Canad. Math. Bull. Vol. **41** (2), 1998 pp. 166–177

PERCOLATION ON PENROSE TILINGS

A. HOF

ABSTRACT. In Bernoulli site percolation on Penrose tilings there are two natural definitions of the critical probability. This paper shows that they are equal on almost all Penrose tilings. It also shows that for almost all Penrose tilings the number of infinite clusters is almost surely 0 or 1. The results generalize to percolation on a large class of aperiodic tilings in arbitrary dimension, to percolation on ergodic subgraphs of $\mathbb{Z}^d$, and to other percolation processes, including Bernoulli bond percolation.

1. **Introduction.** Penrose tilings [2, 3, 10, 4] are tilings of the plane that are aperiodic in the sense that no Penrose tiling coincides with itself after any translation. They are built out of two kinds of tiles, thick rhombs and thin rhombs. Figure 1 shows part of a Penrose tiling. There are uncountably many different Penrose tilings (where different means that they cannot be made to coincide by a translation and/or a rotation). Yet they all look alike in the sense that one cannot decide from any bounded part whether two Penrose tilings are different. Penrose tilings have become a standard two-dimensional model of the kind of aperiodic long-range order found in quasicrystals [24]. This is a reason for recent interest in models on Penrose tilings (see [8, 11] and references contained therein).

Percolation on Penrose tilings (*i.e.*, on the graph formed by the vertices and edges of the rhombs) has been studied numerically in [13, 27, 23]. The results suggest that percolation on Penrose tilings is in the same universality class as percolation on the square lattice. The critical probability for Bernoulli bond percolation on a Penrose tiling is 0.483 ± 0.005 [13], compared to $\frac{1}{2}$ on $\mathbb{Z}^2$. Lu and Birman [13] point out that the average coordination number of the vertices in a Penrose tiling is 4, the same as for $\mathbb{Z}^2$.

In the theory of percolation on $\mathbb{Z}^2$ (see [9]) translation invariance of events and the periodicity of $\mathbb{Z}^2$ play an important role. Since Penrose tilings are aperiodic, the obvious question arises how results can be generalized, how techniques can be extended, and whether any new phenomema occur. Since there are many different Penrose tilings, one should also ask whether results depend on the Penrose tiling under consideration. A technical difficulty here is that there is no natural identification between the sets of configurations on Penrose tilings that are not translates of each other. We are not aware of any rigorous results about percolation on Penrose tilings.

In this paper we set up a formalism for treating percolation on Penrose tilings rigorously. The main result is the construction of an ergodic measure that describes percolation on all Penrose tilings simultaneously. The construction of this measure allows us to discuss and settle two problems. As in percolation on $\mathbb{Z}^d$, a critical probability can be defined

Received by the editors November 13, 1996.
Work partially supported by NSERC.
AMS subject classification: 60K35, 82B43.

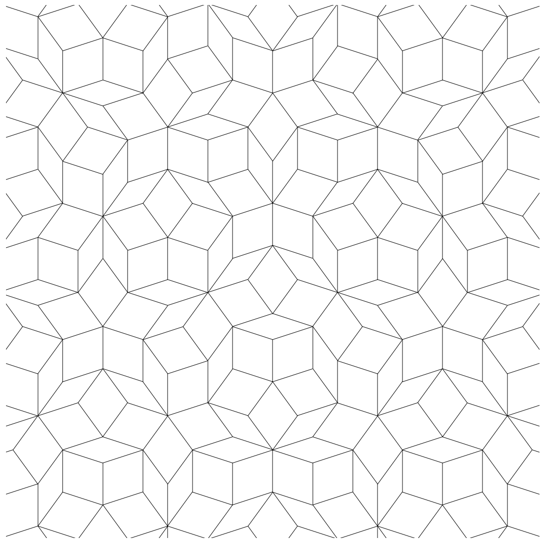

FIGURE 1: PART OF A PENROSE TILING.

in terms of the probability that there is an infinite cluster and in terms of the density of the infinite cluster. We show that these define the same number on almost all Penrose tilings. (The only way they can fail to be the same on *all* Penrose tilings is if there is a Penrose tiling on which (the union of) the infinite cluster(s) has zero density for p in an interval of strictly positive length.) The second problem is to show that the number of infinite clusters is almost surely 0 or 1. We generalize the argument of Burton and Keane [5] to prove that the number of infinite clusters is almost surely 0 or 1 on almost every Penrose tiling. Here the 'almost all' is with respect to a probability measure on the set of all Penrose tilings and the 'almost surely' with respect to the percolation measure on each tiling.

It should be noted that the generalization of the Burton-Keane argument in [7] does not cover percolation on aperiodic tilings.

The results extend to percolation on large classes of aperiodic tilings in arbitrary dimension and to percolation on a class of ergodic subgraphs of $\mathbb{Z}^d$ (described in Section 6). The results also generalize to more percolation processes more general than Bernoulli percolation.

The paper is organized as follows. In Section 2 we recall that Penrose tilings give rise to a uniquely ergodic dynamical system with respect to translation. We actually prove the unique ergodicity because this will indicate how to prove, in Secion 3, ergodicity of measures describing percolation on all Penrose tilings simultaneously. Section 4 shows that the critical probabilities are the same on almost all Penrose tilings. Section 5 proves that the number of infinite clusters in almost surely 0 or 1, for almost all Penrose tilings. Section 6 explains how the results generalize to percolation on other aperiodic tilings and to percolation on ergodic subgraphs of $\mathbb{Z}^d$.

