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THE f-DECOMPOSITION OF ARTINIAN MODULES OVER HYPERFINITE GROUPS

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A $\mathbb{Z}G$-module A is said to have an f-decomposition if $A = A^f \oplus A^{\bar{f}}$ in which A^f is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^f as an abelian group is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ has no finite irreducible $\mathbb{Z}G$-factors. In this paper, we prove that: if G is a hyperfinite group then any artinian $\mathbb{Z}G$-module A has an f-decomposition, which gives a positive answer to the question raised by D.I. Zaitzev in 1986.

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If G is a locally soluble hyperfinite group, then it was known that any minimal (or artinian), maximal (or noetherian), or minimax $\mathbb{Z}G$-module A has an f-decomposition, that is, $A = A^f \oplus A^{\bar{f}}$ in which A^f is a $\mathbb{Z}G$-submodule of A such that each irreducible $\mathbb{Z}G$-factor of A^f as an abelian group is finite and the $\mathbb{Z}G$-submodule $A^{\bar{f}}$ of A has no finite irreducible $\mathbb{Z}G$-factors (see [**1**, **2**, and **3**]). In the above results, is the locally soluble condition necessary? This question was asked by Zaitzev in 1986 [**3**]; we give a positive answer for the artinian case. That is, we prove the following

Theorem. *If G is a hyperfinite group, then any artinian $\mathbb{Z}G$-module A has an f-decomposition.*

In order to prove the theorem, we need some lemmas.

Lemma 1. *Let G be a group, x an element of G, A a $\mathbb{Z}G$-module with $pA = 0$ for some prime p, and B a nonzero subset of A. If also x is of order p, then $B(x-1) \neq B$.*

Proof. Suppose $B(x-1) = B$ then for $0 \neq a \in B$ we have

$$a = a_1(x-1) = a_2(x-1)^2 = \cdots = a_m(x-1)^p$$
$$= a_m\left[\sum_{i=0}^{p} \binom{p}{i}(-1)^i x^{p-i}\right] = a_m(x^p - 1) = 0,$$

a contradiction.

The following lemma, though easy, is the key to our removal of the solubility hypothesis. We emphasise that S is assumed only to be a normal *subset*; that is, $S^g = S$ for all $g \in G$.

Lemma 2. *Let G be a group, S a normal subset of G and A a $\mathbb{Z}G$-module. Then $C_A(S)$ and $[A,S] = \sum_{x \in S} A(x-1)$ are $\mathbb{Z}G$-submodules of A.*

The proofs of Lemma 2 and the well-known Lemma 3 are left to the reader.

Lemma 3. *Let G be a group, x an element of G, and H a subgroup of G contained in the centralizer $C_G(x)$ of x in G. If A is a $\mathbb{Z}G$-module, then for any $\mathbb{Z}H$-submodule B of A we have $B(x-1)$ and $C_B(x)$ are $\mathbb{Z}H$-submodules of A and $B/C_B(x) \cong_{\mathbb{Z}H} B(x-1)$.*

Lemma 4. *Let G be a hyperfinite group and A an artinian $\mathbb{Z}G$-module such that $pA = 0$ for some prime p. If the irreducible $\mathbb{Z}G$-factors of A as abelian groups are all finite, then A is finite.*

Proof. Suppose A is infinite, then by the artinian condition we may assume each proper $\mathbb{Z}G$-submodule of A is finite. We may also assume that G acts faithfully on A. If G were finite then A would be artinian as a $\mathbb{Z}$-module and so would be finite, therefore we may assume that G is infinite.

Let B be a proper $\mathbb{Z}G$-submodule of A and let $H = C_G(B)$, then H is a normal subgroup of finite index in G. So H contains a nontrivial finite subgroup K which is normal in G (by G being hyperfinite). Let q be a prime factor of the order of K and let $S = \{x \in K;\ x \text{ is of order } q\}$. For $G_1 = C_G(S)$, since $G_1 \geqq C_G(K)$ and $G/C_G(K)$ is finite so G_1 is a subgroup of finite index in G. Thus A is an infinite artinian $\mathbb{Z}G_1$-module. Therefore A contains a least $\mathbb{Z}G_1$ $-$submodule A_1 such that A_1 is infinite, $B < A_1$, and each proper $\mathbb{Z}G_1$-submodule of A_1 as an abelian group is finite. Since $C_A(S)$ is a $\mathbb{Z}G$-submodule of A (Lemma 2) and $B \leqq C_A(S) < A$ so $C_A(S)$ as an abelian group is finite. Thus $C_{A_1}(S) = A_1 \cap C_A(S)$ is a proper $\mathbb{Z}G_1$-submodule of A_1 and $B \leqq C_{A_1}(S)$. Hence there exists $x_0 \in S$ such that $B \leqq C_{A_1}(x_0) < A_1$. Let $G_2 = C_G(x_0)$, then $G_2 \geqq G_1$ and so G_2 is of finite index in G. Clearly there exists a least $\mathbb{Z}G_2$-submodule A_2 of A such that A_2 is an infinite artinian $\mathbb{Z}G_2$-module, each proper $\mathbb{Z}G_2$-submodule of A_2 as an abelian group is finite, and A_1 is contained in A_2. For $x_0 \in G_2$, if $|x_0| = q \neq p$ then, by Fitting's lemma, $A_2 = [A_2, <x_0>] + C_{A_2}(<x_0>)$. Since $B \leqq C_{A_1}(x_0) < A_1$ *so* $B \leqq C_{A_2}(<x_0>) < A_2$ and then $[A_2, <x_0>]$ and $C_{A_2}(<x_0>)$ are nonzero proper $\mathbb{Z}G_2$-submodules of A_2. Thus $[A_2, <x_0>)$ are finite and so A_2 is finite, a contradiction. Therefore $|x_0| = q = p$. Since $B \leqq C_{A_1}(x_0) < A_1$ and $A_2(x_0 - 1) < A_2$ (Lemma 1), we have $B \leqq C_{A_2}(x_0) < A_2$ and so both $C_{A_2}(x_0)$ and $A_2/C_{A_2}(x_0) \cong_{\mathbb{Z}G_2} (A_2(x_0 - 1))$ are finite. Hence A_2 is finite, a contradiction again. Thus the result holds.

