embedding with monotonicity). Let $p' > 1$ and suppose Ω is a domain of double revolution with monotonicity, $n \leq m$ and

$$1 \leq d < \frac{(n+1)p'}{(n+1)-2p'} = \max\left\{\frac{(m+1)p'}{(m+1)-2p'}, \frac{(n+1)p'}{(n+1)-2p'}\right\}.$$

In addition, let

$$K := \left\{0 \leq u = u(s,t) \in W^{2,p'}_G(\Omega) \cap W^{1,p'}_0(\Omega) : su_t - tu_s \leq 0 \text{ a.e. in } \widehat{\Omega}\right\}.$$

Then the embedding $K \hookrightarrow L^d(\Omega)$ is compact with the obvious interpretation if $(n+1) - 2p' \leq 0$.

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Proof. We begin by proving (i). Assume that $N = m+n$. Then, expressing in terms of s and t, i.e., $u(x) = u(s,t)$, we obtain

$$\int_\Omega |u|^d dx = c\int_{\widehat{\Omega}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt.$$

Take δ small enough so that $t \geq \delta$ if and only if $s \leq \delta$. So

$$\int_{\widehat{\Omega}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt = \int_{\{\widehat{\Omega}, t\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt + \int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt. \tag{27}$$

Looking at the first term on the right hand side of (27),

$$\int_{\{\widehat{\Omega}, t\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt \leq c_1 \int_{\widehat{\Omega}} |u(s,t)|^d s^{m-1} ds dt.$$

Let $u(s,t) = u(y,z)$ where $s = |y|$ and $t = |z|$. Then by change of variables,

$$\int_{\widehat{\Omega}} |u(s,t)|^d s^{m-1} ds dt = c_0 \int_{\Omega_1} |u(y,t)|^d dy dt,$$

where $\Omega_1 = \{(y,t) : (|y|,t) \in \widehat{\Omega}\} \in \mathbb{R}^m \times \mathbb{R}$. Note that $\Omega_1 \subset \mathbb{R}^{m+1}$. If

$$d < \frac{(m+1)\mathcal{P}}{(m+1) - 2\mathcal{P}},$$

then by Theorem 3.2,

$$\begin{aligned}
\left(\int_{\Omega_1} |u(y,t)|^d dy dt\right)^{\mathcal{P}/d} &\leq c_2 \|u\|^{\mathcal{P}}_{W^{2,\mathcal{P}}(\Omega_1)} \\
&\leq c_3 \int_{\Omega_1} \left(|D^2 u(y,t)|^{\mathcal{P}} + |\nabla u(y,t)|^{\mathcal{P}} + |u(y,t)|^{\mathcal{P}}\right) t^{n-1} dy dt \\
&\leq c_4 \int_{\Omega} \left(|D^2 u(y,z)|^{\mathcal{P}} + |\nabla u(y,z)|^{\mathcal{P}} + |u(y,z)|^{\mathcal{P}}\right) dy dz \\
&= c_4 \|u\|^{\mathcal{P}}_{W^{2,\mathcal{P}}(\Omega)}.
\end{aligned}$$

So we have the compact embedding $W^{2,\mathcal{P}}_G(\Omega) \hookrightarrow L^d(\Omega)$ for

$$d < \frac{(m+1)\mathcal{P}}{(m+1) - 2\mathcal{P}}.$$

For the second term on the right hand side of (27) we have that

$$\begin{aligned}
\int_{\{\widehat{\Omega}, s\geq\delta\}} |u(s,t)|^d s^{m-1} t^{n-1} ds dt &\leq c_1' \int_{\widehat{\Omega}} |u(s,t)|^d t^{n-1} ds dt \\
&= c_1'' \int_{\Omega_2} |u(s,z)|^d ds dz,
\end{aligned}$$

where $\Omega_2 = \{(s,z) : (s,|z|) \in \widehat{\Omega}\} \in \mathbb{R}^n \times \mathbb{R}$. Note that $\Omega_2 \subset \mathbb{R}^{n+1}$. If