2**. The Penrose dynamical system.** There are several ways to define Penrose tilings: matching rules (see, *e.g.*, [10]), a substitution rule (or 'inflation') on the tiles (see, *e.g.*, [10, 16, 4]) and De Bruijn's pentagrid construction [2, 3]. We will not be concerned with these descriptions and simply state the properties we need. This section contains no new results. We assume that Penrose tilings have edges parallel to the x-axis; this amounts to identifying Penrose tilings that differ only by a rotation.

A finite set of tiles is called a patch. Patches are called equivalent if they are translates of each other. The equivalence class of a patch modulo translation is called a pattern. For instance, every thick rhomb is a patch and every thick rhomb is a copy of one of ten patterns (corresponding to the ten possible orientations of the tile). We only consider patches and patterns that actually occur in Penrose tilings. For every $R > 0$ the number of different patterns P with $\operatorname{diam}(P) < R$ is finite, since the tiles in a Penrose tilings match 'edge-to-edge'. For any Penrose tiling t and any bounded $\Lambda \subset \mathbb{R}^2$, the (Λ, t)-patch, or the Λ-patch of t, is the set of tiles of t that have non-empty intersection with Λ (recall that tiles are closed sets).

Let $C_L := \{x \in \mathbb{R}^2 \mid |x_i| \leq L/2\}$ be the square of side L centered around 0. Let t be a Penrose tiling. For any pattern P and $\Lambda \subset \mathbb{R}^2$ let $N_P(\Lambda)$ denote the number of copies of P that occurs in C_L in t. Penrose tilings have the property [16] (see also [8, 25]) that for every pattern P there is a $n_P > 0$ such that

$$n_P = \lim_{L \to \infty} L^{-2} N_P(C_L + a) \quad \text{uniformly in } a \in \mathbb{R}^2 \tag{1}$$

for all Penrose tilings t.

Let $\mathcal{T}$ denote the set of all Penrose tilings and B_r the open disk $\{x \in \mathbb{R}^2 \mid \|x\| < r\}$. Let $\mathbb{R}^2$ act on $\mathcal{T}$ by translation: $T_x t := t + x$. Define a metric [22, 20] on $\mathcal{T}$ by

$$d(t, t') := \min(1, \epsilon),$$

where ϵ is the smallest number such that $T_x t = t'$ on $B_{1/\epsilon}$ for some $x \in \mathbb{R}^2$ with $\|x\| \leq \epsilon$. The space $\mathcal{T}$ is compact in this metric and $\mathbb{R}^2$ acts continuously on it. This is the Penrose dynamical system. It is minimal with respect to translations: the orbit $\{T_x t\}_{x \in \mathbb{R}^2}$ is dense in $\mathcal{T}$ for every $t \in \mathcal{T}$. (This is a consequence of the fact that the frequencies n_P exist uniformly in a, are independent of t and $n_P > 0$.) The topological dynamics of this system has been analyzed in [21].

We can now construct the uniquely ergodic probability measure on $\mathcal{T}$. We will use the construction in the next section. (Recall that a dynamical system is called unique ergodic if it admits only one invariant probability measure—this measure is ergodic.)

For any bounded Lebesgue-measurable set $U \subset \mathbb{R}^2$ and any patch P define the cylinder set

$$X_{P,U} := \{t \in \mathcal{T} \mid P \text{ is a patch in } T_x t \text{ for some } x \in U\};$$

(*cf.* [25]). By cutting U into pieces we can assume that $\operatorname{diam}(U) < \kappa$, where κ is less than the smallest distance between vertices. If $\operatorname{diam}(V) \leq \kappa$ then every $t \in \mathcal{T}$ has at most one vertex in V. It follows that for all $t \in \mathcal{T}$

$$\Big|L^{-2}\int_{C_L+a} 1_{X_{P,U}}(T_x t)\,dx - L^{-2}N_P(C_L+a)|U|\Big| \le AL^{-1}$$

uniformly in $a \in \mathbb{R}^2$, where $|U|$ denotes the Lebesgue measure of U. The constant A depends on P and U but not on t. Hence

$$\lim_{L\to\infty} L^{-2}\int_{C_L+a} 1_{X_{P,U}}(T_x t)\,dx = n_P|U|, \tag{2}$$

uniformly in $t \in \mathcal{T}$.

With respect to the supremum norm, linear combinations of indicator functions of cylinder sets are dense in the continuous functions on $\mathcal{T}$. (Because there are finitely many patterns of diameter less than R for all R, one can partition $\mathcal{T}$ into cylinder sets of diameter less than or equal to $1/R$ [25, Lemma 1.5].) Hence

$$\lim_{L\to\infty} L^{-2}\int_{C_L} \phi(T_x t)\,dx$$

exists uniformly in $t \in \mathcal{T}$ for every continuous function ϕ on $\mathcal{T}$. This proves that $\mathcal{T}$ is uniquely ergodic (see, *e.g.*, Section 6.5 in [26] for the analogous statement for the action of $\mathbb{Z}$ by action a continuous map on a compact metric space; the proof easily generalizes to a continuous action of $\mathbb{R}^d$). Denote the uniquely ergodic measure by μ. Clearly, $\mu(X_{P,U}) = n_P|U|$.

