THE NUMBER OF COLOURED GRAPHS

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1. Let $F_n(k)$ denote the total number of k-coloured graphs on n labelled nodes and let $M_n(k)$ denote the number of graphs on n nodes that are coloured in *at most* k colours; also let $f_n(k)$ denote the number of *connected* k coloured graphs on n nodes. Read **(3)** has proved the following formulas:

$$\sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \left\{ \sum_{s=1}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!} \right\}^k, \tag{1.1}$$

$$\sum_{n=0}^{\infty} 2^{-\frac{1}{2}n^2} M_n(k) \frac{x^n}{n!} = \left\{ \sum_{s=0}^{\infty} 2^{-\frac{1}{2}s^2} \frac{x^s}{s!} \right\}^k, \tag{1.2}$$

$$1 + \sum_{n=1}^{\infty} 2^{-\frac{1}{2}n^2} F_n(k) \frac{x^n}{n!} = \exp\left\{ \sum_{n=1}^{\infty} f_n(k) \frac{x^n}{n!} \right\}. \tag{1.3}$$

In a recent paper **(4)** Wright has proved some asymptotic formulas for $F_n(k)$, $M_n(k)$, $f_n(k)$.

In the present paper we discuss some arithmetic properties of these numbers. We shall show that if p is an odd prime and w is a positive integer such that $p^{e-1}(p-1) \mid w$, then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} F_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}}, \tag{1.4}$$

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} M_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}}, \tag{1.5}$$

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}}, \tag{1.6}$$

where in (1.4) and (1.5) $n \geqslant re$, in (1.6) $n > re$ and r_1 is the greatest integer $\leqslant (r+1)/2$. Some additional properties of $F_n(k)$ and $f_n(k)$ are described in Theorems 4 and 5 below.

The results just quoted do not hold for $p = 2$. However in this case we have some auxiliary congruences described in Theorem 6. In particular we find that

$$M_n(k) \equiv k \pmod{2^n} \qquad (n > 2),$$

from which it follows that $M_n(k)$ is odd if and only if k is odd.

2. A series of the type

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \tag{2.1}$$

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where a_n are rational integers, is called a Hurwitz series, or briefly an H-series. It is easily verified that the sum, difference, and product of two H-series are again H-series; also the derivative and integral of H-series are H-series. If $a_0 = \pm 1$, the reciprocal of (2.1) is also an H-series.

If

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

is a second H-series, the statement

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \equiv \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \pmod m$$

means

$$a_n \equiv b_n \pmod m \qquad (n = 0, 1, 2, \ldots).$$

If $a_0 = 0$ we have the useful property

$$\left(\sum_{n=1}^{\infty} a_n \frac{x^n}{n!}\right)^k \equiv 0 \pmod{k!}, \tag{2.2}$$

where k is an arbitrary positive integer.

If

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_p^{r-s} a_{n+s(p-1)} \equiv 0 \pmod{p^r} \tag{2.3}$$

for all $n \geqslant r \geqslant 0$, where p is a prime, we say that (2.1) satisfies Kummer's congruence. In some cases we have in place of (2.3) the weaker congruence

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_p^{r-s} a_{n+s(p-1)} \equiv 0 \pmod{p^{r_1}} \qquad (n \geqslant r), \tag{2.4}$$

where

$$r_1 = [(r+1)/2]. \tag{2.5}$$

If we put

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad D = \frac{d}{dx},$$

it is evident that (2.3) is equivalent to

$$(D^p - a_p D) f(x) \equiv 0 \pmod{p^r},$$

while (2.4) is equivalent to

$$(D^p - a_p D) f(x) \equiv 0 \pmod{p^{r_1}}.$$

We shall require the following preliminary results.