$$d < \frac{(n+1)\mathcal{P}}{(n+1) - 2\mathcal{P}},$$

then by Theorem 3.2,

$$\begin{aligned}
\left(\int_{\Omega_2} |u(s,z)|^d ds dz\right)^{\mathcal{P}/d} &\leq c_2' \|u\|^{\mathcal{P}}_{W^{2,\mathcal{P}}(\Omega_2)} \\
&\leq c_3' \int_{\Omega_2} \left(|D^2 u(s,z)|^{\mathcal{P}} + |\nabla u(s,z)|^{\mathcal{P}} + |u(s,z)|^{\mathcal{P}}\right) s^{m-1} ds dz \\
&\leq c_4' \int_{\Omega} \left(|D^2 u(y,z)|^{\mathcal{P}} + |\nabla u(y,z)|^{\mathcal{P}} + |u(y,z)|^{\mathcal{P}}\right) dy dz \\
&= c_4' \|u\|^{\mathcal{P}}_{W^{2,\mathcal{P}}(\Omega)}.
\end{aligned}$$

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So we have the embedding $W^{2,\mathcal{P}}_G(\Omega) \hookrightarrow L^d(\Omega)$ is compact for

$$d < \frac{(n+1)\mathcal{P}}{(n+1)-2\mathcal{P}}.$$

Taking

$$\min\left\{\frac{(m+1)\mathcal{P}}{(m+1)-2\mathcal{P}}, \frac{(n+1)\mathcal{P}}{(n+1)-2\mathcal{P}}\right\},$$

we obtain the desired result in part (i). Now, we proceed with proving part (ii). Let $1 \leq n \leq m$ and

$$d < \frac{(n+1)p'}{(n+1)-2p'}.$$

Using polar coordinates with $s = r\cos(\theta)$ and $t = r\sin(\theta)$ we obtain

$$\int_{\widehat{\Omega}} u(s,t)^d s^{m-1} t^{n-1} dsdt = \int_0^{\pi/2}\int_{g_1}^{g_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^d r dr d\theta.$$

For $\theta \in [\pi/3, \pi/2]$ we have that $\sin(\theta) \leq c\sin(\theta - \pi/4)$ for some constant $c > 0$. According to the monotonicity properties of g_1, g_2 and $\theta \mapsto u(r,\theta)$ we obtain that

$$\begin{aligned}
&\int_{\pi/3}^{\pi/2}\int_{g_1(\theta)}^{g_2(\theta)} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1} u(r,\theta)^d r dr d\theta \\
&\leq c^{n-1}\int_{\pi/3}^{\pi/2}\int_{g_1(\theta-\pi/4)}^{g_2(\theta-\pi/4)} r^{m-1}\cos^{m-1}(\theta-\pi/4) r^{n-1}\sin^{n-1}(\theta-\pi/4) u(r,\theta-\pi/4)^d r dr d\theta \\
&= c^{n-1}\int_{\pi/12}^{\pi/4}\int_{g_1(\theta)}^{g_2(\theta)} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^d r dr d\theta.
\end{aligned}$$

Thus, there is a constant $C_1 > 0$ such that

$$\int_0^{\pi/2}\int_{g_1}^{g_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^d r dr d\theta \leq C_1\int_0^{\pi/3}\int_{g_1}^{g_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^d r dr d\theta.$$

On the other hand,

$$\int_0^{\pi/3}\int_{g_1}^{g_2} r^{m-1}\cos^{m-1}(\theta) r^{n-1}\sin^{n-1}(\theta) u(r,\theta)^d r dr d\theta = \int_{\{\widehat{\Omega}, s\geq\beta\}} u(s,t)^d s^{m-1} t^{n-1} dsdt$$

for some positive constant β. Hence,

$$\Big(\int_{\{\widehat{\Omega}, s\geq\beta\}} u(s,t)^d s^{m-1} t^{n-1} dsdt\Big)^{p'/d} \leq C_2\Big(\int_{\{\widehat{\Omega}, s\geq\beta\}} u(s,t)^d t^{n-1} dsdt\Big)^{p'/d}.$$

