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THE SECOND CONJUGATES OF CERTAIN BANACH ALGEBRAS

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1. Introduction. Let A be a Banach algebra and A^{**} its second conjugate space. Arens has defined two natural extensions of the product on A to A^{**}. Under either Arens product, A^{**} becomes a Banach algebra. Let A be a semi-simple Banach algebra which is a dense two-sided ideal of a B^*-algebra B and R^{**} the radical of $(A^{**}, \circ)$. We show that $A^{**} = Q \oplus R^{**}$, where Q is a closed two-sided ideal of $(A^{**}, \circ)$. This was inspired by Alexander's recent result for simple dual A^*-algebras (see [**1**, p. 573, Theorem 5]). We also obtain that if A is commutative, then A is Arens regular. As an application of this result, we show that if A is commutative and $B = C_0(M_A)$, then the following statements are equivalent:

(i) A is a modular annihilator algebra.

(ii) For each maximal modular ideal M of $(A^{**}, \circ)$ such that $M \not\supset \pi_A(A)$, M is weakly closed.

(iii) For each F in A^{**}, F belongs locally to $\hat{A}$ at each point of M_A.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [**7**].

Let A be a semi-simple commutative Banach algebra with carrier space M_A. Then $C_0(M_A)$ will denote the algebra of complex-valued functions on M_A, which vanish at infinity and $\hat{A}$ the function algebra on M_A isomorphic to A in the Gelfand theory.

For any subset E of a Banach algebra A, let $l_A(E)$ and $r_A(E)$ denote the left and right annihilators of E in A, respectively. Then A is called a modular annihilator algebra if, for every maximal modular left ideal I and for every maximal modular right ideal J we have $r_A(I) = (0)$ if and only if $I = A$ and $l_A(J) = (0)$ if and only if $J = A$. It is well-known that a semi-simple commutative Banach algebra A is a modular annihilator algebra if and only if its carrier space M_A is discrete (see [**4**] and [**9**]).

Let A be a Banach algebra, A^* and A^{**} the conjugate and second conjugate spaces of A, respectively. The two Arens' products on A^{**} are defined in stages according to the following rules (see [**2**]). Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$.

(a) Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.

(b) Define $G \circ f$ by $(G \circ f)(x) = G(f \circ x)$. Then $G \circ f \in A^*$.

(c) Define $F \circ G$ by $(F \circ G(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.

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A^{**} with the Arens product $\circ$ is denoted by $(A^{**}, \circ)$.

(a') Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.

(b') Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.

(c') Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ' G \in A^{**}$.

A^{**} with the Arens product $\circ'$ is denoted by $(A^{**}, \circ')$.

Each of these products extends the original multiplication on A when A is canonically embedded in A^{**}. In general, $\circ$ and $\circ'$ are distinct on A^{**}. If they coincide on A^{**}, then A is called Arens regular.

Notation. Let A be a Banach algebra. The mapping π_A will denote the canonical embedding of A into A^{**}.

In this paper, all algebras and linear spaces under consideration are over the field C of complex numbers.

3. The Algebra $(A^{**}, \circ)$**.** In this section, let A be a semi-simple Banach algebra which is a dense two-sided ideal of a B^*-algebra B. We write $\|\cdot\|$ for the norm on A and $|\cdot|$ for the norm on B. By [**3**, p. 3, Proposition 2.2], there exists a constant K such that $K\|\cdot\| \geqq |\cdot|$. Hence by [**3**, p. 3, Theorem 2.3], there exists a constant M such that

$$(3.1) \quad \|ab\| \leqq M\|a\|\,|b| \quad \text{and} \quad \|ba\| \leqq M\|a\|\,|b|,$$

for all a in A and b in B. For each $g \in B^*$, let g_A denote the restriction of g to A. Then $g_A \in A^*$. For each $F \in A^{**}$, define $\theta(F)$ on B^* by $\theta(F)(g) = F(g_A)$ $(g \in B^*)$. Then $\theta(F) \in B^{**}$.

For all $f \in A^*$ and $y \in B$, define

$$(f \circ y)(x) = f(yx) \quad (x \in A).$$

Then by (3.1), $f \circ y \in A^*$ and $\|f \circ y\| \leqq M\|f\|\,|y|$.