THEOREM A. *If the sequence* $\{a_n\}$ *satisfies* (2.4) *and the sequence* $\{b_n\}$ *satisfies*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} b_p^{r-s} b_{n+s(p-1)} \equiv 0 \pmod{p^{r_1}} \qquad (n \geqslant r) \tag{2.6}$$

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and $a_p \equiv b_p \pmod{p}$, *then the sequence* $\{c_n\}$, *where*

$$c_n = \sum_{s=0}^{n} \binom{n}{s} a_s b_{n-s}, \tag{2.7}$$

satisfies

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} k^{r-s} c_{n+s(p-1)} \equiv 0 \pmod{p^{r1}} \qquad (n \geqslant r), \tag{2.8}$$

where

$$k \equiv a_p \equiv b_p \pmod{p}. \tag{2.9}$$

For the proof of this theorem see **(1)**.

As a corollary we have

THEOREM B. *Let* $\{a_n\}$ *satisfy* (2.4) *and define the sequence* $\{a_n^{(k)}\}$ *by means of*

$$\left\{\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right\}^k = \sum_{n=0}^{\infty} a_n^{(k)} \frac{x^n}{n!}, \tag{2.10}$$

where k *is an integer* $\geqslant 1$. *Then* $\{a_n^{(k)}\}$ *satisfies*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_p^{r-s} a_{n+s(p-1)}^{(k)} \equiv 0 \pmod{p^{r1}} \tag{2.11}$$

for $n \geqslant r$.

THEOREM C. *Let* $a_0 = 1$ *and define the sequence* $\{b_n\}$ *by means of*

$$\sum_{s=0}^{n} \binom{n}{s} a_s b_{n-s} = \begin{cases} 1 & (n = 0) \\ 0 & (n > 0). \end{cases} \tag{2.12}$$

Then if $\{a_n\}$ *satisfies* (2.4) *it follows that* $\{b_n\}$ *satisfies*

$$\sum_{s=0}^{n} (-1)^{r-s} \binom{r}{s} a_p^{r-s} b_{n+s(p-1)} \equiv 0 \pmod{p^{r1}} \qquad (n \geqslant r). \tag{2.13}$$

We shall also require

THEOREM D. *Let* $\{a_n\}$ *satisfy* (2.4) *and let* $\{b_n\}$ *satisfy a similar condition. Then we have*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (a_p b_p)^{r-s} a_{n+s(p-1)} b_{n+s(p-1)} \equiv 0 \pmod{p^{r1}} \tag{2.14}$$

for $n \geqslant r$.

To prove this theorem put

$$\Delta^r a_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_p^{r-s} a_{n+s(p-1)},$$

$$\Delta^r b_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} b_p^{r-s} b_{n+s(p-1)}.$$

Then it is easily verified that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (a_p b_p)^{r-s} a_{n+s(p-1)} b_{n+s(p-1)} = \sum_{j=0}^{r} \binom{r}{j} b_p^{r-j} \Delta^j b_n \cdot \Delta^{r-j} a_{n+j(p-1)}.$$

Since

$$\Delta^j b_n \equiv 0 \pmod{p^{[(j+1)/2]}} \qquad (n \geqslant j),$$
$$\Delta^{r-j} a_{n+j(p-1)} \equiv 0 \pmod{p^{[(r-j+1)/2]}} \qquad (n \geqslant r-j),$$

and

$$[(j+1)/2] + [(r-j+1)/2] \geqslant [(r+1)/2],$$

(2.14) follows at once.

3. In order to apply the above results we require the following lemma.

LEMMA. *If p is an odd prime, then*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-\frac{1}{2}(n+s(p-1))^2} \equiv 0 \pmod{p^{r_1}} \tag{3.1}$$

for all $n \geqslant 0$.