Thus, by part (i), we have

$$\begin{aligned}
\Big(\int_{\{\widehat{\Omega}, s\geq\beta\}} u(s,t)^d t^{n-1} dsdt\Big)^{p'/d} &\leq C_3\int_{\{\widehat{\Omega}, s\geq\beta\}}\Big(|D^2u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\Big) t^{n-1} dsdt \\
&\leq C_4\int_{\{\widehat{\Omega}, s\geq\beta\}}\Big(|D^2u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\Big) t^{n-1} s^{m-1} dsdt \\
&\leq C_5\int_{\widehat{\Omega}}\Big(|D^2u(s,t)|^{p'} + |\nabla u(s,t)|^{p'} + |u(s,t)|^{p'}\Big) t^{n-1} s^{m-1} dsdt \\
&= C_6\int_{\Omega}\Big(|D^2u|^{p'} + |\nabla u|^{p'} + |u|^{p'}\Big) dx \\
&= C_6\|u\|^{p'}_{W^{2,p'}(\Omega)}.
\end{aligned}$$

This completes the proof. $\square$

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Remark 3.4. *Let $p > 1$ and let p' be the conjugate of p, that is,*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Suppose Ω is an annular domain with monotonicity, and $n \leq m$. We can rewrite the condition in Theorem 3.3 (ii) given by

$$1 \leq d < \frac{(n+1)p'}{(n+1)-2p'} = \max\left\{\frac{(m+1)p'}{(m+1)-2p'}, \frac{(n+1)p'}{(n+1)-2p'}\right\}, \quad \text{for } n > 2p'-1,$$
$$1 \leq d < \infty, \quad \text{for } n \leq 2p'-1,$$

as follows:

$$\frac{1}{p} + \frac{1}{d} > 1 - \frac{2}{n+1} = \min\left\{1 - \frac{2}{m+1}, 1 - \frac{2}{n+1}\right\}, \quad \text{for } n > \frac{p+1}{p-1},$$
$$\text{no lower bound condition imposed on } \frac{1}{p} + \frac{1}{d}, \quad \text{for } n \leq \frac{p+1}{p-1}.$$

Proof. By Theorem 3.3 (ii), we have compactness when

$$1 \leq d < \frac{(n+1)p'}{(n+1)-2p'} = \max\left\{\frac{(m+1)p'}{(m+1)-2p'}, \frac{(n+1)p'}{(n+1)-2p'}\right\}, \quad \text{for } n+1-2p' > 0,$$

and

$$1 \leq d < \infty, \quad \text{for } n+1-2p' \leq 0.$$

Equivalently,

$$1 \leq d < \frac{(n+1)p}{(n+1)(p-1)-2p}, \quad \text{for } (n+1)(p-1)-2p > 0,$$

and

$$1 \leq d < \infty, \quad \text{for } (n+1)(p-1)-2p \leq 0.$$

Simplifying, we obtain

$$\frac{1}{p} + \frac{1}{d} > 1 - \frac{2}{n+1}, \quad \text{for } (n+1)(p-1)-2p > 0,$$

and with no lower bound condition on $1/p + 1/d$ for $(n+1)(p-1) - 2p \leq 0$. On the other hand,

$$(n+1)(p-1) - 2p \leq 0 \iff n \leq \frac{p+1}{p-1}.$$

Therefore, we conclude that

$$\frac{1}{p} + \frac{1}{d} > 1 - \frac{2}{n+1} = \min\left\{1 - \frac{2}{m+1}, 1 - \frac{2}{n+1}\right\}, \quad \text{for } n > \frac{p+1}{p-1},$$
$$\text{no lower bound condition imposed on } \frac{1}{p} + \frac{1}{d}, \quad \text{for } n \leq \frac{p+1}{p-1}.$$

□

We require the following proposition arising from Cowan and Moameni in [12].