For each $F \in A^{**}$ and $f \in A^*$, define

$$(F * f)(y) = F(f \circ y) \quad (y \in B).$$

Then $F * f \in B^*$. For any $H \in B^{**}$, define

$$H * F(f) = H(F * f) \quad (f \in A^*,\ F \in A^{**}).$$

Then $H * F \in A^{**}$.

Let R_1^{**} (respectively R_2^{**}) be the radical of $(A^{**}, \circ)$ (respectively $(A^{**}, \circ')$). B^{**} with the Arens product will be denoted by $(B^{**}, \cdot)$. It is well-known that $(B^{**}, \cdot)$ is a B^*-algebra.

LEMMA 3.1. *Let A be a semi-simple Banach algebra which is a dense two-sided ideal of a B^*-algebra B. Then*

(i) *R_1^{**} is the left and right annihilator of $(A^{**}, \circ)$,*

(ii) *R_1^{**} coincides with R_2^{**}.*

Proof. (i) Put $\theta(R_1^{**}) = \{\theta(R) : R \in R_1^{**}\}$. For any $H \in B^{**}$ and $R \in R_1^{**}$, we show that $H * R \in R_1^{**}$. Suppose this is not so. Then by the proof of [**7**, p. 55, Theorem (2.3.2) (iii)], we can choose some $F \in A^{**}$ such that $F \circ (H * R)$ is not quasi-regular in $(A^{**}, \circ)$. Easy calculations show that $F \circ (H * R) = F_H \circ R$, where F_H is defined as in the proof of [**10**, p. 446, Theorem 5.7 (ii)]. Since $F_H \circ R \in R_1^{**}$ is quasi-regular in $(A^{**}, \circ)$, so is $F \circ (H * R)$; a contradiction. Therefore $H * R \in R_1^{**}$ and consequently $\theta(H * R) \in \theta(R_1^{**})$. Hence $H \cdot \theta(R) = \theta(H * R) \in \theta(R_1^{**})$. It is easy to see that $\theta(R_1^{**})$ is a quasi-regular left ideal of $(B^{**}, \cdot)$. Since $(B^{**}, \cdot)$ is a B^*-algebra, it follows that $\theta(R_1^{**}) = (0)$. Since $\theta(R_1^{**}) = (0)$, we have

$$(R \circ F)(f) = \theta(R)(F * f) = 0 \quad (f \in A^*,\ F \in A^{**},\ R \in R_1^{**}).$$

Consequently $R_1^{**} \circ A^{**} = (0)$. For each $x \in A$ and $f \in A^*$, define $f * x$ on B by $(f * x)(y) = f(xy)$ $(y \in B)$. Then it follows from (3.1) that $f * x \in B^*$. Since $\theta(R_1^{**}) = (0)$, we have

$$(3.2) \quad (\pi_A(x) \circ R)(f) = R(f \circ x) = \theta(R)(f * x) = 0 \quad (R \in R_1^{**}).$$

Since $\pi_A(A)$ is weakly dense in A^{**}, it follows from (3.2) that $A^{**} \circ R_1^{**} = (0)$. Hence it is now easy to see that R_1^{**} is equal to the left and right annihilator of $(A^{**}, \circ)$ and this proves (i).

(ii) By a similar argument as in (i), we can show that R_2^{**} is the left and right annihilator of $(A^{**}, \circ')$. Since by (i), $R_1^{**} \circ \pi_A(A) = R_1^{**} \circ' \pi_A(A) = (0)$, it follows that $R_1^{**} \circ' A^{**} = (0)$. Hence $R_1^{**} \subset R_2^{**}$. Similarly $R_2^{**} \subset R_1^{**}$ and so they are equal. This completes the proof.

Notation. Let $R^{**} = R_1^{**} = R_2^{**}$.

THEOREM 3.2. *Let A be a semi-simple Banach algebra which is a dense two-sided ideal of a B^*-algebra B. Then*

(i) $A^{**} = Q \oplus R^{**}$, *where Q is a closed two-sided ideal of* $(A^{**}, \circ)$.

