6
LOCALIZATION

We consider two types of localization in this text, one defined directly in terms of rings and their elements, the other based on category theory. In the first type of localization, we take a suitable set Σ of elements of a ring R and adjoin the inverses of the members of Σ to R, thus obtaining a new ring R_Σ whose elements are fractions of the form r/σ. We can then localize an R-module M by extending scalars from R to R_Σ. Some modules become the zero module on localization, while others embed in their localization. This distinction defines two important categories of R-module, the Σ-torsion modules and the Σ-torsion-free modules respectively, and provides the starting point for the second type of localization. In this approach, we are given a subcategory $\mathcal{C}$ of a category $\mathcal{A}$ and the aim of the localization procedure is to annihilate the category $\mathcal{C}$; in other words, the objects in $\mathcal{C}$ are to play the role of the torsion modules.

The motive for localization is that it provides a technique for simplifying a problem by comparing a given ring, or category, with one or more localized rings, or categories, over which the problem is more amenable. In the next chapter, we shall see this technique exploited to obtain some results on the module theory of orders over Dedekind domains.

This chapter is organised into three sections. The first deals with the technicalities of the construction of the localization of a ring, and the behaviour of modules under localization is considered in the second. The third section is devoted to the categorical approach to localization.

6.1 LOCALIZATION FOR RINGS

Our main objective in this section is to construct, from a ring R and a suitable subset Σ of R, a ring R_Σ in which the elements of Σ have inverses. This process is called localization at Σ, and the resulting ring R_Σ is the ring of fractions (of R with respect to Σ). The set Σ is required to satisfy a condition, the *Ore* condition, which guarantees that the elements of R_Σ are indeed fractions, and that the properties of R and R_Σ are reasonably well connected with one another; in particular, R_Σ is a flat R-module.

Since we work with noncommutative rings, we are obliged to distinguish between right fractions and left fractions. An example in which this distinction becomes manifest is given in (6.2.21) below.

6.1.1 Ore sets

Let R be a ring. A subset Σ of R is said to be *multiplicatively closed* (or just *multiplicative*) if whenever σ and τ are in Σ, so also is $\sigma\tau$. For convenience, we make the further, harmless, assumption that $1 \in \Sigma$. In other words, Σ is a submonoid of R. For ease of exposition, we confine the discussion to the case that Σ consists of *non-zerodivisors*: given σ in Σ, the equations $\sigma x = 0 = y\sigma$ have no solution in R other than $x = y = 0$. Some texts call such σ *regular*. (The extension of our results to allow zerodivisors is indicated in Exercises 6.1.8 and 6.2.13.)

The multiplicative set Σ is called a *right Ore set* if, for any $\sigma \in \Sigma$ and $a \in R$,

$$a\Sigma \cap \sigma R \neq \emptyset;$$

that is, there are elements $\nu \in \Sigma$ and $c \in R$ with

$$a\nu = \sigma c.$$

Such a set is sometimes called a *right denominator set*.

In particular, if a multiplicative set Σ is *central*, that is, $\sigma r = r\sigma$ for all σ in S and r in R, then it is automatically a right Ore set. We record immediately a fundamental property of Ore sets. Two elements σ and τ in Σ are said to have a *common right multiple* in Σ if there are elements x and y in R such that

$$\sigma x = \tau y \in \Sigma.$$

6.1.2 Lemma

Any two elements σ and τ in a right Ore set Σ have a common right multiple in Σ.

Proof

By definition, $\tau\Sigma \cap \sigma R \neq \emptyset$, so $\tau\nu = \sigma c$ for some ν in Σ and c in R. □

6.1.3 Examples

Here are some examples which serve to motivate our general discussion. First, let $\mathcal{O}$ be a commutative domain and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$; that is, if x and y are elements of $\mathcal{O}$ and $xy \in \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Write $\Sigma = \mathcal{O} \setminus \mathfrak{p}$, the set of elements of $\mathcal{O}$ that are not in $\mathfrak{p}$. Then Σ is multiplicative. In this case, we usually write the localization of $\mathcal{O}$ at Σ as $\mathcal{O}_{\mathfrak{p}}$ rather than $\mathcal{O}_{\Sigma}$, and speak of the *localization of* $\mathcal{O}$ *at* $\mathfrak{p}$.

The zero ideal 0 of $\mathcal{O}$ is itself prime, and the corresponding ring of fractions $\mathcal{O}_0$ is simply the field of fractions $\mathcal{K}$ of $\mathcal{O}$. (A direct construction of $\mathcal{K}$ can be found in [BK: IRM] (1.1.12).) For a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, the ring $\mathcal{O}_{\mathfrak{p}}$ is easily obtained as a subring of $\mathcal{K}$ ([BK: IRM] (6.2.4)).

More generally, suppose that R is an $\mathcal{O}$-order, where $\mathcal{O}$ is again a commutative domain. By definition, $\mathcal{O}$ is in the centre of R, no element of $\mathcal{O}$ is a zerodivisor in R, and R is a finitely generated $\mathcal{O}$-module.

Thus, for a prime ideal $\mathfrak{p}$ of $\mathcal{O}$, $\Sigma = \mathcal{O} \setminus \mathfrak{p}$ is a central multiplicative subset of R. We write $R_{\mathfrak{p}}$ for the localization of R at Σ. It is straightforward ([BK: IRM] (1.2.23)) to construct a vector space $\mathcal{K}R$ over $\mathcal{K}$ which is spanned by R, and it is clear that $\mathcal{K}R$ can be identified as the ring R_0. Again, the localizations $R_{\mathfrak{p}}$ can be considered to be subrings of R_0.

Some non-central examples of Ore sets are provided by skew polynomial rings $A[T, \alpha]$. Recall that, given a ring A and a ring endomorphism α of A, the ring $A[T, \alpha]$ is the set of all (formal) 'right' polynomials

$$a_0 + Ta_1 + \cdots + T^k a_k, \text{ where } a_0, a_1, \ldots, a_k \in A, k \geq 0,$$

with addition given by the expected rule, but with multiplication twisted by the rule

$$a \cdot T = T \cdot \alpha(a).$$

The set $\Sigma = \langle T \rangle = \{1, T, T^2, \ldots\}$ is easily seen to be a right Ore set in $A[T, \alpha]$. If α is injective, then $\langle T \rangle$ consists of non-zerodivisors, regardless of whether or not A is a domain. The corresponding ring of fractions $A[T, T^{-1}, \alpha]$ is called the *skew Laurent polynomial ring* over A.

The general construction of a ring of fractions is first given in [Grell 1927] for commutative domains and in [Ore 1933] for noncommutative domains. The systematic study of commutative local rings begins with [Krull 1938].

6.1.4 Basic properties of rings of fractions

We summarize the properties of the ring of right fractions R_{Σ} before we give the construction. Let Σ be a right Ore set of non-zerodivisors in R.

The elements of R_Σ are to be written as fractions a/σ with a in R and σ in Σ, and equality of fractions is given by the rule that

RF1: $a/\sigma = b/\tau$ if and only if $ax = by$ and $\sigma x = \tau y \in \Sigma$ for some x and y in R.