Proposition 3.3. *Suppose $\Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ is an annular domain with monotonicity (see Definition 2.2), and assume that $\kappa(x)$ satisfies $(\mathcal{A})$ as in Definition 2.3. Let $0 \leq \tilde{u} \in H^1_{0,G}(\Omega) \cap L^{\mathcal{P}}(\Omega)$ with $s\tilde{u}_t - t\tilde{u}_s \leq 0$ a.e. on $\widehat{\Omega}$ where $\mathcal{P} > 2$, and*

$$H^1_{0,G}(\Omega) := \left\{u \in H^1_0(\Omega) : gu = u, \quad \forall g \in G = O(m) \times O(n)\right\}.$$

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Suppose that $\tilde{v}$ is the solution of

$$\begin{cases} -\Delta \tilde{v} = \kappa(x)\tilde{u}^{\mathcal{P}-1} & \text{in } \Omega, \\ \tilde{v} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $0 \le \tilde{v} \in H^1_{0,G}(\Omega) \cap L^{\mathcal{P}}(\Omega)$ *with* $s\tilde{v}_t - t\tilde{v}_s \le 0$ *a.e. on* $\widehat{\Omega}$.

Now, we can prove the first main result of the paper.

Proof of Theorem 1.1. First, we recall the convex cone $K := K(m,n)$ as in (26), namely,

$$K = K(m,n) := \left\{0 \le u = u(s,t) \in W^{2,p'}_G(\Omega) \cap W^{1,p'}_0(\Omega) : su_t - tu_s \le 0 \text{ a.e. in } \widehat{\Omega}\right\},$$

where $W^{2,p'}_G(\Omega) := \{u \in W^{2,p'}(\Omega) : gu = u, \quad \forall g \in G\}$ where $G := O(m) \times O(n)$, and where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ with $gu(x) := u(g^{-1}x)$. By Theorem 3.3 (ii), we have that the embedding $K \hookrightarrow L^q(\Omega)$ is compact for

$$\begin{aligned} 1 \le q < \frac{(n+1)-p'}{(n+1)-2p'} \quad & \text{if } (n+1)-2p' > 0 \\ 1 \le q < \infty \quad & \text{if } (n+1)-2p' \le 0. \end{aligned}$$

By Remark 3.4, this can be rewritten as

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n+1} = \min\left\{1 - \frac{2}{m+1}, 1 - \frac{2}{n+1}\right\}, \quad \text{for } n > \frac{p+1}{p-1}$$

with no condition on the lower bound of

$$\frac{1}{p} + \frac{1}{q}, \quad \text{for } 1 \le n \le \frac{p+1}{p-1}.$$

It follows from Proposition 3.1 that I_K has a critical point $\bar{u}$ in K with $I_K(\bar{u}) = c$ where $c > 0$ is the critical value characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0, \gamma(1) = e, I_K(\gamma(1)) \le 0.\}$ Since $I_K(\bar{u}) > 0$, it follows that $\bar{u}$ is non-zero. Now, we want to show that there exists $\tilde{u} \in K$ and $\tilde{v} \in W^{2,q'}(\Omega) \cap W^{1,q}_0(\Omega)$ satisfying

$$\begin{cases} -\Delta \tilde{u} = a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta \tilde{v} = b(x)|\bar{u}|^{q-2}\bar{u}, \end{cases}$$

so that we can conclude by Proposition 3.2 that $(\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} -\Delta u = a(x)|v|^{p-2}v \\ -\Delta v = b(x)|u|^{q-2}u. \end{cases}$$

Indeed, it follows from Proposition 3.3 that there exists $\tilde{v} \in K$ such that

$$-\Delta \tilde{v} = \tilde{b}(x)|\bar{u}|^{q-2}\bar{u}.$$

Applying Proposition 3.3 once again, there exists $\tilde{u} \in K$ satisfying

$$-\Delta \tilde{u} = \tilde{a}(x)|\tilde{v}|^{p-2}\tilde{v}.$$

Thus, $(\tilde{u}, \tilde{v})$ satisfies the equation

$$\begin{cases} -\Delta \tilde{u} = a(x)|\tilde{v}|^{p-2}\tilde{v} \\ -\Delta \tilde{v} = b(x)|\bar{u}|^{q-2}\bar{u}, \end{cases}$$

and by Proposition 3.2, we conclude that $(\tilde{u}, \tilde{v})$ is a solution of

$$\begin{cases} -\Delta u = a(x)|v|^{p-2}v \\ -\Delta v = b(x)|u|^{q-2}u. \end{cases}$$

Note that both $\tilde{u}$ and $\tilde{v}$ are nonzero and non-negative. It now follows from the strong maximum principle [Theorem 8.19, [16]] that both $\tilde{u}$ and $\tilde{v}$ are strictly positive. ☐

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4 Non-radial solutions when Ω is an annulus.