(ii) *There exists a continuous algebraic homomorphism θ of* $(A^{**}, \circ)$ *into* $(B^{**}, \cdot)$ *such that the restriction of θ to Q is an isomorphism.*

Proof. (i) Let Λ be the collection of all finite subsets of A ordered by inclusion. Since A is a dense two-sided ideal of B, by the proof of [**7**, p. 245, Theorem (4.8.14)], we can show that there exists an approximate identity $\{e_\lambda : \lambda \in \Lambda\}$ for B such that $\{e_\lambda : \lambda \in \Lambda\} \subset A$. Let $F \in A^{**}$. Since $\|\pi_A(e_\lambda) \circ F\| \leqq M\|F\|$, there exists a subnet $\{e_\alpha\}$ of $\{e_\lambda\}$ and $F_1 \in A^{**}$ such that $\pi_A(e_\alpha) \circ F \to F_1$ weakly. Then for all f in A^{**}, we have

$$(3.3) \quad \begin{aligned} F_1(f) &= \lim_\alpha \pi_A(\mathrm{e}_\alpha) \circ F(f) = \lim_\alpha \pi_A(e_\alpha)(F \circ f) \\ &= \lim_\alpha \pi_B(e_\alpha)(F * f) = I_B(F * f), \end{aligned}$$

where I_B denotes the identity of $(B^{**}, \cdot)$ (see [**5**, p. 855, Lemma 3.8]). Now it follows easily from (3.3) that F_1 is the unique limit point of $\{\pi_A(e_\lambda) \circ F\}$ in

A^{**}. For all a in A and f in A^*, by (3.1) we have

$$\begin{aligned}|\pi_A(ae_\lambda) \circ F(f) - \pi_A(a) \circ F(f)| &= |F(f \circ (ae_\lambda - a))| \\ &\leqq M\|F\| \, \|f\| \, |ae_\lambda - a|.\end{aligned}$$

Hence $\pi_A(ae_\lambda) \circ F \to \pi_A(a) \circ F$ weakly in A^{**} and so $\pi_A(a) \circ F_1 = \pi_A(a) \circ F$. Consequently $A^{**} \circ (F_1 - F) = (0)$ and therefore by Lemma 3.1, $F - F_1 \in R^{**}$. Let

$$Q = \{F_1 : F \in A^{**}\}.$$

If there exists some F in A^{**} such that $F_1 \in R^{**}$, then by Lemma 3.1, $\pi_A(e_\lambda) \circ F = \pi_A(e_\lambda) \circ F_1 = 0$ and consequently $F_1 = 0$. Therefore $Q \oplus R^{**} = A^{**}$. We show that Q is a closed two-sided ideal of $(A^{**}, \circ)$. Let $F, G \in A$. Then

$$(3.4) \quad F \circ G = (F_1 + (F - F_1)) \circ (G_1 + (F - G_1)) = F_1 \circ G_1.$$

Also

$$(3.5) \quad (F \circ G)_1 = \lim_\lambda \pi_A(e_\lambda) \circ (F \circ G) = F_1 \circ G = F_1 \circ G_1.$$

It follows from (3.4) and (3.5) that Q is a two-sided ideal of $(A^{**}, \circ)$. It is easy to see that Q is closed and this proves (i).

(ii) We show that the mapping $\theta : F \to \theta(F)$ $(F \in A^{**})$ is such a mapping. In fact, for all $F, G \in A^{**}$ and $g \in B^*$, we have

$$(3.6) \quad (\theta(F) \cdot \theta(G))(g) = F(G \circ g_A) = \theta(F \circ G)(g).$$

Hence by (3.6), $\theta(F) \cdot \theta(G) = \theta(F \circ G)$. Therefore we see easily that θ is an algebraic homomorphism from $(A^{**}, \circ)$ into $(B^{**}, \cdot)$. Since $\|g_A\| \leqq K|g|$, we have $|\theta(F)| \leqq K\|F\|$ and consequently θ is continuous. It remains to show that the restriction of θ to Q is an isomorphism. Suppose $F \in Q$ and $\theta(F) = 0$. Then for all f in A^*, we have

$$F(f) = \lim_\lambda (\pi_A(e_\lambda) \circ F)(f) = \lim_\lambda \theta(F)(f * e_\lambda) = 0.$$

Therefore $F = 0$ and so θ is an isomorphism. This completes the proof of the theorem.