To write down addition, we recall that any two elements σ and τ in Σ have a common right multiple $\sigma x = \tau y$ in Σ. Then

RF2: $a/\sigma + b/\tau = (ax + by)/\sigma x$.

For multiplication, we use the fact that for given b and σ, we have $b\sigma' = \sigma b'$ for some b' and σ'; then

RF3: $(a/\sigma)(b/\tau) = ab'/\tau\sigma'$.

The above rule has the superficially surprising consequence that

$$(1/\sigma)(1/\tau) = 1/\tau\sigma \;\text{ for } \sigma, \tau \in \Sigma;$$

however, this identity merely reflects the fact that in a noncommutative situation, the inverse of a product is the product of the inverses in the reverse order.

The zero element in R_Σ is $0/1$ and the identity is $1/1$, and it is easy to check that the mapping $\iota : R \to R_\Sigma$, $\iota(a) = a/1$, is a ring homomorphism.

Moreover, $a/1 = 0$ in R_Σ only if $a = 0$ in R, since Σ contains no zerodivisors. Thus ι is an injection, and in practice we often regard R as a subring of R_Σ through ι.

For $\sigma \in \Sigma$, the equality $\sigma/\sigma = 1/1$ shows that σ has acquired a twosided inverse after being embedded in R_Σ.

6.1.5 The construction of the ring of fractions

The most direct method is to use the desired criterion for equality, RF1 above, to define an equivalence relation on pairs (a, σ) so that the equivalence class of the pair becomes the fraction a/σ. This method works well enough in the commutative case ([BK: IRM] (1.1.12)), but, in general, it requires much complicated checking (see [Rowen 1988] §3.1 for details). We therefore prefer to give a direct limit construction for R_Σ that circumvents some of the checking and makes the functorial properties of R_Σ more transparent. The starting-point for the construction is obtained by setting $y = 1$ in RF1. Then, whenever $\sigma x \in \Sigma$, we have

$$a/\sigma = ax/\sigma x.$$

So introduce a relation $\leq$ on Σ by the rule that $\sigma \leq \tau$ if and only if $\sigma x = \tau$ for some x in R. Since any two elements of R have a common right multiple (6.1.2), Σ is a directed set. Notice that the element x must be unique if it exists, because Σ contains no zerodivisors.

We note a useful result which is a simple consequence of the fact that Σ is directed.

6.1.6 Lemma

Any finite set of elements $\{\sigma_1, \ldots, \sigma_k\}$ of Σ has a common right multiple, that is, there is an element σ of Σ and elements $\{x_1, \ldots, x_k\}$ of R with $\sigma_i x_i = \sigma$ for all i. □

6.1.7 Definition of the ring of fractions

We now associate to Σ a direct system of left R-modules, each isomorphic to R itself, as follows.

For each $\sigma \in \Sigma$, let (R, σ) be a copy of the ring R, with elements denoted (a, σ). (As with the formation of the disjoint union (Exercise 1.4.9), we use the elements σ as markers to distinguish different copies of R.)

Whenever $\sigma, \nu \in \Sigma$ and $\sigma \leq \nu$, define

$$\phi^{\sigma\nu} : (R, \sigma) \longrightarrow (R, \nu)$$

by

$$(a, \sigma)\phi^{\sigma\nu} = (ax, \nu), \quad \text{where } \sigma x = \nu.$$

Since the element x is unique, this rule defines the map $\phi^{\sigma\nu}$ unambiguously, and it is clear that each $\phi^{\sigma\nu}$ is a homomorphism of left R-modules.

The *ring of fractions* R_Σ of R with respect to Σ is defined to be

$$R_\Sigma = \operatorname{dir\,lim}_\Sigma R$$

(see (5.1.4) and (5.1.17)). It is immediate from the definition that R_Σ is a left R-module. The fact that it is also a ring under the stated rules for addition and multiplication, RF2, RF3 of (6.1.4) above, is proved in stages.

We write a/σ for the class $[(a, \sigma)]$ in R_Σ of the element (a, σ) of (R, σ). From the definition, for any x in R, $a/\sigma = ax/\sigma x$ provided $\sigma x \in \Sigma$. More generally, the pairs (a, σ) and (b, τ) give the same fraction if and only if there is some common right multiple $\nu = \sigma x = \tau y$ in Σ such that $(a, \sigma)\phi^{\sigma\nu} = (b, \tau)\phi^{\tau\nu}$, that is,

$$ax = by \text{ and } \sigma x = \tau y \in \Sigma \text{ for some } x, y \in R,$$

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which is condition RF1 of (6.1.4).

The rule for addition is easily seen to be

RF2: $a/\sigma + b/\tau = (ax + by)/\sigma x$,

since we must compute the sum using a common right multiple $\sigma x = \tau y$ of σ and τ, and the left R-module structure is given by

$$r \cdot (a/\sigma) = (ra)/\sigma \text{ for } r \in R.$$

The final step in showing that R_Σ is a ring requires two preliminary results, the first of which is immediate from the definition and (5.2.2).

6.1.8 Theorem

R_Σ is a flat left R-module. □

The above theorem is due to [Serre 1956], where it is exploited to construct modules over the ring of fractions.

6.1.9 Proposition

Let $\sigma \in \Sigma$. Then

(i) $(R/\sigma R) \otimes_R R_\Sigma = 0$;

(ii) *the left multiplication homomorphism $\sigma_\bullet : R \to R$, $a \mapsto \sigma a$, induces an automorphism of R_Σ (as an additive group).*

Proof

(i) Write members of $R/\sigma R$ as $\overline{a}$ where a is in R. Given $a, x \in R$ and $\tau \in \Sigma$, we have

$$\overline{a} \otimes (x/\tau) = \overline{ax} \otimes (1/\tau),$$

so we can write elements of the tensor product as sums of elements of the form $\overline{a} \otimes (1/\tau)$. The right Ore condition shows that $a\nu = \sigma c$ for some $c \in R$ and $\nu \in \Sigma$. Then

$$\overline{a} \otimes (1/\tau) = \overline{a} \otimes (\nu/\tau\nu) = \overline{a\nu} \otimes (1/\tau\nu) = \overline{\sigma c} \otimes (1/\tau\nu) = 0.$$

(ii) There is a short exact sequence

$$0 \longrightarrow R \xrightarrow{\sigma_\bullet} R \longrightarrow R/\sigma R \longrightarrow 0$$

of right R-modules, the first map being injective since Σ contains no zerodivisors. Because R_Σ is left flat, the sequence

$$0 \longrightarrow R \otimes_R R_\Sigma \xrightarrow{\sigma_\bullet \otimes id} R \otimes_R R_\Sigma \longrightarrow R/\sigma R \otimes_R R_\Sigma \longrightarrow 0$$

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is also exact, so the result follows from (i) together with the natural isomorphism $R \otimes_R R_\Sigma \cong R_\Sigma$ (3.1.4). □

6.1.10 R_Σ is a ring

In order to confirm that R_Σ is a ring, we construct an injective additive homomorphism θ from R_Σ to the ring $\mathrm{End}(R_\Sigma) = \mathrm{End}_{\mathbb{Z}}(R_\Sigma)$, and then transfer the multiplication in $\mathrm{End}(R_\Sigma)$ to R_Σ. Notice that we work with the ring of *all* additive homomorphisms of R_Σ. The construction of θ exploits the definition of R_Σ as a direct limit of copies (R, σ) of R.