Proof. Put

$$f(x) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} x^{as+bs^2}, \tag{3.2}$$

where a, b are non-negative integers. We shall show that

$$f(x) = (x-1)^{r_1} g(x), \tag{3.3}$$

where $g(x)$ is a polynomial with integral coefficients. Consider the jth derivative

$$D^j f(1) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \prod_{i=0}^{j-1} (as + bs^2 - i).$$

We may put

$$\prod_{i=0}^{j-1} (as + bs^2 - i) = A_0 + A_1 s + A_2 s(s-1) + \ldots + A_{2j} s(s-1)\ldots(s-2j+1),$$

where the A_i are integers. Then we have

$$\begin{aligned} D^j f(1) &= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{i=0}^{2j} A_i s(s-1)\ldots(s-i+1) \\ &= \sum_{i=0}^{2j} A_i r(r-1)\ldots(r-i+1) \sum_{s=i}^{r} (-1)^{r-s} \binom{r-i}{s-i}, \end{aligned}$$

so that

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$$D^j f(1) = 0 \qquad (0 \leqslant 2j < r). \tag{3.4}$$

Clearly (3.4) implies (3.3).

In the next place we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-\frac{1}{2}(n+s(p-1))^2} = 2^{-\frac{1}{2}n^2} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-ns(p-1)-\frac{1}{2}s^2(p-1)^2}$$
$$= 2^{-\frac{1}{2}n^2} f(2^{1-p}),$$

with $a = n$, $b = (p-1)/2$. Thus by (3.3)

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-\frac{1}{2}(n+s(p-1))^2} = 2^{-\frac{1}{2}n^2}(2^{1-p} - 1)^{r_1} g(x) \equiv 0 \pmod{p^{r_1}}.$$

This completes the proof of the lemma.

4. We now make use of (1.1), (3.1), and Theorem B. The fact that the summation in the right member of (1.1) begins at $n = 1$ causes no difficulty. We get

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-\frac{1}{2}(n+s(p-1))^2} F_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \tag{4.1}$$

for $n \geqslant r \geqslant 1$. Similarly it follows from (1.2) that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{-\frac{1}{2}(n+s(p-1))^2} M_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \tag{4.2}$$

for all $n \geqslant r \geqslant 1$.

The factor $2^{-\frac{1}{2}(n+s(p-1))2}$ occurring in (4.1) and (4.2) is removed by means of Theorem D together with

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} 2^{\frac{1}{2}(n+s(p-1))^2} \equiv 0 \pmod{p^{r_1}}. \tag{4.3}$$

The proof of (4.3) is exactly like the proof of (3.1). We may therefore state the following congruences:

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} F_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \qquad (n \geqslant r), \tag{4.4}$$

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} M_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \qquad (n \geqslant r). \tag{4.5}$$

These results may be stated in a more general form by making use of the following theorem (cf. **2**, § 3).

THEOREM E. *Let $\{a_n\}$ satisfy* (2.4) *and let $p^{e-1}(p-1) \mid w$. Then we have*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a_p^{(r-s)w/(p-1)} a_{n+sw} \equiv 0 \pmod{p^{r(e-1)+r_1}} \tag{4.6}$$

for $n \geqslant re$.

Applying Theorem E to (4.4) and (4.5) we obtain the following results.

THEOREM 1. *Let p be an odd prime and let w be a positive integer such that*

$$p^{e-1}(p-1) \mid w. \tag{4.7}$$

Then we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} F_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}}, \tag{4.8}$$

where $r_1 = [(r+1)/2]$, $n \geqslant re$, $k \geqslant 1$.

THEOREM 2. *With the hypotheses of Theorem* 1 *we have*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} M_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}} \tag{4.9}$$

for $n \geqslant re$, $k \geqslant 1$.

We remark that (4.8) holds for negative k also.

5. Turning next to $f_n(k)$, we make use of (1.3). As remarked in **(4)** the series occurring in (1.3) are divergent; however, the formal identity is sufficient for our purpose. It follows from (1.3) by logarithmic differentiation that

$$\sum_{n=0}^{\infty} f_{n+1}(k) \frac{x^n}{n!} = \sum_{n=0}^{\infty} F_{n+1}(k) \frac{x^n}{n!} \Big/ \left(1 + \sum_{n=1}^{\infty} F_n(k) \frac{x^n}{n!}\right). \tag{5.1}$$

If we put

$$\left\{1 + \sum_{n=1}^{\infty} F_n(k) \frac{x^n}{n!}\right\}^{-1} = \sum_{n=0}^{\infty} G_n(k) \frac{x^n}{n!},$$

it follows from (4.4) and Theorem B that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} G_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \tag{5.2}$$

for $n \geqslant r$. Therefore by (5.1), (5.2), and Theorem A we get

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f_{n+s(p-1)}(k) \equiv 0 \pmod{p^{r_1}} \tag{5.3}$$

for $n > r$. Now applying Theorem E we get the following theorem.