By using the proofs of Theorem 3.2 and [**10**, p. 446, Theorem 5.7 (ii)], we have the following result:

COROLLARY 3.3. *A^{**}/R^{**} is a semi-simple Banach algebra which is a dense two-sided ideal of some B^*-algebra.*

THEOREM 3.4. *Let A be a semi-simple Banach algebra which is a dense two-sided ideal of a B^*-algebra B. Then A is Arens regular if any of the following conditions holds:*

(i) *A is a modular annihilator algebra.*

(ii) *A is a commutative algebra.*

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Proof. We show that

$$(3.7) \quad F \circ \pi_A(a) \circ G = F \circ' \pi_A(a) \circ' G \quad (F, G \in A^{**}, a \in A).$$

In fact, this is true if condition (i) holds, because $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$ by Theorem 3.4 in [**11**]. Now suppose A is commutative. Then for all f in A^*, we have

$$\begin{aligned}(F \circ \pi_A(a) \circ G &- F \circ' \pi_A(a) \circ' G)(f) \\ &= (\pi_A(a) \circ (F \circ G - F \circ' G))(f) \\ &= (F \circ G - F \circ' G)(f \circ a) = (\theta(F) \cdot \theta(G) - \theta(F) \cdot \theta(G))(f * a) \\ &= 0.\end{aligned}$$

Therefore (3.7) holds. Now by using the proofs of Theorem 3.2 and [**10**, p. 446, Theorem 5.7], we can show that $F \circ G = F \circ' G$. Therefore A is Arens regular.

We remark that condition (i) in Theorem 3.3 does not imply that A is an annihilator algebra. For example, the algebra $\mathfrak{S}_{\mathrm{II}}$ given in [**6**, p. 141, Theorem 14.1] is a modular annihilator A^*-algebra which is a dense two-sided ideal of the dual B^*-algebra $\mathfrak{S}_\infty$. However $\mathfrak{S}_{\mathrm{II}}$ is not an annihilator algebra. Therefore Theorem 3.3 (i) is a generalization of [**10**, p. 446, Theorem 5.7 (i)].

We believe that Theorem 3.4 is true without conditions (i) or (ii).

4. A characterization of modular annihilator algebras. In this section, we shall give an application of Theorem 3.4. Unless otherwise stated A will be a commutative Banach algebra which is a dense ideal of $C_0(M_A)$ where M_A is the carrier space of A. Then by [**7**, p. 57, Corollary (2.3.7)], A is semi-simple. Hence by Theorem 3.4, A is Arens regular and so $(A^{**}, \circ)$ is a commutative Banach algebra.

LEMMA 4.1. *Let $f \in M_A$ and $\{f_\alpha\}$ a net in M_A such that $f_\alpha \to f$ and $f_\alpha \neq f$ for all α. Then there exists an element F in A^{**} such that $F(f) \neq 0$ and $F(f_\alpha) = 0$ for all α.*

Proof. This follows easily from the proof of [**9**, p. 829, Lemma 5.1].

For each ϕ in M_A, let ϕ' be the multiplicative linear functional on A^{**} such that $\phi'(F) = F(\phi)$ for all F in A^{**} (see [**5**, p. 854, Lemma 3.6]).

For each maximal modular ideal M in A^{**}, let f_M be the multiplicative linear functional on A^{**} such that $M = \{F \in A^{**} : f_M(F) = 0\}$.