For each $\sigma \in \Sigma$, define an additive homomorphism

$$\theta^\sigma : (R, \sigma) \longrightarrow \mathrm{End}(R_\Sigma)$$

by

$$(a, \sigma)\theta^\sigma = a_\bullet(\sigma_\bullet)^{-1},$$

where, for $c/\gamma \in R_\Sigma$ with $c \in R$ and $\gamma \in \Sigma$, we have $a_\bullet(c/\gamma) = ac/\gamma$ and $\sigma_\bullet(c/\gamma) = \sigma c/\gamma$; $\sigma_\bullet$ was shown to be invertible in the previous proposition.

The relation

$$(a, \sigma)\theta^\sigma(\sigma/1) = a/1$$

shows that each θ^σ is an injective map.

Next, we check the coherence of the system $\{\theta^\sigma\}$. Suppose that $\sigma \leq \nu$, with $\sigma x = \nu$. Then

$$\begin{aligned}((a, \sigma)\phi^{\sigma\nu}\theta^\nu)(\nu_\bullet(b/\tau)) &= ((ax, \sigma x)\theta^{\sigma x})((\sigma x)_\bullet(b/\tau)) \\ &= (ax)_\bullet((\sigma x)_\bullet)^{-1}(\sigma x)_\bullet(b/\tau) \\ &= axb/\tau \\ &= a_\bullet(\sigma_\bullet)^{-1}(\sigma x)_\bullet(b/\tau) \\ &= ((a, \sigma)\theta^\sigma)(\nu_\bullet(b/\tau)).\end{aligned}$$

Because $\nu_\bullet$ is an automorphism, the endomorphisms $(a, \sigma)\phi^{\sigma\nu}\theta^\nu$ and $(a, \sigma)\theta^\sigma$ coincide, as required.

Since the system $\{\theta^\sigma\}$ is coherent, we have an induced additive homomorphism $\theta : R_\Sigma \to \mathrm{End}(R_\Sigma)$ as desired. As each θ^σ is injective, so also is θ, by (5.1.8).

It remains to show that $\mathrm{Im}\,\theta$ is a subring of $\mathrm{End}(R_\Sigma)$, for which we need to verify that $id_{R_\Sigma} \in \mathrm{Im}\,\theta$ and that each product $(a/\sigma)\theta \cdot (b/\tau)\theta$ is in $\mathrm{Im}\,\theta$. We have $id_{R_\Sigma} = (\sigma, \sigma)\theta^\sigma$ for all σ, so that $id_{R_\Sigma} = (1/1)\theta$. To handle the product, we first note that θ is multiplicative on R, since $a_\bullet b_\bullet = (ab)_\bullet$ always. In the general case, given $b \in R$ and $\sigma \in \Sigma$, choose σ' and b' with $b\sigma' = \sigma b'$. Then

$$(\sigma_\bullet)^{-1}b_\bullet = b'_\bullet(\sigma'_\bullet)^{-1} \text{ in } R_\Sigma,$$

whence

$$\begin{aligned}((a/\sigma)\theta)\cdot((b/\tau)\theta) &= (a_\bullet)(\sigma_\bullet)^{-1}(b_\bullet)(\tau_\bullet)^{-1}\\ &= (ab')_\bullet(\sigma'_\bullet)^{-1}(\tau_\bullet)^{-1}\\ &= (ab')_\bullet((\tau\sigma')_\bullet)^{-1}\\ &= (ab'/\tau\sigma')\theta.\end{aligned}$$

We have shown that θ is an additive injective homomorphism on R_Σ and $\mathrm{Im}(\theta)$ is a ring, from which we can see ([BK: IRM] Lemma 1.1.7) that there is a uniquely defined multiplication on R_Σ such that θ becomes a ring homomorphism, and that the product rule is

RF3: $(a/\sigma)(b/\tau) = ab'/\tau\sigma'$, whenever $b\sigma' = \sigma b'$.

Next we exhibit the universal property of localization. To conform with our most recent notation, it is more convenient here to regard the category $\mathcal{R}$ING of rings-with-identity as a left category, rather than as a right category as is our usual custom. This is because we have constructed R_Σ as a direct limit in the left category ${}_R\mathcal{M}$OD. (Strictly speaking, we are replacing $\mathcal{R}$ING with its mirror $\mathcal{R}\text{ING}^\odot$.)

We say that a ring homomorphism $\alpha : R \to S$ is *Σ-inverting* if $\sigma\alpha$ has a twosided inverse in S for each element σ in Σ.

6.1.11 Theorem

R_Σ is the universal ring in which the elements of Σ become invertible.

More precisely, the canonical embedding $\iota : R \to R_\Sigma$ is Σ-inverting, and if $\alpha : R \to S$ is any Σ-inverting ring homomorphism, then there is a unique induced homomorphism $\overline{\alpha} : R_\Sigma \to S$ with $\iota\overline{\alpha} = \alpha$.

Proof

Since $1/\sigma$ is a twosided inverse of σ in R_Σ, ι is Σ-inverting. Given α, define $\overline{\alpha}$ by $(a/\sigma)\overline{\alpha} = (a\alpha)\cdot(\sigma\alpha)^{-1}$. The verification that this definition does actually give a ring homomorphism is much the same as the construction of the map θ in the preceding proof. Clearly $\overline{\alpha}$ is then the unique homomorphism with $\iota\overline{\alpha} = \alpha$. □

When we identify R with its canonical image in R_Σ, Σ is again an Ore set, and the universal property has the following consequence.

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6.1.12 Corollary

There is a canonical ring isomorphism $R_\Sigma \cong (R_\Sigma)_\Sigma$. □

6.1.13 Local rings

A ring R is said to be *local* if it has an ideal $\mathfrak{a}$ which is simultaneously the unique maximal left, right and twosided ideal of R. Then $\mathfrak{a}$ is the set of nonunits of R; it is also the Jacobson radical $\mathrm{rad}(R)$ of R, and $R/\mathfrak{a}$ is a division ring ([BK: IRM] (4.3.23)).

For a commutative domain, and in particular, for a Dedekind domain, a case important for the next chapter, we can now link the term 'localization' with local rings. First observe that in a commutative domain $\mathcal{O}$ any Ore set Σ extends to

$$\overline{\Sigma} = \{x \in \mathcal{O} \mid xy \in \Sigma \text{ for some } y \in \mathcal{O}\}\,.$$

This is also an Ore set, with $\overline{\Sigma} = \mathcal{O} \cap U(\mathcal{O}_\Sigma)$, and $\overline{\overline{\Sigma}} = \overline{\Sigma}$. Since $\mathcal{O}_{\overline{\Sigma}}$ and its subring $\mathcal{O}_\Sigma$ are both Σ-inverting and $\overline{\Sigma}$-inverting, we have, by universality, $\mathcal{O}_\Sigma = \mathcal{O}_{\overline{\Sigma}}$.