3. **Construction of ergodic measures.** Let $t \in \mathcal{T}$ be a Penrose tiling. The set of vertices of t is denoted by V_t. The configuration space for site percolation on t is $\Omega_t := \prod_{v\in V_t}\{0,1\}$. We mainly consider Bernoulli percolation, *i.e.*, vertices are occupied independently with probability p. Then Ω_t carries the probability measure $\lambda^t := \prod_{v\in V_t}\lambda_p$, where $\lambda_p\big(\omega(v)=1\big) = p$ and $\lambda_p\big(\omega(v)=0\big) = 1-p$. Note that Ω_t and $\Omega_{t'}$ can be identified if and only if t and t' are translates of each other.

In order to be able to use ergodic theory, we have to consider percolation on all Penrose tilings simultaneously. Define

$$\Sigma := \big\{(t,\omega) \;\big|\; t \in \mathcal{T}, \quad \omega \in \Omega_t\big\};$$

this is the set of all Penrose tilings with all possible configurations. We will think of Σ as a set of tilings in which all vertices have been 'coloured' 0 or 1. We will sometimes write t^ω instead of (t,ω). Translations act on Σ by $T_x(t,\omega) = (T_x t,\omega)$. So if a Penrose tiling is shifted, the configuration it carries is shifted along. The space Σ becomes a compact metric space when we give it the metric d' that is defined analogously to d (with t replaced by t^ω). Let $\mathcal{C}(\Sigma)$ denote the Banach space of continuous functions on Σ with supremum norm $\|\cdot\|_\infty$.

For $(t,\omega) \in \Sigma$ and $\Lambda \subset \mathbb{R}^2$ let $\omega_\Lambda := \{\omega_i\}_{i\in V_t\cap\Lambda}$ and $\lambda^\Lambda := \prod_{v\in V_t\cap\Lambda}\lambda_p$. For any patch P let V_P be the set of vertices in P and let $\lambda^P := \prod_{v\in V_P}\lambda_p$, which we consider as a

https://doi.org/10.4153/CMB-1998-026-0 Published online by Cambridge University Press

probability measure on $\Omega_P := \prod_{v\in V_P}\{0,1\}$. Note that for every copy of P in $t \in \mathcal{T}$ there is a copy of Ω_P embedded in Ω_t.

For any patch P, any Lebesgue-measurable $U \subset \mathbb{R}^2$ and any $\eta \in \Omega_P$ define the cylinder set

$$X^\eta_{P,U} := \left\{(t,\omega) \in \Sigma \;\middle|\; P \text{ is a patch in } T_x t \text{ for some } x \in U \text{ and } \omega_{P-x} = \eta\right\};$$

Below we will always assume that $\operatorname{diam}(U) < \kappa$.

The following lemma is the essential step in the construction of an ergodic measure on Σ.

LEMMA 3.1. *For every $t \in \mathcal{T}$*

$$\lim_{L\to\infty} L^{-2}\int_{C_L} 1_{X^\eta_{P,U}}(T_x t^\omega)\,dx = \mu(X_{P,U})\lambda^P(\eta) \quad \text{for } \lambda^t\text{-a.e } \omega \in \Omega_t.$$

PROOF. If all copies of P in t are disjunct then the realizations of ω on the copies of P are independent and the Lemma follows from the Strong Law of Large Numbers and the definition of μ.

Now suppose that there are copies of P that overlap, *i.e.*, share at least one vertex. Let $R > 2\operatorname{diam}(P)$. Then $x, y \in V_t$ do not belong to overlapping copies of P if $\|x-y\| > R$. For $x \in V_t$, let the L-environment $E_L(x)$ of x be the $(B_L + x)$-patch of t. We claim that there is an $L > 0$ such that for all $x, y \in V_t$ with $\|x-y\| < R$ one has that $E_L(x) - x \neq E_L(y) - y$. For otherwise there would be a sequence $L_j \longrightarrow \infty$ and points $x_j, y_j \in V_t$ with $\|x_j - y_j\| < R$ such that $E_L(x_j) - x_j = E_L(y_j) - y_j$ for all j. Since $x_j - y_j$ can take only finitely many values, we can take a subsequence along which $x_j - y_j$ is constant, say a. But then it follows that there is a $t' \in \mathcal{T}$ with $t' = T_a t'$ contradicting the aperiodicity of the Penrose tilings.

Choose a vertex w in P. There are finitely many possibilities, say $E_1, \ldots, E_q$, for the L-environment of w in the copies of P. To every copy of E_i there corresponds a copy of P. The realizations of ω on copies of P are independent if those copies have different kinds of environments E_i. Each of the E_i occurs with a well-defined frequency n_{E_i} and $n_P = \sum_{i=1}^q n_{E_i}$. Applying the Strong Law of Large Numbers to the copies of P corresponding to each of the E_i proves the lemma. ∎

THEOREM 3.1. *There exists an ergodic Borel probability measure ν on Σ satisfying:*

(i) $\nu(X^\eta_{P,U}) = \mu(X_{P,U})\lambda^P(\eta)$.

(ii) *For every $t \in \mathcal{T}$ and every $\phi \in C(\Sigma)$, and every ϕ that is a linear combination of cylinder functions,*

$$\lim_{L\to\infty} L^{-2}\int_{C_L} \phi(T_x t^\omega)\,dx = \int \phi\,d\nu \quad \text{for } \lambda^t\text{-a.e. } \omega \in \Omega_t. \tag{3}$$

(iii) *For every $\phi \in \mathrm{L}^1(\Sigma,\nu)$*

$$\lim_{L\to\infty} L^{-2}\int_{C_L} \phi(T_x t^\omega)\,dx = \int \phi\,d\nu \quad \text{for } \nu\text{-a.e. } t^\omega \in \Sigma. \tag{4}$$

(iv) *For every $\phi \in \mathrm{L}^1(\Sigma,\nu)$*

$$\int \phi\,d\nu = \int\left[\int \phi(t,\omega)\,d\lambda^t(\omega)\right]d\mu(t). \tag{5}$$

https://doi.org/10.4153/CMB-1998-026-0 Published online by Cambridge University Press

PROOF. We first show that there is an invariant Borel probability measure ν on Σ satisfying (i) and (ii). Then we prove (iii), which is equivalent to the statement that ν is ergodic. Statement (iv) will follow from the construction of ν.