LEMMA 4.2. *Let M be a maximal modular ideal of $(A^{**}, \circ)$ such that $M \not\supseteq \pi_A(A)$. Then the following statements are equivalent:*

(i) *M is weakly closed in A^{**}.*

(ii) *$M = \{F \in A^{**} : F(\phi) = 0\}$ for some ϕ in M_A.*

(iii) *$f_M = \phi'$ for some ϕ in M_A.*

Proof. (i) $\Rightarrow$ (ii). Suppose M is weakly closed in A^{**}. Let ϕ be the restriction of f_M to $\pi_A(A)$. Then $\phi \neq 0$ and $\phi \in M_A$. Clearly $\{\pi_A(x) \in \pi_A(A) : \phi(x) = 0\} \subset M$. Let N be the weak closure of $\{\pi_A(x) \in \pi_A(A) : \phi(x) = 0\}$ in A^{**}. Then $N \subset M$. Also by the proof of [**5**, p. 865, Theorem 5.3], N is a maximal modular ideal of A^{**} and $N = \{F \in A^{**} : F(\phi) = 0\}$. Hence it follows from the maximality of N that $M = N$ and this proves (ii).

(ii) $\Rightarrow$ (iii). Suppose $M = \{F \in A^{**} : F(\phi) = 0\}$ for some ϕ in M_A. Then f_M and ϕ' have the same null space and so $f_M = \phi'$.

(iii) $\Rightarrow$ (i). Suppose $f_M = \phi'$ for some ϕ in M_A. Then $M = \{F \in A^{**} : F(\phi) = 0\}$ and consequently M is weakly closed in A^{**}. This completes the proof.

Let A be a commutative Banach algebra with carrier space M_A. A function f on M_A is said to belong locally to $\hat{A}$ at p in M_A if there exists a neighborhood V of p and a function $\hat{x}$ in $\hat{A}$ such that $f|_V = \hat{x}|V$.

We now have the main result of this section.

THEOREM 4.3. *Let A be a commutative Banach algebra which is a dense ideal of $C_0(M_A)$. Then the following statements are equivalent:*

(i) *A is a modular annihilator algebra.*

(ii) *For each maximal modular ideal M of A^{**} such that $M \not\supset \pi_A(A)$, M is weakly closed in A^{**}.*

(iii) *For each F in A^{**}, F belongs locally to $\hat{A}$ at each point of M_A.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) holds. Since $M \not\supset \pi_A(A)$ is a maximal modular ideal of $\pi_A(A)$, it follows from [**12**, p. 38, Lemma 3.3] that there exists some minimal idempotent e in A such that $\pi_A(e) \notin M$. By Theorem 3.4 in [**11**], $\pi_A(A)$ is an ideal of $(A^{**}, \circ)$. Since $\pi_A(e) \circ A^{**} = \pi_A(eA) = C\pi_A(e)$, where C is the field of complex numbers, $\pi_A(e) \circ M \subset \pi_A(e) \circ A^{**} \cap M = (0)$. Hence $M \subset (1 - \pi_A(e)) \circ A^{**}$ and so by the maximality of M, $M = (1 - \pi_A(e)) \circ A^{**}$. It follows that M is weakly closed and this gives (ii).

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Let $\phi \in M_A$ and let $\{\phi_\alpha\} \subset M_A$ be a net converging to ϕ in M_A. Since $\{\phi_\alpha'\}$ are multiplicative linear functionals on A^{**}, by Alaoglu's Theorem, we can assume that there exists some f' in A^{***} such that $\phi_\alpha'(F) \to f'(F)$ for all F in A^{**}. It is easy to see that f' is a multiplicative linear functional on A^{**} and $f'|\pi_A(A) = \phi$. Therefore by Lemma 4.2, $f' = \phi'$. Hence $F(\phi_\alpha) \to F(\phi)$ for all F in A^{**}. It now follows from Lemma 4.1 that M_A is discrete and so A is a modular annihilator algebra.

(i) $\Rightarrow$ (iii). This is clear because M_A is discrete.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $\phi \in M_A$ and let $\{\phi_\alpha\} \subset M_A$ be a net converging to ϕ in M_A. Then by (iii), we can assume that $F(\phi_\alpha) \to F(\phi)$ for all F in A. Therefore it follows from Lemma 4.1 that M_A is discrete and this completes the proof of the theorem.

Theorem 4.3 (iii) is a generalization of [**8**, p. 532, Theorem 4.2 (4)].

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