6.1.14 Proposition

Let $\mathcal{O}$ be a commutative domain, with Σ an Ore set in $\mathcal{O}$. Then the localization $\mathcal{O}_\Sigma$ is a local ring if and only if $\overline{\Sigma}$ is the complement of a prime ideal $\mathfrak{p}$ of $\mathcal{O}$, in which case $\mathcal{O}_\Sigma = \mathcal{O}_\mathfrak{p}$.

Proof

It is clear from the definition above that the commutative ring $\mathcal{O}_\Sigma$ is local if and only if it has a unique maximal ideal, which necessarily consists of the nonunit elements in $\mathcal{O}_\Sigma$. If $\overline{\Sigma} = \mathcal{O} \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of $\mathcal{O}$, then $\mathfrak{p}_\Sigma = \{a/\sigma \mid a \in \mathfrak{p}\}$ is clearly the set of nonunits in $\mathcal{O}_\Sigma$. Conversely, if $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_\Sigma$, then $\mathfrak{p} = \mathfrak{m} \cap \mathcal{O}$ is a prime ideal of $\mathcal{O}$, and $\overline{\Sigma} = \mathcal{O} \setminus \mathfrak{p} = \mathcal{O} \setminus \mathfrak{m}$. □

The above proposition tells us that localizations of commutative domains at prime ideals always give rise to local commutative domains. In combination with the universal property of localization and (6.1.12), it also gives a converse that every local commutative domain is a localization – of itself!

6.1.15 Corollary

(a) *Let $\mathcal{O}$ be a commutative domain, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$, and let*

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$\overline{\mathfrak{p}} = \mathfrak{p}\mathcal{O}_\mathfrak{p}$ *be the extension of* $\mathfrak{p}$ *to* $\mathcal{O}_\mathfrak{p}$. *Then there is a canonical ring isomorphism*

$$\mathcal{O}_\mathfrak{p} \cong (\mathcal{O}_\mathfrak{p})_{\overline{\mathfrak{p}}}.$$

(b) *If, moreover,* $\mathcal{O}$ *is already local and* $\mathfrak{m}$ *is the unique maximal ideal of* $\mathcal{O}$, *then*

$$\mathcal{O} \cong \mathcal{O}_\mathfrak{m}.$$ □

It is often useful to have an 'internal' description of R_Σ. Let R/σ denote all fractions of the form a/σ with a in R. When $\sigma \leq \nu$, with $\sigma x = \nu$, R/σ is a subset of R/ν since $a/\sigma = ax/\sigma x$ for any a. Then

6.1.16 Lemma

(i) $R_\Sigma = \bigcup_\Sigma R/\sigma$;

(ii) R/σ *is isomorphic to* R *as a left* R*-module.* □

6.1.17 The maximal ring of fractions

As we have already noted, a commutative domain $\mathcal{O}$ has a field of fractions $\mathcal{K}$ that contains every localization $\mathcal{O}_\Sigma$ of $\mathcal{O}$. In the noncommutative case, a domain need not have a division ring of fractions, as can be seen from Exercise 6.1.4. More seriously, an example due to Mal'cev [Mal'cev 1937] ([BK: IRM] Exercise 1.1.7) exhibits a domain which cannot be embedded in any division ring, while the example given in (6.2.21) below shows that a domain may have a division ring of right fractions that is not a ring of left fractions. Nevertheless, any ring has a maximal ring of right fractions with respect to right Ore sets of non-zerodivisors. (We are grateful to Paul Cohn for some helpful comments on this point.)

6.1.18 Theorem

Let R *be a ring. Then the following hold.*

(i) R *has a maximal right Ore set* Ω *of non-zerodivisors.*

(ii) *If* σ *is any right Ore set of non-zerodivisors in* R, *then there is a canonical injective ring homomorphism of* R_Σ *to* R_Ω.

Proof

Clearly (ii) is immediate from (i). To prove (i), it suffices to show that the multiplicative submonoid Ω of R generated by all the right Ore sets of non-zerodivisors is again a right Ore set of non-zerodivisors. Since any member of Ω is a product of terms from only a finite set of Ore sets, it is enough to prove the assertion for a finite number of Ore sets, and then, by induction, we can reduce to the case that $\Omega = \langle \Sigma, \Sigma' \rangle$ is generated by two Ore sets. Again using induction, we need only show that

$$r\Omega \cap \sigma\sigma' R \neq \emptyset \quad \text{for } r \in R,\ \sigma \in \Sigma \ \text{ and } \sigma' \in \Sigma'.$$

But $r\sigma_1 = \sigma r_1$ for some σ_1 in Σ and r_1 in R, and then $r_1\sigma_1' = \sigma' r'$ for some σ_1' in Σ' and r' in R, which gives

$$r\sigma_1\sigma_1' = \sigma r_1 \sigma_1' = \sigma\sigma' r'.$$

Since any product of non-zerodivisors is again a non-zerodivisor, the assertion follows. □

The maximal Ore set in a domain need not be very big, as Exercise 6.1.4 shows.

Exercises

6.1.1 Let $\alpha : R \to S$ be an injective ring homomorphism such that each element of S has the form $\alpha(a)\alpha(\sigma)^{-1}$ for some a and σ in R. Show that

$$\Sigma = \{\sigma \in R \mid \alpha(\sigma) \text{ is a unit in } S\}$$

is a right Ore set of non-zerodivisors in R and that $S \cong R_\Sigma$.

Thus the Ore conditions are necessary as well as sufficient for the construction of the ring of fractions. The case when α need not be injective is characterized in [Gabriel 1962]. See Exercise 6.1.8 below.

6.1.2 Let Ω be an Ore subset of an Ore set Σ in the ring R.

Show that the following statements are equivalent.

(i) Ω is cofinal in Σ (as a directed set).

(ii) Given $\sigma \in \Sigma$, there exist $a \in R$ and $\omega \in \Omega$ with $\sigma a = \omega$.

Prove that if Ω is cofinal in Σ, then $R_\Omega = R_\Sigma$.

6.1.3 Show that if a ring R is right Artinian, so also is R_Σ, where Σ is a right Ore set of non-zerodivisors in R.

6.1.4 Let $\mathcal{K}$ be a field and let $R = \mathcal{K}\langle X, Y \rangle$ be the free associative $\mathcal{K}$-algebra (noncommutative poynomial ring) in two variables X and Y. Verify that R is a domain.

Let R^* be the set of nonzero elements of R. Show that, given any nonconstant polynomial f, there is a polynomial g with

$$fR^* \cap gR^* = \emptyset;$$

in fact, g can always be taken to be X or Y.

Deduce that any Ore set in R is a subset of $\mathcal{K}$ and hence that R is its own maximal ring of fractions.

6.1.5 Let Σ be a right Ore set of non-zerodivisors in a ring R, and let $\mathfrak{a}$ be a right ideal of R.

Show that any element of $\mathfrak{a}_\Sigma = \mathfrak{a}R_\Sigma$ can be written in the form a/σ with a in $\mathfrak{a}$ and σ in Σ. Deduce that $\mathfrak{a}_\Sigma = R_\Sigma$ if and only if $\mathfrak{a} \cap \Sigma \neq \emptyset$.