In this section we discuss the case when $a(x) = b(x) = 1$, and Ω is an annulus, that is $\Omega = \{x : R_1 < |x| < R_2\}$,

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \Omega, \\ -\Delta v = u^{q-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

We shall prove that the solution obtained in Theorem 1.1 is non-radial when radii R_1, R_2 satisfy certain conditions. We first begin with the following general result for positive solutions of (28).

Theorem 4.1. *Let $q \geq p \geq 2$. Assume that (u, v) is a positive solution of (28). The following assertion hold*

$$\inf_{0 \neq \eta \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla \eta|^2 \, dx}{\int_\Omega \eta^2 v(x)^{\frac{p-2}{2}} u(x)^{\frac{q-2}{2}} \, dx} \leq \sqrt{\frac{q}{p}}. \tag{29}$$

Proof. We first prove that

$$qv(x)^p \geq pu(x)^q, \qquad \forall x \in \Omega. \tag{30}$$

Let $\sigma = p/q \in (0, 1]$ and $e = \sigma^{-\frac{1}{q}}$. Define $z(x) = u(x) - ev(x)^\sigma$. It follows that

$$\begin{aligned} \Delta z &= \Delta u - e\sigma v^{\sigma-1}\Delta v - e\sigma(\sigma - 1)v^{\sigma-2}|\nabla v|^2 \\ &\geq \Delta u - e\sigma v^{\sigma-1}\Delta v \\ &= v^{\sigma-1}\left(\frac{u^{q-1}}{e^{q-1}} - v^{\sigma(q-1)}\right), \end{aligned}$$

from which we obtain that $\Delta z \geq 0$ on the set

$$\{x \in \Omega : z(x) \geq 0\}.$$

Take $\epsilon > 0$. It follows that

$$(z - \epsilon)^+ \Delta z \geq 0,$$

and therefore

$$\int_\Omega |\nabla (z - \epsilon)^+|^2 \, dx \leq 0.$$

This implies that $z \leq \epsilon$, and since ϵ is arbitrary the inequality (30) follows.
We shall now prove inequality (29). It follows from inequality (30) that

$$v \geq \left(\frac{p}{q}\right)^{\frac{1}{p}} u^{\frac{q}{p}}.$$

Therefore,

$$v^{\frac{p-2}{2}} u^{\frac{q-2}{2}} v^2 = u^{\frac{q-2}{2}} v^{\frac{p}{2}} v \geq \sqrt{\frac{p}{q}} u^{\frac{q-2}{2}} u^{\frac{q}{2}} v = \sqrt{\frac{p}{q}} u^{q-1} v.$$

It then follows that

$$\inf_{0 \neq \eta \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla \eta|^2 \, dx}{\int_\Omega \eta^2 v(x)^{\frac{p-2}{2}} u(x)^{\frac{q-2}{2}} \, dx} \leq \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 v(x)^{\frac{p-2}{2}} u(x)^{\frac{q-2}{2}} \, dx} \leq \frac{\int_\Omega u^{q-1} v \, dx}{\int_\Omega \sqrt{\frac{p}{q}} u^{q-1} v \, dx} = \sqrt{\frac{q}{p}}.$$

$\square$

Remark 4.2. *We would like to remark that inequalities of the type (30) were first developed to study Liouville type theorems for stable Lane-Emden systems and Hardy-Hénon elliptic systems on $\mathbb{R}^N$. We refer the interested reader to [11, 25, 29].*

