where the product runs over all prime divisors of d and an empty product is taken to be 1.

L. Moser

P 15. (Conjecture) Every point on the perimeter of an ellipse is a vertex of an inscribed triangle of maximum area. There are other closed convex curves with this property, e.g. parallelograms. Is it true that the ellipse is the only closed strictly convex (no proper segments) plane curve with this property [cf. R.P. Bambah, Proc. Nat. Inst. Sci. India, Part A 23 (1957), 540-543]?

H. Helfenstein

SOLUTIONS

P 2. Put $S_k(n) = \Sigma_{a \leqslant n}\, a^k$, $T_k(n) = \Sigma a^k$, the second sum extending over all $a \leqslant n$ such that $(a,n) \neq 1$ and a does not divide n. Let $\mu(n)$ denote Moebius' function. Prove that

(i) $T_k(n) = -\Sigma_{d|n,\, d>1}\,(1 + \mu(d)S_k(n/d))$,

(ii) n divides $S_1(n)$ (that is, n is multiply perfect) if and only if

$$T_1(n) \equiv 1 (\mathrm{mod}\ n) \text{ if n is odd}, (1+n/2)(\mathrm{mod}\ n) \text{ if n is even.}$$

J.C. Hayes and P. Scherk

Solution by the proposers. Let $\varphi_k(n) = \Sigma_{a \leqslant n,\,(a,n)=1}\, a^k$. Then $S_k(n) = \Sigma_{d|n}\, \Sigma_{a \leqslant n,\, (a,n)=d}\ a^k = \Sigma_{d|n}\ d^k \varphi_k(n/d)$ and by Moebius inversion $\varphi_k(n) = \Sigma_{d|n}\, \mu(d) d^k S_k(n/d)$. Now put $\sigma_k(n) = \Sigma_{d|n}\, d^k$ Then

$$(1) \quad T_k(n) = S_k(n) - \varphi_k(n) - \sigma_k(n) + 1$$

$$= S_k(n) - \Sigma_{d|n}\, \mu(d) d^k S_k(n/d) - \Sigma_{d|n}\, d^k + 1.$$

This yields the first assertion $T_k(n) = -\Sigma_{d|n,\, d>1}\, d^k (1 + \mu(d)\, S_k(n/d))$.

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The number n is multiply perfect if and only if $\sigma_1(n) \equiv 0 \pmod{n}$. By (1) this is equivalent to

$$(2) \qquad T_1(n) \equiv S_1(n) - \varphi_1(n) + 1 \pmod{n}.$$

The right hand side of (2) is congruent to

$$-\sum_{d|n, d>1} \mu(d) d S_1(n/d) + 1 \equiv -\sum_{d|n, d>1} \mu(d) n \tfrac{1}{2}(1+n/d) + 1 \pmod{n}.$$

If n is odd, each $1 + n/d$ is even and $n | n\frac{1}{2}(1+n/d)$. Thus an odd n is multiply perfect if and only if $T_1(n) \equiv 1 \pmod{n}$.

Now let $n = \prod_{p|n} p^{\alpha}$ be even. Correcting the statement of our problem we have to assume $n \neq 2$. We wish to show that n is multiply perfect if and only if $T_1(n) \equiv 1 + n/2 \pmod{n}$. Thus we have to show $\sum_{d|n, d>1} \mu(d) n \frac{1}{2}(1+n/d) \equiv n/2 \pmod{n}$ or $\sum_{d|n, d>1} \mu(d)(1+n/d) + 1 \equiv 0 \pmod{2}$. This is equivalent to

$$(4) \qquad 2 \mid \Sigma = \sum_{d|n} \mu(d)(1+n/d).$$

But $\Sigma = \sum_{d|n} \mu(d)(n/d) + \sum_{d|n} \mu(d) = \sum_{d|n} \mu(d)(n/d)$

$$= \varphi(n) = \prod_{p|n} (p^{\alpha} - p^{\alpha-1}).$$

Thus Σ is even unless n = 2. This proves (4).

P 3. Let F be a finite field of characteristic p. Let V_n be an n-dimensional vector space over F. In V_n a symmetric bilinear form (a, b) is given. Let $n \geq 2$ if p = 2 and $n \geq 3$ if p is odd. Show that there is a vector $a \neq 0$ in V_n such that (a, a) = 0.

P. Scherk

Solution by the proposer. Let $F = \{\xi, \eta, \ldots\}$ be a finite field of characteristic p. Let G denote the multiplicative group of all the squares $\neq 0$. If p = 2, $\xi^2 = \eta^2$ if and only if $\xi = \eta$. Thus the mapping of the elements $\neq 0$ of F onto G is one-one and G is the multiplicative group of F. If p > 2, this mapping is two-one and G is a subgroup of index two in the multiplicative group of F. Let $\bar{G}$ denote the complement of G in this group.

If 1 + G = G, $1 \in G$ would successively imply $2, 3, \ldots, p-1 \in G$ and finally $p = 0 \in G$. Thus

$$(1) \qquad 1 + G \neq G.$$

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