6.1.6 Let $\mathcal{O}$ be a Dedekind domain and let $\mathbf{Q}$ be a subset of the set of nonzero prime ideals of $\mathcal{O}$. Let $\Sigma = \bigcap\{\mathcal{O} \setminus \mathfrak{q} \mid \mathfrak{q} \in \mathbf{Q}\}$.

Show that $\mathcal{O}_\Sigma = \bigcap_{\mathbf{Q}} \mathcal{O}_\mathfrak{q}$.

6.1.7 Let $\mathcal{O}$ be a Dedekind domain with an infinite set $\mathbf{P}$ of nonzero prime ideals $\mathfrak{p}$. Show that $\bigcap_{\mathbf{P}} \mathfrak{p} = 0$ (see (2.3.20)), and deduce that the field of fractions $\mathcal{K}$ of $\mathcal{O}$ is the coproduct of the inclusions $\mathcal{O} \to \mathcal{O}_\mathfrak{p}$ as $\mathfrak{p}$ ranges over $\mathbf{P}$.

6.1.8 **Localization with zerodivisors**

Suppose that a multiplicative set Σ in a ring R may contain zero-divisors. Then a ring of fractions R_Σ can be constructed if Σ satisfies the conditions in (6.1.1), together with the following requirement.

If $\sigma r = 0$ with $\sigma \in \Sigma$, $r \in R$, then $r\tau = 0$ for some $\tau \in \Sigma$.

Let $T = \{r \in R \mid r\sigma = 0 \text{ for some } \sigma \in \Sigma.\}$ Using (6.1.1) and (6.1.2), show that T is a twosided ideal in R.

Put $\overline{R} = R/T$ and let $\overline{\Sigma}$ be the image of Σ in $\overline{R}$. Verify that $\overline{\Sigma}$ is a right Ore set of non-zerodivisors in $\overline{R}$.

Show also that, if $f : R \longrightarrow S$ is Σ-inverting (see sentence preceding Theorem 6.1.11), then $T \subseteq \operatorname{Ker}(f)$.

Thus we define $R_\Sigma = \overline{R}_{\overline{\Sigma}}$. For the flatness of the localization, and the extension of the definition to modules, see Exercise 6.2.13.

6.2 LOCALIZATION FOR MODULES

We now look at the effect of the localization functor $- \otimes_R R_\Sigma$ on modules and on categories of modules. We find that a module has a torsion submodule, which is annihilated by localization, and a torsion-free quotient module, which embeds in its localization. We see that localization is an exact functor, and that any module over the localization R_Σ arises from an R-module through localization.

This section also contains a treatment of Ore domains; such a domain is one that does have a division ring of right fractions. In contrast to the commutative case, a noncommutative domain need not be an Ore domain.

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6.2.1 Localization and torsion

Let Σ be a right Ore set in the ring R and let M be a right R-module. The *localization* M_Σ of M at Σ is defined to be the tensor product $M \otimes_R R_\Sigma$, which is a right R_Σ-module (and so also a right R-module).

There is a natural R-module homomorphism τ from M to M_Σ, given by $\tau m = m \otimes 1$. The *Σ-torsion submodule* $T_\Sigma(M)$ of M is defined to be the kernel of τ, and M is said to be a *Σ-torsion* module if $M = T_\Sigma(M)$. If $T_\Sigma(M) = 0$, then M is said to be *Σ-torsion-free.*

Some special cases have their own traditional notation.

When R is a commutative ring and Σ is the complement $R \setminus \mathfrak{p}$ of a prime ideal $\mathfrak{p}$, we write $M_\mathfrak{p}$ for the localization M_Σ of an R-module, and refer to it as the *localization at* $\mathfrak{p}$.

In the case that $\mathcal{O}$ is a commutative domain, the zero ideal 0 is itself a prime ideal of $\mathcal{O}$, $\mathcal{O} \setminus 0$ is an Ore set, and the localization $\mathcal{O}_0$ is the field of fractions $\mathcal{K}$ of $\mathcal{O}$. The 0-torsion submodule of a module M is dignified with the title of *the torsion submodule* $T(M)$ of M, and we recover the definitions made in (2.3.20): if $M = T(M)$, then M is torsion, while if $T(M) = 0$, M is torsion-free.

When M is torsion-free, the localization M_0 is a vector space $\mathcal{K}M$ over the field of fractions $\mathcal{K}$.

This notation and terminology extends in the obvious way to modules over an $\mathcal{O}$-order R.

We return to the general situation to describe the elementary properties of torsion modules.

6.2.2 Proposition

Let Σ be a right Ore set in a ring R and let M be a right R-module. Then the following hold.

(i) $T_\Sigma(M) = \{m \in M \mid m\sigma = 0 \text{ for some } \sigma \in \Sigma\}$;

(ii) $T_\Sigma(M)$ *is Σ-torsion;*

(iii) $M/T_\Sigma(M)$ *is Σ-torsion-free.*

Proof

Assertion (ii) is immediate from (i); likewise, (iii) follows quickly from (i) because Σ is multiplicative. To establish (i), we use the realization of R_Σ as the union $\bigcup R/\sigma$, where R/σ is the set of all fractions a/σ with a in R (6.1.16). Regarding this union as the direct limit $\operatorname{dir\,lim}_\Sigma R/\sigma$, we have $M_\Sigma = \operatorname{dir\,lim}_\Sigma(M \otimes_R R/\sigma)$ by part (i) of (5.1.19).

Therefore, $m \otimes 1 = 0$ in M_Σ precisely when $m \otimes 1 = 0$ in some $M \otimes_R R/\sigma$, by (5.1.8). But $R/\sigma \cong R$ as a left R-module (using right multiplication by σ), so this means that $m \otimes \sigma = 0$ in $M \otimes_R R$. Finally, the canonical isomorphism $M \otimes_R R \cong M$ (3.1.4) shows that this is the same thing as $m\sigma = 0$. □

6.2.3 Corollary

Let R be a commutative ring and let M be an R-module.

(a) *The canonical R-homomorphism from M to $\prod_{\mathfrak{m}} M_{\mathfrak{m}}$ ($\mathfrak{m}$ ranging over all the maximal ideals of R) is injective.*

(b) *The following assertions are equivalent.*

   (i) $M = 0$.

   (ii) $M_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$ *of* R.

   (iii) $M_{\mathfrak{m}} = 0$ *for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof

In part (b), the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious, while the implication (iii) $\Rightarrow$ (i) follows from (a). To prove (a), take any element m in the kernel of the canonical map from M to $\prod_{\mathfrak{m}} M_{\mathfrak{m}}$, and let $\mathfrak{a}$ be the annihilator ideal of m in R, that is, $r \in \mathfrak{a}$ if and only if $mr = 0$. By (i) of the preceding proposition, $\mathfrak{a}$ contains an element of the complement $R \setminus \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R, and so $\mathfrak{a}$ is not contained in any maximal ideal of R. Hence, by ([BK: IRM] (1.2.22)), $\mathfrak{a} = R$ and so $m = 0$. □

6.2.4 Corollary

Let R be a commutative ring and let $\alpha : M \to N$ be a right R-module homomorphism, with $\alpha_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ the induced $R_{\mathfrak{p}}$-module homomorphism for a prime ideal $\mathfrak{p}$ of R. Then the following hold.