Let C be the set of linear combinations of characteristic functions of the sets $X^{\eta}_{P,U}$. For measurable functions ψ on Σ and $L > 0$ define the averaged function $M_L\psi$ by $M_L\psi(t^{\omega}) := L^{-2}\int_{C_L}\psi(T_x t^{\omega})\,dx$. By Lemma 3.1 we have that for every $\psi \in C$ there is a constant m_ψ such that for all $t \in \mathcal{T}$

$$\lim_{L\to\infty} M_L\psi = m_\psi \quad \text{for } \lambda^t\text{-a.e. } \omega \in \Omega_t.$$

Every $\phi \in \mathcal{C}(\Sigma)$ can be approximated in the supremum norm by a sequence $\psi_n \in C$. If $\|\phi - \psi_n\|_\infty \le \epsilon$ then $\|M_L\phi - M_L\psi_n\|_\infty \le \epsilon$ for all L. Hence there is a linear functional M on $\mathcal{C}(\Sigma)$ satisfying

$$M\phi = \lim_{L\to\infty} M_L\phi \qquad \text{for all } t \in \mathcal{T} \text{ and } \lambda^t\text{-a.e. } \omega \in \Omega_t.$$

Also, $M\phi \ge 0$ if $\phi \ge 0$ and $M1 = 1$. By the Riesz representation theorem there exists a Borel probability measure ν on Σ such that

$$M\phi = \int \phi\,d\nu \quad \text{for all } \phi \in \mathcal{C}(\Sigma).$$

This proves (3) for $\phi \in \mathcal{C}(\Sigma)$. It is clear ν is invariant and satisfies (i).

It suffices to prove (iii) for $\phi \ge 0$. Since ν is invariant, the (pointwise) ergodic theorem gives that for every $\phi \in \mathrm{L}^1(\Sigma,\nu)$ there is an invariant $\phi^* \in \mathrm{L}^1(\Sigma,\nu)$ such that

$$\lim_{L\to\infty} M_L\phi = \phi^* \quad \text{for } \nu\text{-a.e. } t^{\omega} \in \Sigma, \tag{6}$$

with $\int\phi\,d\nu = \int\phi^*\,d\nu$. To prove (iii) we have to show that, ν-almost surely, $\phi^* = \int\phi\,d\nu$. The following argument allows to prove (4) from (ii), first for indicator functions of open sets, then for indicator functions of Borel measurable sets and finally, via simple functions, for positive $\phi \in \mathrm{L}^1(\Sigma,\nu)$.

Suppose $\phi_n, \phi \in \mathrm{L}^1(\Sigma,\nu)$ such that $0 \le \phi_1 \le \cdots \le \phi_n \le \phi_{n+1}\cdots \le \phi$ and $\phi_n \longrightarrow \phi$ pointwise. Suppose that (4) holds for all ϕ_n, so that $\phi_n^* = \int\phi_n\,d\nu$, ν-a.e. Then (4) also holds for ϕ. This follows from $0 \le (\phi-\phi_n)^* = \phi^* - \int\phi_n\,d\nu$ and $\int(\phi-\phi_n)^*\,d\nu = \int(\phi-\phi_n)\,d\nu$, which tends to 0 as $n \longrightarrow \infty$ by the dominated convergence theorem.

If $\phi = 1_V$ for an open $V \subset \Sigma$, then we can take ϕ_n to be continuous:

$$\phi_n(x) := \begin{cases} 0 & x \notin V \\ n\,\mathrm{dist}(x,\partial V) & \text{if } x \in V \text{ and } \mathrm{dist}(x,V) \le \frac{1}{n} \\ 1 & \text{otherwise,} \end{cases}$$

where ∂V denotes the boundary of V. Thus we get (4) for characteristic sets of open sets. Regularity of Borel measures on metric spaces gives (4) for characteristic functions of Borel sets.

https://doi.org/10.4153/CMB-1998-026-0 Published online by Cambridge University Press

Statement (iv) follows from

$$\begin{aligned}\int 1_{X^\eta_{P,U}}\, d\nu &= \nu(X^\eta_{P,U}) = \mu(X_{P,U})\lambda^P(\eta)\\ &= \int_T 1_{X_{P,U}}(t)\lambda^P(\eta)\, d\mu(t)\\ &= \int_T \Big[\int_{\Omega_t} 1_{X^\eta_{P,U}}(t,\omega)\, d\lambda^t(\omega)\Big]\, d\mu(t).\end{aligned}$$ ■

The construction of ν can be generalized to give other ergodic measures on Σ. What is essential is that there is an $R > 0$ such that for any patch P the collections ω_P and $\omega_{P'}$ are independent if $E_R(P)$ and $E_R(P')$ are disjunct translates of each other. Here $E_R(P)$, the R-environment of P, is the patch generated by the set $\{x \in \mathbb{R}^2 \mid \mathrm{dist}(x, P) \le R\}$. For instance, one could let $\lambda(v)$ depend on $E_R(v)$ and take $\lambda^t = \prod_{v\in V_t} \lambda(v)$.