THEOREM 3. *With the hypotheses of Theorem* 1 *we have*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} f_{n+sw}(k) \equiv 0 \pmod{p^{r(e-1)+r_1}} \tag{5.4}$$

for $n > re$, $k \geqslant 1$.

6. The results obtained above hold for all $k \geqslant 1$ and arbitrary primes $p \geqslant 3$. If we make use of the fact that

$$F_n(k) = 0 \qquad (1 \leqslant n < k), \tag{6.1}$$

which is implied by (1.1), we get some additional congruences.

In (4.8) take $r = 1$, $w = tp^{e-1}(p-1)$, so that

$$F_{n+w}(k) \equiv F_n(k) \pmod{p^e} \tag{6.2}$$

for $n \geqslant e$. If $1 \leqslant n < k$, it therefore follows from (6.1) that

$$F_{n+w}(k) \equiv 0 \pmod{p^e}. \tag{6.3}$$

Changing the notation slightly, we may state the following theorem.

THEOREM 4. *If p is an odd prime and*

$$n = m + tp^{e-1}(p-1) \qquad (e \leqslant m < k), \tag{6.4}$$

then we have

$$F_n(k) \equiv 0 \pmod{p^e}. \tag{6.5}$$

In this connection it should be recalled that by a general property of Hurwitz series without a constant term (see (2.2) above)

$$F_n(k) \equiv 0 \pmod{p^\nu}, \tag{6.6}$$

where p^ν is the highest power of p that divides $k!$. This congruence holds for all $n \geqslant 1$.

For $f_n(k)$ we have the following theorem.

THEOREM 5. *If p is an odd prime and*

$$n = m + tp^{e-1}(p-1) \qquad (e < m < k), \tag{6.7}$$

then

$$f_n(k) \equiv 0 \pmod{p^e}. \tag{6.8}$$

7. The case $p = 2$ requires special treatment. To begin with we take the formula **(3, 4)**

$$M_n(k) = \sum \frac{n!}{s_1! \ldots s_k!} T\left(\frac{1}{2}n^2 - \frac{1}{2}\sum_{i=1}^{k} s_i^2\right), \tag{7.1}$$

where the summation is over all non-negative s_i such that

$$s_1 + \ldots + s_k = n \tag{7.2}$$

and $T(n) = 2^n$. Clearly

$$E = \frac{1}{2}n^2 - \frac{1}{2}\sum_{i=1}^{k} s_i^2 = \sum s_1 s_2 \geqslant 0.$$

Indeed the minimum value is attained when

$$s_1 = \ldots = s_{k-1} = 0, \qquad s_k = n,$$

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and for any permutation of these values. Moreover for all other choices of s_i satisfying (7.2) we have

$$E \geqslant n - 1. \tag{7.3}$$

This second minimum occurs when

$$s_1 = \ldots = s_{k-2} = 0, \qquad s_{k-1} = 1, \qquad s_k = n - 1,$$

and of course for all permutations of these values; for $n > 2$ the number of permutations is $k(k - 1)$.