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Let $w(x) = w(s,t)$ be a function of (s,t). If we write w in terms of polar coordinates (recall we have $s = r\cos(\theta), t = r\sin(\theta)$), we obtain that $w(x) = w(r,\theta)$. Writing the Laplace operator in polar coordinates gives

$$-\Delta w(x) = -w_{rr} - \frac{(N-1)w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \frac{w_\theta}{r^2}h(\theta) \tag{31}$$

where

$$h(\theta) = (m-1)\tan(\theta) - \frac{(n-1)}{\tan(\theta)}. \tag{32}$$

Let (μ_1, ψ_1) be the second eigenpair of the following eigenvalue problem

$$\begin{cases} -\psi_1''(\theta) + \psi_1'(\theta)h(\theta) = \mu_1\psi_1(\theta) & \text{in } (0, \frac{\pi}{2}), \\ \psi'(\theta) > 0 & \text{in } (0, \frac{\pi}{2}), \\ \psi_1'(0) = \psi_1'(\frac{\pi}{2}) = 0, \end{cases} \tag{33}$$

and note that the first eigenpair is given by $(\mu_0, \psi_0) = (0,1)$. Note in this problem one can find an explicit solution given by

$$\mu_1 = 2N, \quad \psi_1(\theta) = \frac{m-n}{N} - \cos(2\theta),$$

and note we can apply Sturm–Liouville theory and count the number of zeros of ψ_1 to see it is in fact the second pair.

Proof of Theorem 1.2. Let us assume the solution (u,v) of (28) obtained in Theorem 1.1 is radial. Let (λ_1, φ) be the first eigenpair of the following eigenvalue problem

$$\begin{cases} -\varphi''(r) - \frac{(N-1)\varphi'(r)}{r} + \frac{2N\varphi(r)}{r^2} = \lambda_1 v(r)^{\frac{p-2}{2}}u(r)^{\frac{q-2}{2}}\varphi(r) & r \in (R_1, R_2), \\ \varphi(r) = 0 & r \in \{R_1, R_2\}. \end{cases}$$

Set $w(x) = \varphi(r)\psi_1(\theta)$ and note that

$$\begin{aligned} -\Delta w(x) &= -w_{rr} - \frac{(N-1)w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \frac{w_\theta}{r^2}h(\theta) \\ &= -\varphi_{rr}(r)\psi_1(\theta) - \frac{(N-1)\varphi_r(r)\psi_1(\theta)}{r} - \frac{\varphi(r)\psi_1''(\theta)}{r^2} + \frac{\varphi(r)\psi_1'(\theta)}{r^2}h(\theta) \\ &= -\varphi_{rr}(r)\psi_1(\theta) - \frac{(N-1)\varphi_r(r)\psi_1(\theta)}{r} + \frac{2N\varphi(r)\psi_1(\theta)}{r^2} \\ &= \lambda_1 v(|x|)^{\frac{p-2}{2}}u(|x|)^{\frac{q-2}{2}}w(x) \end{aligned}$$

Recall that $I_K(u) = c > 0$ where the critical value c is characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{\tau\in[0,1]} I_K[\gamma(\tau)],$$

where

$$\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0 \neq \gamma(1), I_K(\gamma(1)) \leq 0\}.$$

For the sake of simplifying the notations, we use I instead of I_K in the rest of the proof.

Set $\gamma_\sigma(\tau) = \tau(u+\sigma w)l$, where $l > 0$ is chosen in such a way that $I\big((u+\sigma w)l\big) \leq 0$ for all $|\sigma| \leq 1$. Note that $\gamma_\sigma \in \Gamma$. We shall show that there exists $\sigma > 0$ such that for every $\tau \in [0,1]$ one has $I(\gamma_\sigma(\tau)) < I(u)$, and therefore,

$$c \leq \max_{\tau\in[0,1]} I(\gamma_\sigma(\tau)) < I(u),$$

which leads to a contradiction since $I(u) = c$. Note first that there exists a unique twice differentiable real function g on a small neighbourhood of zero with $g'(0) = 0$ and $g(0) = 1/l$ such that $\max_{\tau\in[0,1]} I(\gamma_\sigma(\tau)) = I\big(g(\sigma)(u+\sigma w)l\big)$. We now define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(\sigma) = I\big(g(\sigma)(u+\sigma w)l\big) - I(u).$$