4. **Critical probabilities.** In percolation on $\mathbb{Z}^2$ the percolation probability $\theta(p)$, which is the probability that a given lattice point belongs to an infinite cluster, is the same for all lattice points by translation invariance. Denoting the cluster of 0 by C, one has $\theta(p) = P(|C| = \infty)$. By the ergodic theorem $\theta(p)$ is also equal, with probability one, to the density of the points belonging to an infinite cluster [18]. The critical probability p_c is defined as $\sup\{p \mid \theta(p) = 0\}$.

In percolation on a Penrose tiling $t \in \mathcal{T}$ it is less obvious how to define the percolation probability and the critical probability. Since vertices have different environments, the probability $\theta(p, v, t)$ that $v \in V_t$ belongs to an infinite cluster will depend on v. However, it follows from the FKG inequality [6] that if $\theta(p, v, t) > 0$ for some $v \in V_t$ then $\theta(p, u, t) > 0$ for all $u \in V_t$ ([15], Section 4.1): denoting by $\lambda^t(\{u \leftrightarrow \infty\})$ the probability that the cluster of u is infinite, we have

$$\begin{aligned}\lambda^t(\{u \leftrightarrow \infty\}) &\ge \lambda^t\big(\{u \leftrightarrow v, \quad v \leftrightarrow \infty\}\big)\\ &\ge \lambda^t\big(\{u \leftrightarrow v\}\big)\lambda^t\big(\{v \leftrightarrow \infty\}\big)\end{aligned} \tag{7}$$

and $\lambda^t(\{u \leftrightarrow v\}) > 0$ for any pair of vertices u, v. (This also follows from 'finite energy', defined in Section 5). So a critical probability for percolation on $t \in \mathcal{T}$ can be defined by $p_c(t) = \sup\{p \mid \theta(p, v, t) = 0\}$. Since $p_c(t)$ is invariant under translation it is μ-a.s. constant by the ergodicity of the Penrose dynamical system. (It is not hard to show that $p_c(t)$ is not 0 or 1; indeed, there exist numbers $a > 0$ and $b < 1$ such that $a \le p_c(t) \le b$ for all $t \in \mathcal{T}$. The Peierls argument does not depend on periodicity, *cf.* [19].)

Another critical probability p_c^d—independent of $t \in \mathcal{T}$—can be defined as the supremum of the p's for which either there is no infinite cluster or there is an infinite cluster of density zero. We can make sense of this density by considering percolation on all Penrose tilings simultaneously and using Theorem 3.1. We will show that $p_c(t) = p_c^d$ for μ-a.e. $t \in \mathcal{T}$. For this we need to introduce some notation.

Let $D \subset \mathbb{R}^2$ be Lebesgue measurable with $\operatorname{diam}(D) < \eta$, so that every $t \in \mathcal{T}$ has at most one vertex in D. For $t \in \mathcal{T}$ and $v \in V_t$ denote by C_v^t the open cluster of v. For $k = 0, 1, 2, \ldots$ define

$$A_k := \left\{(t, \omega) \;\middle|\; t \text{ has a vertex } v \text{ in } D \text{ and } |C_v^t| = k\right\},$$

where $|C_v^t|$ denotes the number of vertices in C_v^t. Since for some $R > 0$ all possible clusters of size k lie within a disk of radius R of v, the set 1_{A_K} can be written as a finite union of sets $X_{P,U}^{\eta}$. Hence there exist by Lemma 3.1 numbers $d_k \geq 0$ such that for all $t \in \mathcal{T}$

$$d_k = \lim_{L\to\infty} L^{-2}|D|^{-1} \int_{C_L} 1_{A_k}(T_x t^{\omega})\, dx \quad \text{for } \lambda^t\text{-a.e. } \omega \in \Omega_t. \tag{8}$$

Thus d_k is the density of vertices that belong to a cluster of size k and d_0 is the density of the closed vertices. Let α denote the density of vertices; this number is the same for all $t \in \mathcal{T}$. Then $\sum_{k=0}^{\infty} d_k \leq \alpha$ and the density of vertices belonging to an infinite cluster is given by $d_\infty(p) := \alpha - \sum_{k=0}^{\infty} d_k$. Note that $d_\infty(p)$ is independent of $t \in \mathcal{T}$. Therefore, if there is one $t \in \mathcal{T}$ for which λ^t-almost surely (the union of) the infinite cluster(s) has positive density, then in all $t' \in \mathcal{T}$ (the union of) the infinite cluster(s) has the same density. We can define a second critical probability by $p_c^d := \sup\{p \mid d_\infty(p) = 0\}$. Clearly, $p_c(t) \leq p_c^d$ for all $t \in \mathcal{T}$. Note that $d_\infty(p)$ is also given by (8), with $k = \infty$, for all $t \in \mathcal{T}$.

Let $\phi \geq 0$ be a continuous function with support in D such that $\int \phi(x)\, dx = 1$. Denote by $\{|C_v| = \infty\}_t$ the event that in t the cluster of v is infinite. Define

$$\tilde{\psi}(t^{\omega}, p) := \begin{cases} 1_{\{|C_v|=\infty\}_t}(t^{\omega})\phi(v) & \text{if } t \text{ has a vertex } v \text{ in } D \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(t, p) := \begin{cases} \theta(p, v, t)\phi(v) & \text{if } t \text{ has a vertex } v \text{ in } D \\ 0 & \text{otherwise.} \end{cases}$$

We can now prove the following theorem.