To prove the assertion concerning (7.3) suppose that the s_i are numbered so that

$$s_1 \geqslant 1, \ldots, s_r \geqslant 1, \quad s_{r+1} = \ldots = s_k = 0.$$

Then we have

$$\begin{aligned}
E &= \sum_{1\leqslant i<j\leqslant r} s_i s_j \sum_{1\leqslant i<j\leqslant r} (s_i + s_j - 1) \\
&= \sum_{i=1}^{r-1} (r - i)s_i + \sum_{j=2}^{r} (j - 1)s_j - \sum_{1\leqslant i<j\leqslant r} 1 \\
&= \sum_{i=1}^{r} (r - 1)s_i - \tfrac{1}{2}r(r - 1) = (r - 1)n - \tfrac{1}{2}r(r - 1) \\
&= \tfrac{1}{2}(r - 1)(2n - r).
\end{aligned}$$

Thus if $r > 1$ it follows that

$$E \geqslant \tfrac{1}{2}(r - 1)(2n - r) \geqslant \tfrac{1}{2}(r - 1 + 2n - r - 1) = n - 1.$$

Moreover, since $\frac{1}{2}(r - 1)(2n - r) = n - 1$ implies

$$(r - 2)(2n - r - 1) = 0$$

and $n \geqslant r$, we conclude that $r = 2$ and the statement made about (7.3) follows at once.

As a consequence of this and of the fact that $k(k - 1)$ is even we have

$$M_n(k) \equiv k \pmod{2^n} \qquad (n > 2). \tag{7.4}$$

We remark that

$$M_1(k) = k, \qquad M_2(k) = 2k^2 - k.$$

In the next place we take

$$F_n(k) = \sum \frac{n!}{s_1! \ldots s_k!} T\left(\frac{1}{2} n^2 - \frac{1}{2} \sum_1^k s_i^2\right), \tag{7.5}$$

where the summation is over all *positive* s_i that satisfy (7.2). Then in the present instance

$$E = \sum s_1 s_2 \geqslant \tfrac{1}{2}(k - 1)(2n - k);$$

the minimum is attained when

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$$s_1 = \ldots = s_{k-1} = 1, \qquad s_k = n - k + 1.$$

The proof is exactly as before.

The second minimum occurs when

$$s_1 = \ldots = s_{k-2} = 1, \qquad s_{k-1} = 2, \qquad s_k = n - k,$$

provided $n \geqslant k + 2$.

The proof is a slight refinement of the previous proof. We have for the second minimum

$$E \geqslant \tfrac{1}{2}(k-2)(k-3) + (k-2)(n-k+2) + 2(n-k);$$

the difference between the minima is equal to $n - k - 1$. As before, the number of permutations associated with the second minimum is $k(k-1)$, provided $n > k + 2$. We have therefore

$$F_n(k) = 2^{\frac{1}{2}(k-1)(2n-k)} F_n{}'(k), \tag{7.6}$$

where $F_n{}'(k)$ is an integer such that

$$F_n{}'(k) \equiv k \pmod{2^{n-k}} \qquad (n > k+2). \tag{7.7}$$

When $n = k + 2$, this holds at least $(\text{mod } 2^{n-k-1})$.

By making use of (7.6) and the formula **(3)**

$$F_n(k) = \sum_{r=1}^{n-k} \binom{n-1}{r-1} F_{n-r}(k) f_r(k) + f_n(k),$$

it is easy to show that

$$f_n(k) \equiv 0 \pmod{2^{\frac{1}{2}k(k-1)}}. \tag{7.8}$$

It is not clear whether a congruence similar to (7.7) can be found for $f_n(k)$.

We may now state the following theorem.

THEOREM 6. *The number $M_n(k)$ satisfies*

$$M_n(k) \equiv k \pmod{2^n} \qquad (n > 2).$$

In particular $M_n(k)$ is even if and only if k is even. The number $F_n(k)$ satisfies (7.6) *and* (7.7). *The number $f_n(k)$ satisfies* (7.8).

REFERENCES

1. L. Carlitz, *A note on Kummer's congruence*, Arch. der Math., *7* (1956), 441–445.
2. L. Carlitz and J. Riordan, *The number of labeled two-terminal series-parallel networks*, Duke Math. J., *23* (1956), 435–445.
3. R. C. Read, *The number of k-coloured graphs on labelled nodes*, Can. J. Math., *12* (1960), 410–414.
4. E. M. Wright, *Counting coloured graphs*, Can. J. Math., *13* (1961), 683–693.

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