(a) *For each prime ideal* $\mathfrak{p}$ *of* R,

   (i) $\operatorname{Ker} \alpha_{\mathfrak{p}} = (\operatorname{Ker} \alpha)_{\mathfrak{p}}$;

   (ii) $\operatorname{Im} \alpha_{\mathfrak{p}} = (\operatorname{Im} \alpha)_{\mathfrak{p}}$;

   (iii) $\operatorname{Cok} \alpha_{\mathfrak{p}} = (\operatorname{Cok} \alpha)_{\mathfrak{p}}$.

(b) *The following are equivalent statements.*

   (i) α *is injective;*

   (ii) $\alpha_{\mathfrak{p}}$ *is injective for every prime ideal* $\mathfrak{p}$ *of* R;

   (iii) $\alpha_{\mathfrak{m}}$ *is injective for every maximal ideal* $\mathfrak{m}$ *of* R.

(c) *As* (b), *with 'injective' replaced throughout by 'surjective', or by 'bijective', or by 'zero'.*

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(d) *Let*

$$L \xrightarrow{\beta} M \xrightarrow{\alpha} N$$

be a sequence of R-modules. Then the following are equivalent statements.

(i)

$$L \xrightarrow{\beta} M \xrightarrow{\alpha} N$$

is exact at M;

(ii)

$$L_{\mathfrak{p}} \xrightarrow{\beta_{\mathfrak{p}}} M_{\mathfrak{p}} \xrightarrow{\alpha_{\mathfrak{p}}} N_{\mathfrak{p}}$$

is exact at $M_{\mathfrak{p}}$ *for every prime ideal* $\mathfrak{p}$.

(iii)

$$L_{\mathfrak{m}} \xrightarrow{\beta_{\mathfrak{m}}} M_{\mathfrak{m}} \xrightarrow{\alpha_{\mathfrak{m}}} N_{\mathfrak{m}}$$

is exact at $M_{\mathfrak{m}}$ *for every maximal ideal* $\mathfrak{m}$.

Proof

(a) Applying $-\otimes_R R_{\mathfrak{p}}$ to the short exact sequences

$$0 \longrightarrow \operatorname{Ker}\alpha \longrightarrow M \longrightarrow \operatorname{Im}\alpha \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im}\alpha \longrightarrow M \longrightarrow \operatorname{Cok}\alpha \longrightarrow 0$$

gives the result, since $R_{\mathfrak{p}}$ is R-flat by (6.1.8).

(b) Combine (b) of the previous corollary with (a)(i) above.

(c) Combine (b) of the previous corollary with (a)(ii) for 'zero', with (a)(iii) for 'surjective', and with both (a)(i) and (a)(iii) for 'bijective'.

(d) By (c), $\alpha\beta = 0$ if and only if $(\alpha\beta)_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$, which shows that $\operatorname{Im}\beta \subseteq \operatorname{Ker}\alpha$ if and only if $\operatorname{Im}\beta_{\mathfrak{p}} \subseteq \operatorname{Ker}\alpha_{\mathfrak{p}}$ for all $\mathfrak{p}$. Regard these inclusions as homomorphisms, say ι and $\iota_{\mathfrak{p}}$, as appropriate. Then the exactness of the sequence

$$L \xrightarrow{\beta} M \xrightarrow{\alpha} N$$

is the same as the surjectivity of ι, and similarly the localized sequences are exact when each $\iota_{\mathfrak{p}}$ is surjective. Appealing to (c) again gives the equivalence of (i) and (ii), and the equivalence of (i) and (iii) is proved in the same way. □

6.2.5 Some categories

Given a ring R and a right Ore set Σ in R, we introduce the following categories of right modules.

$\mathcal{T}\textsc{or}_{\Sigma,R}$ – the category of all Σ-torsion right R-modules;

$\mathcal{T}_{\Sigma,R}$ – the category of all finitely generated Σ-torsion right R-modules;

$\mathcal{TF}_{\Sigma,R}$ – the category of all finitely generated Σ-torsion-free right R-modules.

In each case, the category is to be taken to be the full subcategory of $\mathcal{M}\textsc{od}_R$ with the specified objects. Again, when Σ arises from a prime ideal $\mathfrak{p}$ of a commutative domain $\mathcal{O}$, we replace Σ by $\mathfrak{p}$ in this notation, to obtain $\mathcal{T}\textsc{or}_{\mathfrak{p},\mathcal{O}}$, $\mathcal{T}_{\mathfrak{p},\mathcal{O}}$ and $\mathcal{TF}_{\mathfrak{p},\mathcal{O}}$; when $\mathfrak{p}$ is the zero ideal, we write simply $\mathcal{T}\textsc{or}_{\mathcal{O}}$, $\mathcal{T}_{\mathcal{O}}$ and $\mathcal{TF}_{\mathcal{O}}$.

This notation extends to an $\mathcal{O}$-order R in the expected way, to give $\mathcal{T}\textsc{or}_{\mathfrak{p},R}$, $\mathcal{T}_{\mathfrak{p},R}$ $\mathcal{TF}_{\mathfrak{p},R}$, $\mathcal{T}\textsc{or}_{\mathcal{O},R}$ $\mathcal{T}_{\mathcal{O},R}$ and $\mathcal{TF}_{\mathcal{O},R}$

Similar notation may be used when R is a commutative ring and $\mathfrak{p}$ is a prime ideal of R.

We next investigate how torsion behaves with respect to short exact sequences.

6.2.6 Proposition

Let Σ be a right Ore set in a ring R and let

$$0 \longrightarrow M' \xrightarrow{\ \alpha\ } M \xrightarrow{\ \beta\ } M'' \longrightarrow 0$$

be a short exact sequence of right R-modules.

Then the following assertions hold.

(i) *M is in $\mathcal{T}\textsc{or}_{\Sigma,R}$ if and only if both M' and M'' are in $\mathcal{T}\textsc{or}_{\Sigma,R}$.*

(ii) *If M is in $\mathcal{T}_{\Sigma,R}$, then M'' is in $\mathcal{T}_{\Sigma,R}$.*

(iii) *If both M' and M'' are in $\mathcal{T}_{\Sigma,R}$, then M is in $\mathcal{T}_{\Sigma,R}$.*

(iv) *If both M' and M'' are in $\mathcal{TF}_{\Sigma,R}$, then M is in $\mathcal{TF}_{\Sigma,R}$.*

Proof

Routine checking, using criterion (i) of (6.2.2), gives (i), and (ii) follows since a homomorphic image of a finitely generated module is again finitely generated. For (iii), note that M must be finitely generated (as in [BK: IRM] (3.1.2)).