THEOREM 4.1. *$p_c(t) = p_c^d$ for μ-a.e. $t \in \mathcal{T}$*

PROOF. Since $\int \phi(x)\, dx = 1$, we have that $\int_{C_L} \tilde{\psi}(T_x t^{\omega}, p)\, dx$ is equal to the number of vertices $v \in V_t \cap C_L$ that belong to an infinite cluster of ω (apart from a boundary term). Hence, by Theorem 3.1

$$\begin{aligned} d_\infty(p) &= \int \tilde{\psi}(t^{\omega}, p)\, d\nu(t^{\omega}) \\ &= \int_{\mathcal{T}} \left[\int_{\Omega_t} \tilde{\psi}(t^{\omega}, p)\, d\lambda^t\right] d\mu(t) \\ &= \int_{\mathcal{T}} \psi(t, p)\, d\mu(t) \end{aligned}$$

Suppose that $p < p_c^d$, i.e., $d_\infty(p) = 0$. Then $\psi(t, p) = 0$ for μ-a.e. $t \in \mathcal{T}$ since $\psi(t, p) \geq 0$. Hence $\theta(p, t, v) = 0$ for μ-a.e. $t \in \mathcal{T}$. Since this argument works for any translate of D it follows that $\theta(p, t, v) = 0$ for μ-a.e. $t \in \mathcal{T}$. This means that $p \leq p_c(t)$ for μ-a.e. $t \in \mathcal{T}$. The theorem now follows from the fact that $p_c(t) \leq p_c^d$ for all $t \in \mathcal{T}$. ∎

We expect that for Bernoulli percolation $p_c(t) = p_c^d$ for *all* $t \in \mathcal{T}$, not just μ-almost all. This would follow if ψ would be continuous on $\mathcal{T}$. We have not been able to prove this.

An approach to proving that $p_c(t) = p_c^d$ for all $t \in \mathcal{T}$ would be to try to generalize Menshikov's theorem [17] (see also [9], Section 3.2) on the exponential tail decay of the radius of an open cluster. If one could prove that for every $t \in \mathcal{T}$ there exists a function $\psi(p) > 0$ such that for all $p < p_c(t)$ and all $v \in V_t$ the probability that there is an open path from v to the complement of a ball of radius r about v is bounded by $e^{-r\psi(p)}$ for all $r > 0$, then it would follow that $p_c(t) = p_c^d$ for all $t \in \mathcal{T}$.

Note that the proof of Theorem 4.1 generalizes to any percolation process that (1) connects any two vertices with positive probability, (2) satisfies the FKG inequality and (3) gives rise to an ergodic measure on Σ. In this more general statement the 'for μ-a.e $t \in \mathcal{T}$' probably cannot be improved.

A third way to introduce a critical probability is through $\chi(p, v, t) := E_{\lambda^t}(|C_v^t|)$, the expected size of the occupied cluster of v. The FKG-inequality implies (see [15], Section 4.1) that if $\chi(p, v, t) = \infty$ for one $v \in V_t$ then $\chi(p, u, t) = \infty$ for all $u \in V_t$. Then $p_c^\chi(t) := \sup\{p \mid \chi(p, v, t) < \infty\}$ is a critical probability. The generalization of Menshikov's result would show that $p_c^\chi(t) = p_c(t)$ for all $t \in \mathcal{T}$.

5**. Uniqueness of the infinite cluster.** This section shows that for Bernoulli percolation on Penrose tilings the number of infinite clusters is either 0 ν-almost surely or 1 ν-almost surely. The proof is a generalization of the Burton-Keane argument [5]. It therefore applies to a large class of percolation processes that satisfy the 'finite energy' condition defined as follows.

First consider one $t \in \mathcal{T}$ and a probability measure ρ on Ω_t. Given a finite $K \subset V_t$ and a configuration $\phi \in \{0, 1\}^K$ define $\tilde{\omega} \in \Omega_t$ for $\omega \in \Omega_t$ by

$$\tilde{\omega}_v := \begin{cases} \omega_v & \text{if } v \notin K \\ \phi_v & \text{if } v \in K. \end{cases}$$

For any event $E \subset \Omega_t$ define the event $\tilde{E} \subset \Omega_t$ by $\{\tilde{\omega} \mid \omega \in E\}$. We say that the measure ρ has finite energy if $\rho(E) > 0$ implies $\rho(\tilde{E}) > 0$ for all events $E \subset \Omega_t$, all finite $K \subset V_t$ and all $\phi \in \{0, 1\}^K$. It is clear that λ^t has finite energy for $0 < p < 1$.

Now consider a probability measure ρ on Σ. Let $\Lambda \subset \mathbb{R}^2$ be bounded. Then for every $t \in \mathcal{T}$ the patch defined by Λ in t is a copy of one of finitely many patterns $P_1, \ldots, P_l$. Let V^i denote the set of vertices of P_i and choose $\phi^i \in \{0, 1\}^{V^i}$. For every $t^\omega \in \Sigma$, the Λ-patch of t is a copy of one of the P^i; define $t^{\tilde{\omega}} \in \Sigma$ by setting $\tilde{\omega}_v = \omega_v$ if $v \notin V^i$ and $\tilde{\omega}_v = \phi_v^i$ if $v \in V^i$. Also, for events $E \subset \Sigma$ define the event $\tilde{E} \subset \Sigma$ as $\{t^{\tilde{\omega}} \mid t^\omega \in E\}$. We say that ρ has finite energy if $\rho(E) > 0$ implies $\rho(\tilde{E}) > 0$ for all events $E \subset \Sigma$, all bounded $\Lambda \subset \mathbb{R}^2$ and all $\phi^i \in \{0, 1\}^{V^i}$. The measure ν constructed in Section 3 describing Bernoulli percolation has finite energy for $0 < p < 1$.