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Clearly we have $h(0) = 0$. Note also that $h'(0) = 0$ due to the fact that $I'(u) = 0$. We now show that $h''(0) < 0$. Indeed,

$$\begin{aligned}
h''(0) &= (p'-1)\int_\Omega |\Delta u|^{p'-2}(-\Delta w)^2\,dx - (q-1)\int_\Omega |u|^{q-2}(w)^2\,dx \\
&= (p'-1)\lambda_1^2\int_\Omega |\Delta u|^{p'-2}v(|x|)^{p-2}u(|x|)^{q-2}w^2(x)\,dx - (q-1)\int_\Omega |u|^{q-2}(w)^2\,dx \\
&= (p'-1)\lambda_1^2\int_\Omega \big(v(|x|)^{p-1}\big)^{p'-2}v(|x|)^{p-2}u(|x|)^{q-2}w^2(x)\,dx - (q-1)\int_\Omega |u|^{q-2}(w)^2\,dx \\
&= (p'-1)\lambda_1^2\int_\Omega u(|x|)^{q-2}w^2(x)\,dx - (q-1)\int_\Omega |u|^{q-2}(w)^2\,dx \\
&= \Big((p'-1)\lambda_1^2 - (q-1)\Big)\int_\Omega u(|x|)^{q-2}w^2(x)\,dx.
\end{aligned}$$

Note that

$$(p'-1)\lambda_1^2 - (q-1) < 0 \qquad \text{if and only if} \qquad \lambda_1^2 < (p-1)(q-1).$$

Let λ_H denote the best constant in the Hardy inequality

$$\lambda_H = \inf_{0\neq\eta\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\eta|^2\,dx}{\int_\Omega \frac{|\eta|^2}{|x|^2}\,dx}.$$

It follows that

$$\lambda_1 = \inf_{0\neq\eta\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla\eta|^2\,dx + 2N\int_\Omega \frac{|\eta|^2}{|x|^2}\,dx}{\int_\Omega \eta^2 v(x)^{\frac{p-2}{2}}u(x)^{\frac{q-2}{2}}\,dx} \le \inf_{0\neq\eta\in H_0^1(\Omega)} \frac{\left(1+\frac{2N}{\lambda_H}\right)\int_\Omega |\nabla\eta|^2\,dx}{\int_\Omega \eta^2 v(x)^{\frac{p-2}{2}}u(x)^{\frac{q-2}{2}}\,dx} \le \sqrt{\frac{q}{p}}\left(1+\frac{2N}{\lambda_H}\right),$$

where the last inequality follows from Theorem 4.1. In particular, if

$$\frac{q}{p}\left(1+\frac{2N}{\lambda_H}\right)^2 < (p-1)(q-1)$$

then $(p'-1)\lambda_1^2 - (q-1) < 0$. This implies that $h''(0) < 0$. This in fact shows that

$$\max_{\tau\in[0,1]} I(\gamma_\sigma(\tau)) = I\big(g(\sigma)(u+\sigma v)l\big) < I(u),$$

for small $\sigma > 0$ as desired. □

Recall from (26) that

$$K = K(m,n) := \left\{0 \le u = u(s,t) \in W_G^{2,p'}(\Omega)\cap W_0^{1,p'}(\Omega) : su_t - tu_s \le 0 \text{ a.e. in } \widehat{\Omega}\right\},$$

which corresponds to the decomposition $\mathbb{R}^m\times\mathbb{R}^n$ of the annulus $\Omega = \{x : R_1 < |x| < R_2\}$ in $\mathbb{R}^N$ with $N = m+n$. We have the following result regarding the distinction of solutions for different decompositions of $\mathbb{R}^N$.

Lemma 4.3. *Let $1 < n < n' \le \lfloor\frac{N}{2}\rfloor$ and set $m = N-n$, $m' = N-n'$. Let $u_{m,n}\in K(m,n)$ and $u_{m',n'}\in K(m',n')$ be the solutions obtained in Theorem 1.1 corresponding to the decomposition $\mathbb{R}^m\times\mathbb{R}^n$ and $\mathbb{R}^{m'}\times\mathbb{R}^{n'}$ of $\mathbb{R}^N$ respectively. Then $u_{m,n}\neq u_{m',n'}$ unless they are both radial functions.*