We give an argument for (iv). Suppose that $m \in T_\Sigma(M)$. Then $m\sigma = 0$ for some σ in Σ. Clearly, $\beta m \in T_\Sigma(M'')$, so $\beta m = 0$ and $m = \alpha m'$ for m' in M'. Then $m'\sigma = 0$, so $m' = 0$. □

In the language of (2.4.13), the categories $\mathcal{T}\textsc{or}_{\Sigma,R}$, $\mathcal{T}_{\Sigma,R}$ and $\mathcal{TF}_{\Sigma,R}$ are all closed under extensions. Also $\mathcal{T}\textsc{or}_{\Sigma,R}$ is abelian by Exercise 2.3.9, while

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the category $\mathcal{T}_{\Sigma,R}$ contains the cokernel of any of its homomorphisms – see (1.4.15).

In general, neither $\mathcal{T}_{\Sigma,R}$ nor $\mathcal{TF}_{\Sigma,R}$ can be expected to contain the kernels of all their homomorphisms unless R is right Noetherian. For example, take $\mathcal{O} = \mathcal{K}[X_i \mid i \geq 1]$, the polynomial ring over a field $\mathcal{K}$ in a countably infinite set of variables, and let $\mathfrak{m}$ be the (maximal) ideal in $\mathcal{O}$ generated by the variables. Then neither the torsion-free module $\mathfrak{m}$ nor the $\mathfrak{m}$-torsion submodule $\mathfrak{m}/\mathfrak{m}^2$ of $\mathcal{O}/\mathfrak{m}^2$ is finitely generated.

The category $\mathcal{TF}_{\Sigma,R}$ rarely contains cokernels: trivial counterexamples are provided by quotients $\mathcal{O}/a\mathcal{O}$ where a is a nonzero element of a domain $\mathcal{O}$.

Nevertheless, all these categories share a pleasant property.

6.2.7 Theorem

Let Σ be a right Ore set in a ring R. Then the categories $\mathcal{T}or_{\Sigma,R}$, $\mathcal{T}_{\Sigma,R}$ and $\mathcal{TF}_{\Sigma,R}$ are each full sub-Q-exact categories in $\mathcal{M}od_R$, and hence each is a Q-exact category.

Proof

The final assertion follows immediately from (2.4.15), once we have verified that each category satisfies the axioms SubQ1 and SubQ2 for a sub-Q-exact category, given in (2.4.13).

The first of these is simply the statement that each category is closed under extensions, which we have already noted. The second axiom requires that the specified class of exact sequences in each category comprises all the short exact sequences in $\mathcal{M}od_R$ whose terms belong to the given category. This is obvious. □

6.2.8 Modules over the ring of fractions

The relationship between R-modules and R_Σ-modules, for varying choices of the set Σ, often provides important information about the ring R; an illustration is given in Exercise 6.2.5 below. As a preliminary step towards analysing this relationship, we show that any R_Σ-module N can be obtained as a localization of some R-module. The argument has several steps. We often use the fact that an R_Σ-module is an R-module by restriction of scalars.

First, we show that an R_Σ-module N is always its own localization.

6.2.9 Lemma

Suppose that N is a right R_Σ-module. Then there is a canonical isomorphism $N \otimes_R R_\Sigma \cong N$ of right R_Σ-modules.

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Proof

Let $\eta : N \otimes_R R_\Sigma \to N$ be the surjection induced by multiplication, so that $\eta(n \otimes a/\sigma) = na/\sigma$. To see that η is an injection, regard R_Σ as the direct limit dir lim R/σ. Then $N \otimes_R R_\Sigma \cong \text{dir lim}(N \otimes_R R/\sigma)$ by (5.1.19). For each σ, the homomorphism from $N \otimes_R R/\sigma$ to N induced by η is clearly an isomorphism, and so η is the limit of a family of injections. Thus η is injective by (5.1.8). □

6.2.10 Corollary

Suppose that M is an R-submodule of an R_Σ-module N. Then there is a canonical isomorphism between M_Σ and the R_Σ-submodule MR_Σ of N generated by M.

Proof

Since R_Σ is flat as a left R-module, $M \otimes_R R_\Sigma$ can be regarded as a submodule of $N \otimes_R R_\Sigma \cong N$. □

A second consequence is as follows.

6.2.11 Corollary

Let M be a Σ-torsion-free right R-module. Regard R^k as an R-submodule of $(R_\Sigma)^k$ by the natural inclusion. Then we can regard M as a submodule of M_Σ, and M^k as a submodule of $(M_\Sigma)^k$ for $k \geq 1$. With these identifications,

$$(M^k)_\Sigma = (M_\Sigma)^k. \qquad \square$$

Using the identifications provided by the preceding results, we have a fundamental result.

6.2.12 Theorem

Let Σ be a right Ore set in a ring R and let N be a right R_Σ-module. Then the following hold.

(i) *There is an R-submodule M of N with $M_\Sigma = N$.*
(ii) *If N is finitely generated over R_Σ, then M can be taken to be finitely generated over R.*
(iii) *If N is a finitely generated R_Σ-submodule of the free R_Σ-module $(R_\Sigma)^k$, then M can be taken to be a finitely generated R-submodule of R^k.*

Proof

(i), (ii) Take any set $\{n_\lambda \mid \lambda \in \Lambda\}$ of generators for N and put $M = \sum n_\lambda R$.

(iii) Choose a finite set $n_1, \dots, n_h$ of generators of N. Then each generator has the form

$$n_i = (r_{i1}/\sigma_{i1}, \dots, r_{ik}/\sigma_{ik}) \text{ in } (R_\Sigma)^k,$$

where $r_{ij} \in R$ and $\sigma_{ij} \in \Sigma$ for all i, j. Since the set $\{\sigma_{ij}\}$ has a common right multiple σ in Σ (6.1.6), we can write $n_i = x_i/\sigma$ with $x_i \in R^k$ for $i = 1, \dots, h$. Take $M = x_1 R + \dots + x_h R$. □

This result allows us to see that the Noetherian property is preserved by localization.

6.2.13 Theorem

Let R be a right Noetherian ring and let Σ be a right Ore set in R. Then the ring of fractions R_Σ is also right Noetherian.

Proof

Recall that a ring is right Noetherian if and only if all its right ideals are finitely generated. By the preceding theorem, any right ideal of R_Σ has the form $\mathfrak{a}_\Sigma = \mathfrak{a} R_\Sigma$ for some right ideal $\mathfrak{a}$ of R. Since $\mathfrak{a}$ is finitely generated over R, $\mathfrak{a}_\Sigma$ is finitely generated over R_Σ. □

6.2.14 Corollary

If the ring A is right Noetherian and α is an automorphism of A, then the skew Laurent polynomial ring $A[T, T^{-1}, \alpha]$ is also right Noetherian.

Proof

The Hilbert Basis Theorem ([BK: IRM] (3.2.3)) tells us that the skew polynomial ring $A[T, \alpha]$ is right Noetherian, so the assertion follows from the preceding theorem. □

6.2.15 Corollary

For any natural number n, the iterated Laurent polynomial ring

$$A[T_1, T_1^{-1}, \dots, T_n, T_n^{-1}]$$

is right Noetherian if A is right Notherian. □

Remark. Contrary to expectations that might be raised by the previous results, a localization of a ring R need not inherit invariant basis number from R – see Exercise 6.2.12 below.

