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ON POLYNOMIALS WITH RELATED LEVEL SETS

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If p is a polynomial in one real variable and $p(x)=p(-x)$ then p has only even powers of x and is thus a polynomial in x^2. If p is a polynomial in n variables and $p(x_1,\ldots,x_n)=p(y_1,\ldots,y_n)$ when $x_1^2+\cdots+x_n^2=y_1^2+\cdots+y_n^2$ then p is a polynomial in q where $q(x_1,\ldots,x_n)=x_1^2+\cdots+x_n^2$.

The problem considered in this note is this: For which polynomials q is it true that if $p(x)=p(y)$ whenever $q(x)=q(y)$ then p is a polynomial in q? Such polynomials q will be said to satisfy (*). If the problem is posed for polynomials with complex variables, the answer is simple: any polynomial in n complex variables satisfies (*) (Theorem 1). However the problem is not as simple for polynomials with real variables. We give two classes of polynomials in one variable satisfying (*), neither class containing the other: if q is a polynomial of degree n and q has a level set containing n points, then q satisfies (*) (Theorem 2). If q is a polynomial such that the polynomial $Q(x,y)=[q(x)-q(y)]/(x-y)$ is irreducible and q is not 1:1, then q satisfies (*) (Theorem 3). Of course, x^3, being 1:1, doesn't satisfy (*) and more generally the composition of two polynomials $q_0\circ q_1$ does not satisfy (*) if q_0 is 1:1 on the range of q_1 (of course q_1 not being a constant). Thus x^3+3x^2+3x $(=(x+1)^3-1)$ doesn't satisfy (*) yet x^3+4x^2+3x $(=x(x+1)(x+3))$ does satisfy (*).

THEOREM 1. *Let $q(z_1,\ldots,z_n)$ be a polynomial in n complex variables. Let $p(z_1,\ldots,z_n)$ be another such that p is constant on the level sets of q. Then p is a polynomial in q.*

Proof. We may assume q is not a constant. The function f from the range of q to C (the complex plane) defined by $f(q(z_1,\ldots,z_n))=p(z_1,\ldots,z_n)$ is well defined by hypothesis. We show that it is a polynomial. It is possible to specialize all but one of the variables of q so that q defines a nonconstant polynomial, say q_0, in just one variable. Its range is C. By the same specialization p defines a polynomial p_0. If $q_0'(z)\neq 0$, then f is differentiable at $q_0(z)$ with derivative $p_0'(z)/q_0'(z)$. Since q_0 is open and p_0 continuous, $f=p_0\circ q_0^{-1}$ is continuous on C and since f is analytic except at a finite number of points, f is an entire function. Since f has a pole at ∞, ($q_0(z)\to\infty$ implies $z\to\infty$ implies $p_0(z)\to\infty$), f is a polynomial.

THEOREM 2. *Let q be a polynomial in one real variable and of degree n. If q has some level set containing n points, then q satisfies* (*).

Proof. The hypothesis guarantees that q has an infinite number of level sets with n points. We show by induction on the degree of p that if p is constant on an infinite number of those level sets of q containing n points, then p is a polynomial in q. It is clearly true if p has degree 0. Thus suppose the assertion is known to be true for any polynomial of degree less than the degree of p. Let $S=\{r_1, r_2, \ldots, r_n\}$ be a level set of q on which p is constant. Then $p-p(r_1)$ is divisible by $(x-r_1)(x-r_2)\cdots(x-r_n)$ and therefore by $q-q(r_1)$. Thus $p(x)-p(r_1)=p_1(x)[q(x)-q(r_1)]$, p_1 is of lower degree than p, and p_1 is constant on all those level sets of q, other than S, that p is constant on. By our inductive hypothesis p_1 is a polynomial in q and hence so is p.

THEOREM 3. *Let q be a polynomial in one real variable such that $Q(x, y)=[q(x)-q(y)]/(x-y)$ is irreducible. If q is not* 1:1 *then q satisfies* (*).

Proof. The hypotheses guarantee that $S=\{y: q(x)-q(y)=0$ has a solution other than $x=y\}$ is infinite. Let p be a polynomial which is constant on an infinite number of the level sets of q which meet S. We show by induction on the degree of p that p is a polynomial in q. It is clear if p has degree 0. Suppose p has degree k and that if p_0 has degree less than k and is constant on an infinite number of level sets of q meeting S, then p_0 is a polynomial in q. Let $P(x, y)=[p(x)-p(y)]/(x-y)$. By an application of the Euclidean algorithm one may, as follows, show that Q divides P. (This argument is modeled after one appearing in [1, p. 291].) Define inductively polynomials r_k, s_k, R_k by:

$$\begin{aligned} r_1(y)P(x, y) &= q_1(x, y)Q(x, y)+R_2(x, y) \\ r_2(y)Q(x, y) &= q_2(x, y)R_2(x, y)+R_3(x, y) \\ &\vdots \\ r_{n-1}(y)R_{n-1}(x, y) &= q_{n-1}(x, y)R_n(x, y)+R_{n+1}(x, y) \end{aligned}$$

where d_k, the degree of $R_k(x, y)$ considered as a polynomial in x over the field of rational functions in y, becomes progressively smaller, $d_{n+1}=0$ and $d_n\neq 0$ (let $Q=R_1$). There are infinitely many numbers y such that $q(x)=q(y)$ and $p(x)=p(y)$ have a common solution $x(y)$ not equal to y. Hence $P(x(y), y)$ and $Q(x(y), y)$ both vanish for infinitely many y. This means $R_{n+1}(y)$ has an infinite number of zeros and must be zero. Any irreducible factor of $R_n(x, y)$ which is of positive degree in x must divide both $Q(x, y)$ and $P(x, y)$. Since Q is irreducible, this means Q divides P. Thus $p(x)-p(y)=R(x, y)[q(x)-q(y)]$ and letting $p_1(x)=R(x, 0)$, $p(x)-p(0)=p_1(x)[q(x)-q(0)]$. The degree of p_1 is less than k and p_1 is constant on those level sets of q on which p is constant (other than the level set containing 0) and therefore p_1 is a polynomial in q and consequently p is a polynomial in q.

Let $q(x)=(x^2-1)(x^2-4)$. Then q meets the hypotheses of Theorem 2, but not Theorem 3, since $q(x)-q(y)$ is divisible by x^2-y^2 and so $[q(x)-q(y)]/(x-y)$ is divisible by $x+y$. Let $q(x)=x^4-x$. Then q meets the hypotheses of Theorem 3 but not Theorem 2. For a straightforward calculation shows that $x^3+x^2y+xy^2+y^3-1$ is irreducible.

REFERENCE

1. L. V. Ahlfors, *Complex analysis*, McGraw-Hill, New York (second edition), 1966.

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