Proof. Let $u_{m,n} = u_{m',n'} = u$. We shall show that u must be radial. It follows that $u(x) = f(s,t) = g(s',t')$ for two functions f and g where

$$s^2 := x_1^2+\cdots+x_m^2, \qquad t^2 := x_{m+1}^2+\cdots+x_N^2,$$

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and

$$s'^2 := x_1^2 + \cdots + x_{m'}^2, \qquad t'^2 := x_{m'+1}^2 + \cdots + x_N^2.$$

By assuming $x_i = 0$ for $i \neq x_1, x_m$ we obtain that

$$g\left(|x_1|, |x_m|\right) = g\left(\sqrt{x_1^2 + x_m^2}, 0\right),$$

from which we obtain that g must be a radial function. To show that f is a radial function, we assume that $x_i = 0$ for $i \neq x_{m'+1}, x_N$. Then

$$f\left(|x_{m'+1}|, |x_N|\right) = g\left(0, \sqrt{x_{m'+1}^2 + x_N^2}\right)$$

from which we obtain that f is a radial function. □

Proof of Theorem 1.3. We begin by proving the existence of a positive solution. Afterwards, we show that the positive solution is in fact, non-radial.

Part 1. It follows from Theorem 1.1 that for each $n \leq k$ and $q \geq p > 2$, equation (28) has a solution of the form $(u_{m,n}, v_{m,n}) = (u_{m,n}(s,t), v_{m,n}(s,t))$ where

$$s^2 := x_1^2 + \cdots + x_m^2, \qquad t^2 := x_{m+1}^2 + \cdots + x_N^2,$$

provided

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{n+1}, \qquad \text{for } n > \frac{p+1}{p-1}.$$

Since $n \leq k$, it follows that

$$1 - \frac{2}{n+1} \leq 1 - \frac{2}{k+1}, \qquad \text{for } k > \frac{p+1}{p-1}.$$

Thus, for each $n \leq k$, we have a positive solution provided

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{k+1}.$$

Part 2. If $k \leq (p+1)/(p-1)$, then $n \leq (p+1)/(p-1)$. So, by Theorem 1.1, there exists a positive solution of (28).

Now we proceed to prove that the solution in parts 1 and 2 are non-radial. Indeed, by Theorem 1.2, the solution $(u_{m,n}, v_{m,n})$ is non-radial provided

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right).$$

Thus, for each $n \in \{1, ..., k\}$ we have a non-radial solution $(u_{m,n}, v_{m,n})$. On the other hand, by Lemma 4.3 we have that $u_{m,n} \neq u_{m',n'}$ for all $n \neq n'$. Similarly, by Lemma 4.3, we obtain $v_{m,n} \neq v_{m',n'}$ for all $n \neq n'$. This indeed implies that we have k distinct positive non-radial solutions. □

Proof of Corollary 1.4.

1. For each $k \in \mathbb{N}$ with $1 \leq k \leq \lfloor \frac{p+1}{p-1} \rfloor$, by part 2 of Theorem 1.3, there exists a solution provided

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right).$$

Thus, if

$$(p-1)(q-1)\left(\frac{p}{q}\right) > \left(1 + \frac{2N}{\lambda_H}\right)^2,$$

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then we must have $\lfloor \frac{p+1}{p-1} \rfloor$ positive non-radial solutions.

2. Assuming $k = \left\lfloor \frac{N}{2} \right\rfloor$ in Theorem 1.3 we obtain that there are $\left\lfloor \frac{N}{2} \right\rfloor$ positive non-radial solutions provided that

$$(p-1)(q-1) > \left(1 + \frac{2N}{\lambda_H}\right)^2 \left(\frac{q}{p}\right)$$

and

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{\left\lfloor \frac{N}{2} \right\rfloor + 1}.$$

Now, to obtain

$$(p-1)(q-1) > \frac{q}{p},$$

we want to show that λ_H can be sufficiently large under conditions 2.(a) and 2.(b) and hence, we conclude that there are $\lfloor \frac{N}{2} \rfloor$ positive non-radial solutions. As for the proof of 2.(a) and 2.(b), we refer the interested reader to [12]. □

Data availability statement

Data sharing not applicable to this article as no data-sets were generated or analysed during the current study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment:

We would like to thank the referees for carefully reading our manuscript and for giving us valuable comments that substantially improved the clarity of the original manuscript.

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