THEOREM 5.1. *If ν is an ergodic probability measure on Σ having finite energy, then the number of infinite clusters is either 0 for ν-a.e. t^ω or 1 for ν-a.e. t^ω.*

PROOF. Ergodicity and finite energy imply that the number of infinite clusters is ν-a.s. 0, 1 or ∞. This was shown by Newman and Schulman [18] for percolation on $\mathbb{Z}^d$ and their argument is easily generalized. We present the generalization for the convenience of the reader. The argument by Burton and Keane [5] to exclude ∞ can also be generalized to Penrose tilings, and that will prove the theorem.

Let $A_n := \{t^\omega \mid \text{the number of infinite clusters in } t^\omega \text{ is } n\}$. Each A_n is invariant. Hence, by ergodicity, $\nu(A_n)$ is either 0 or 1. The A_n are disjunct, so there is one N such that $\nu(A_N) = 1$ and $\nu(A_n) = 0$ for $n \neq N$. Suppose that $1 < N < \infty$. Let $W_L := \{t^\omega \mid t^\omega$ contains N infinite clusters each of which has non-empty intersection with $C_L\}$. Then $\nu(W_L) \longrightarrow \nu(A_N)$ as $L \longrightarrow \infty$. So there is an L such that $\nu(W_L) > 0$. If $\tilde{W}_L$ is obtained from W_L by occupying all vertices inside C_L then $\nu(\tilde{W}_L) > 0$ by finite energy. But all t^ω in $\tilde{W}_L$ have one infinite cluster, contradicting that $\nu(A_1) = 0$. The assumption that $1 < N < \infty$ leads to a contradiction, so we must have $N = 0, 1$ or ∞.

Assume that $N = \infty$. A vertex $v \in V_t$ is called an encounter point for t^ω if

1. v belongs to an infinite cluster C of t^ω, and
2. the set $C \backslash \{v\}$ has no finite component and exactly 3 infinite components.

Let $A_L := \{t^\omega \mid \text{at least 3 infinite clusters in } t^\omega \text{ intersect } C_L\}$. Since $N = \infty$, there is an L such that $\nu(A_L) > 0$. Let $P_1, \ldots, P_l$ be the patterns that the C_L-patches give rise to. In at least one P_i there are vertices v_1, v_2, v_3, contained in C_L but connected by an edge to a vertex outside C_L, such that the event

$$B_L := \{t^\omega \mid \text{the } C_L\text{-patch of } t \text{ is a copy of } P_i \text{ and } \omega \text{ contains at least 3 infinite clusters, three of which enter } C_L \text{ at } v_1, v_2 \text{ and } v_3\}$$

has positive probability, $\nu(B_L) > 0$. One can now modify ω inside C_L in such a way that the three infinite clusters entering at v_1, v_2, v_3 become joined at an encounter point; denote the resulting event by $\tilde{B}_L$. By finite energy, $\nu(\tilde{B}_L) > 0$.

Let $D \subset \mathbb{R}^2$ be measurable. Let $E := \{t^\omega \mid t \text{ has a vertex } v \text{ in } D \text{ and } v \text{ is encounter point for } t^\omega\}$. Since $\nu(\tilde{B}_L) > 0$, we have $\nu(E) > 2\epsilon$ for some $\epsilon > 0$. By (4) in Theorem 3.1 there is for ν-a.e. t^ω a K', depending on t^ω, such that for all $K > K'$ the set C_K contains at least ϵK^2 encounter points. But, for every t^ω, by exactly the same argument as in [5], the number of encounter points in C_K is bounded by a constant times K. Thus the assumption that $N = \infty$ leads to a contradiction. ∎

For Bernoulli percolation on Penrose tilings this has the following consequences.

COROLLARY 5.1. *If $0 < p < 1$ then for μ-a.e. $t \in \mathcal{T}$ the number of infinite clusters is 0 λ^t-a.s. or 1 λ^t-a.s.*

COROLLARY 5.2. *For μ-a.e. $t \in \mathcal{T}$ and all $v \in V_t$ the percolation probability $\theta(p, v)$ is continuous for $p \in \big(p_c(t), 1\big]$ (cf. [1]).*

Note that for Bernoulli percolation Theorem 5.1 implies Theorem 4.1. The proof of Theorem 4.1 is simpler, and applies to FKG measures that do not have finite energy (*e.g.*, measures where on some $v \in V_t$, depending on $E_R(v)$, $\lambda(\omega_v = 1) = 1$).

6. **Generalization.** The foregoing results depend on properties of Penrose tilings that are shared by many other tilings, in arbitrary dimension, and the results immediately generalize to these other tilings. Two properties are needed. First, the tilings should be locally finite in the sense that the number of different patterns up to a given diameter is finite. And, second, each pattern P should occur with a frequency $n_P > 0$ that exists uniformly in the position of the cube $C_L \subset \mathbb{R}^d$ (*cf.* (1)).

Tilings of $\mathbb{R}^d$ that have these properties are 'self-similar tilings' [16] and tilings 'generated by the projection method' (see *e.g.* [14]). Penrose tilings belong to both classes. The uniform existence of frequencies for self-similar tilings is proved in [16] (see also [8, 25]) and in [12] for tilings generated by the projection method.