Lemma 5. *If G is a hyperfinite group and A an artinian $\mathbb{Z}G$-module all of whose irreducible $\mathbb{Z}G$-factors are finite, then A as an abelian group is Černikov and $G/C_G(A)$ is finite.*

Proof. This is similar to the proof of Theorem 3 in [2].

Proof of Theorem. Suppose A does not have an f-decomposition, then by the artinian condition we may assume that each proper $\mathbb{Z}G$-submodule of A has one. Since G is locally finite and A is artinian, A is periodic and further A is a p-group for some prime p. Let M be the sum of all proper $\mathbb{Z}G$-submodules of A; then M has an f-decomposition and so $M \neq A$. Thus M is the unique maximal submodule of A. We may assume that G acts faithfully on A and consider the irreducible $\mathbb{Z}G$-image A/M. (1) If A/M is finite then we may assume $M = M^f$ by investigating A/M^f; similarly, (2) if A/M is infinite we assume $M = M^{\bar{f}}$. Also, it is clear that G is infinite.

Case 1. *A/M is finite and $M = M^{\bar{f}}$.*

In this case, let $H = C_G(A/M)$, then G/H is finite and so H contains a nontrivial finite subgroup K which is normal in G. Let q be a prime factor of the order of K for some prime q and let $S = \{x \in K;\ x \text{ is of order } q\}$. For $G_1 = C_G(S)$, we have $G_1 \geqq C_G(K)$ and so G_1 is of finite index in G. Therefore A is an artinian $\mathbb{Z}G_1$-module and then A contains a least $\mathbb{Z}G_1$-submodule A_1 such that A_1 is not contained in M. Then $A_1/A_1 \cap M)(\cong_{\mathbb{Z}G_1}(A_1 + M)/M)$ is a finite irreducible $\mathbb{Z}G_1$-module. For $x \in S$, since $S \leqq K \leqq H = C_G(A/M)$ we have $A_1/C_{A_1}(x)$ $(\cong_{\mathbb{Z}G_1} A_1(x-1) \leqq M)$ has no nonzero finite $\mathbb{Z}G_1$-factors and then $C_{A_1}(x)$ is not contained in M. Thus $C_{A_1}(x) = A_1$ for any $x \in S$ and then $C_{A_1}(S) = A_1$, which shows that $A_1 \leqq C_A(S)$. So the $\mathbb{Z}G$-submodule $C_A(S)$ of A is not contained in M and then $C_A(S) = A$, which is contrary to G being faithful on A. Case 1 is proved.

Case 2. *A/M is infinite and $M = M^f$.*

In this case, M as an abelian group is Černikov and $G/C_G(M)$ is finite, by Lemma 5. Consider A as a $\mathbb{Z}G_1$-module, where $G_1 = C_G(M)$. Then A is artinian and so contains a least $\mathbb{Z}G_1$-submodule A_1 which is not contained in M. As G_1 contains a nontrivial finite subgroup F which is normal in G so we must have $A_1 \neq C_{A_1}(F)$, since otherwise $A_1 \leqq C_A(F)$ would imply that $C_A(F) = A$, contrary to the faithfulness of G. Since $C_{A_1}(F) \geqq M_1(= A_1 \cap M)$ and $A_1/M_1(\cong_{\mathbb{Z}G1}(A_1 + M)/M)$ is irreducible, $C_{A_1}(F) = M_1$. Let $G_2 = C_{G_1}(F)$, then G_1/G_2 is finite and then A_1 is an artinian $\mathbb{Z}G_2$-module. Let A_2 be a least $\mathbb{Z}G_2$-submodule of A_1 not contained in M. By $C_{A_1}(F) = M_1$ we have $C_{A_2}(F) = M_2(= A_2 \cap C_{A_1}(F) = A_2 \cap M_1 = A_2 \cap A_1 \cap M = A_2 \cap M)$, it implies that there exists $x_0 \in F$ such that $A_2 \neq C_{A_2}(x_0)(\geqq M_2)$. Therefore $A_2(x_0 - 1)(\cong_{\mathbb{Z}G_2} A_2/C_{A_2}(x_0))$ is an infinite irreducible $\mathbb{Z}G_2$-submodule of A_2 and then of A. Thus A contains an infinite irreducible $\mathbb{Z}G$-submodule generated by $A_2(x_0 - 1)$ and then A has an f-decomposition, a contradiction. Case 2 is proved.

By the above proof, the theorem is true.

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