Note that the fact that the Penrose dynamical system is uniquely ergodic and minimal has played no role in our proofs, except for statement (ii) in Theorem 3.1. Mere ergodicity (*i.e.*, existence of the limit in (1) for $a = 0$, without requiring it to be strictly positive) would suffice. But of course one cannot expect that $p_c(t)$ is independent of t (instead of a.s. constant) for Bernoulli percolation if the tiling dynamical system is only ergodic, and not minimal and uniquely ergodic. The local finiteness condition could also be relaxed.

Finally, note that these results apply to percolation on 'ergodic subgraphs' of $\mathbb{Z}^d$, by which we mean the following. Let $A \subset \{0, 1\}^{\mathbb{Z}^d}$ be an ergodic subshift (a closed invariant subset of $\{0, 1\}^{\mathbb{Z}^d}$ that carries an ergodic probability measure). Each $a \in A$ defines a graph G with set of vertices $V_a = \{x \in \mathbb{Z}^d \mid a_x = 1\}$ and set of edges $E = \{(x, y) \in \mathcal{E} \mid a_x = a_y = 1\}$, where $\mathcal{E}$ is the set of all nearest-neighbor pairs in $\mathbb{Z}^d$. For the argument in (7) we have to assume that G is connected; for the Burton-Keane argument that there is an upper bound on the size of 0-clusters in a. Of course, for these ergodic subgraphs one takes translations in $\mathbb{Z}^d$ instead of in $\mathbb{R}^d$.

REFERENCES

1. J. van den Berg and M. Keane, *On the continuity of the percolation function.* In: Conference on Modern Analysis and Probability (Eds. R. Beals *et al.*). Contemp. Math. **26**(1984), 61–65.

2. N. G. de Bruijn, *Algebraic theory of Penrose's non-periodic tilings of the plane. I.* Nederl. Akad. Wetensch. Indag. Math. **84**(1981), 39–52.

3. ———, *Algebraic theory of Penrose's non-periodic tilings of the plane. II.* Nederl. Akad. Wetensch. Indag. Math. **84**(1981), 53–66.

4. ———, *Updown generation of Penrose tilings.* Indag. Math. (N.S.) **1**(1990), 201–220.

5. R. M. Burton and M. Keane, *Density and uniqueness in percolation.* Commun. Math. Phys. **121**(1989), 501–505.

6. C. M. Fortuin, J. Ginibre, and P. W. Kasteleyn, *Correlation inequalities on some partially ordered sets.* Commun. Math. Phys. **22**(1971), 89–103.

7. A. Gandolfi, M. S. Keane, and C. M. Newman, *Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses.* Probab. Theory Related Fields **92**(1992), 511–527.

8. C. P. M. Geerse and A. Hof, *Lattice gas models on self-similar aperiodic tilings.* Rev. Math. Phys. **3**(1991), 163–221.

9. G. Grimmett, *Percolation.* Springer, 1989.

10. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*. W. H. Freeman and Company, 1987.

11. A. Hof, *Some remarks on discrete aperiodic Schrödinger operators.* J. Stat. Phys. **72**(1993), 1353–1374.

12. ———, *On diffraction by aperiodic structures.* Commun. Math. Phys. **169**(1995), 25–43.

13. Jian Ping Lu and J. L. Birman, *Percolation and scaling on a quasilattice.* J. Stat. Phys. **46**(1987), 1057–1066.

14. A. Katz and M. Duneau, *Quasiperiodic patterns and icosahedral symmetry.* J. Physique **47**(1986), 181–196.
15. H. Kesten, *Percolation Theory for Mathematicians*. Birkhäuser, 1982.
16. W. F. Lunnon and P. A. B. Pleasants, *Quasicrystallographic tilings.* J. Math. Pures Appl. **66**(1987), 217–263.
17. M. V. Menshikov, *Coincidence of critical points in percolation problems*. Soviet Math. Dokl. **33**(1986), 856–869.
18. C. M. Newman and L. S. Schulman, *Infinite clusters in percolation models.* J. Stat. Phys. **26**(1981), 613–628.
19. C. J. Preston, *Gibbs States on Countable Sets*. Cambridge University Press, 1974.
20. C. Radin and M. Wolf, *Space tilings and local isomorphism.* Geom. Dedicata **42**(1992), 355–360.
21. E. A. Robinson, Jr., *The dynamical properties of Penrose tilings.* Trans. Amer. Math. Soc. **348**(1996), 4447–4469.
22. D. J. Rudolph, *Rectangular tilings of R^n and free R^n actions.* Lecture Notes in Math. **1342**(1988), 653–689.
23. S. Sakamoto, F. Yonezawa, K. Aoki, S. Nosé, and M. Hori, *Percolation in two-dimensional quasicrystals by Monte Carlo simulations.* J. Phys. A **22**(1989), L705–L709.
24. D. Shechtman, I. Blech, D. Gratias, and J.W. Cahn, *Metallic phase with long-range orientational order and no translational symmetry.* Phys. Rev. Lett. **53**(1984), 1951–1953.
25. B. Solomyak, *Dynamics of self-similar tilings.* Ergodic Theory Dynamical Systems, to appear.
26. P. Walters, *An Introduction to Ergodic Theory*. Springer Verlag, 1982.
27. F. Yonezawa *et al.*, *Percolation in Penrose tiling and its dual—in comparison with analysis for Kagomé, dice and square lattices.* J. Non-Cryst. Solids **106**(1988), 262–269.

Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
L8S 4K1