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6.2.16 Ore domains

In contrast to the commutative case, the set of nonzero elements of a noncommutative domain need not form an Ore set; an example is provided by the free associative algebra in two variables (Exercise 6.1.4). If the set of nonzero elements in a domain R does form a right Ore set, then R is called a *right Ore domain.* The ring of right fractions of an Ore domain with respect to its nonzero elements is evidently a division ring.

Our examples of Ore domains come through the following result.

6.2.17 Proposition

Suppose that the domain R is right Noetherian. Then R is a right Ore domain.

Proof

Let Σ be the set of nonzero elements of R. We wish to show that $a\Sigma \cap \sigma R$ is nonempty for any a in R and σ in Σ. If $a = 0$, this is obvious. If $a \neq 0$, we have to show that $aR \cap \sigma R \neq 0$. But, if $aR \cap \sigma R = 0$, we can find (using an easy contradiction argument) an infinite direct sum of right ideals $\bigoplus_i a^i \sigma R$ in R, contrary to the fact that R is right Noetherian (see [BK: IRM] (3.1.13)). □

6.2.18 Corollary

Let $\mathcal{D}$ be a division ring and let α be a ring endomorphism of $\mathcal{D}$. Then $\mathcal{D}[T, \alpha]$ is a right Ore domain.

Proof

By ([BK: IRM] (3.2.10)), $\mathcal{D}[T, \alpha]$ is a principal right ideal domain and hence right Noetherian. □

The ring of right fractions of $\mathcal{D}[T, \alpha]$ is a division ring, which is written $\mathcal{D}(T, \alpha)$ and called a *ring of skew rational functions* over $\mathcal{D}$.

In particular, take α to be the identity map on $\mathcal{D}$. We have shown that the ordinary polynomial ring $\mathcal{D}[T]$ is a (right) Ore domain. The localization of $\mathcal{D}[T]$ with respect to its nonzero elements is called the *ring of rational functions* $\mathcal{D}(T)$.

6.2.19 Corollary

Let $\mathcal{D}$ be a division ring. Then $\mathcal{D}(T)$ is also a division ring. □

6.2.20 Left–right symmetry

It is evident that all our definitions and results on right localization with respect to right Ore sets have left-handed versions.

Thus, a *left Ore set* is a multiplicatively closed subset Σ of non-zerodivisors in the ring R, with the property that

$$\Sigma a \cap R\sigma \neq \emptyset \text{ for any } \sigma \in \Sigma \text{ and } a \in R.$$

Given such a set Σ, there is a ring of left fractions ${}_{\Sigma}R$ which is flat as a right R-module.

Examples of left Ore sets arise from the left-handed analogues $A_\ell[T,\alpha]$ of the skew polynomial rings $A[T,\alpha]$. The definition (see (6.1.3)) is modified in the obvious way: left polynomials are used in place of right polynomials, and the rule for multiplication is $T \cdot a = (\alpha a) \cdot T$. (A formal method of passing from right to left polynomials using the opposite ring is indicated in [BK: IRM] Exercise 3.2.6.)

By symmetry, the set $\langle T \rangle$ is left Ore in $A_\ell[T,\alpha]$. Furthermore, if $\mathcal{D}$ is a division ring, then $\mathcal{D}_\ell[T,\alpha]$ is a left Ore domain.

6.2.21 An asymmetric example

For an example of a right Ore domain which is not left Ore, we use the ring given in [McConnell & Robson 1987], §1.2.11.

Let $\mathcal{K}$ be a field and let $A = \mathcal{K}(X)$ be the ring of rational functions over $\mathcal{K}$ in a variable X (thus $\mathcal{K}(X)$ is the usual ring of fractions of the unskewed polynomial ring $\mathcal{K}[X]$). The substitution $X \mapsto X^2$ defines an injective endomorphism α of A which is not surjective. Take $R = \mathcal{K}(X)[T,\alpha]$. By (6.2.17), R is a right Ore domain and the localization R_Σ of R at its set of nonzero elements Σ is a division ring.

On the other hand, it is not hard to verify that $RT \cap RTX = 0$, (see [BK: IRM] (3.2.12)), from which we see that Σ is not left Ore.

It is also the case that R_Σ is not flat as a right R-module, although it is flat as a left R-module by (6.1.8). It is this point which explains why we must distinguish between left fractions and right fractions, and why the ring R_Σ of right fractions cannot in general be viewed as a ring of left fractions, despite the fact that the elements of Σ do have twosided inverses in R_Σ.

To show that R_Σ is not right flat, we note that, since $RT \cap RTX = 0$, a straightforward argument as in the proof of (6.2.17) shows that R contains an infinite direct sum of left ideals, namely

$$RTXT + RTXT^2 + RTXT^3 + \cdots = \bigoplus_i RTXT^i.$$

Thus for any exponent $h > 1$, there is an injection of the left R-module $RTXT \oplus \cdots \oplus RTXT^h$ into R. If R_Σ were right flat, tensoring on the left with R_Σ would give an injection of R_Σ^h into R_Σ, which cannot happen, since R_Σ is a division ring.

6.2.22 More symmetry

There are of course many circumstances in which a right Ore set is also a left Ore set. For instance, this is trivially true if the right Ore set Σ is central.

For a slightly deeper example, suppose that the defining ring endomorphism α of the skew polynomial ring $A[T, \alpha]$ is an isomorphism. In this case, we have the identity $A_\ell[T, \alpha^{-1}] = A[T, \alpha]$, since the multiplication rule $a \cdot T = T \cdot \alpha(a)$ can be re-written $\alpha^{-1}(a) \cdot T = T \cdot a$. Thus $\langle T \rangle$ is left and right Ore, as is the set of all nonzero elements of the polynomial ring if A is a division ring.

When the set Σ is both left and right Ore, then the left and right rings of fractions can be identified with one another, because, by (6.1.11), they share the same universal property.

6.2.23 Proposition

Suppose that Σ is both a left and a right Ore set in the ring R. Then there is a ring isomorphism from ${}_\Sigma R$ to R_Σ which is the identity on R. □

6.2.24 Corollary

If Σ is both left and right Ore, R_Σ is both left and right flat as an R-module. □

Exercises

6.2.1 **Tor and torsion**

Let Σ be a right Ore set in a ring R. By considering the short exact sequence

$$0 \longrightarrow R \longrightarrow R_\Sigma \longrightarrow R_\Sigma/R \longrightarrow 0$$

of left R-modules, show that $\mathrm{Tor}_1^R(M, R_\Sigma/R) \cong T_\Sigma(M)$.

Hint. (3.2.6).

6.2.2 Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in the commutative domain $\mathcal{O}$ such that $\mathfrak{p} \cap \mathfrak{q}$ contains no nonzero prime ideals. Show that $(\mathcal{O}_\mathfrak{p})_\mathfrak{q} = \mathcal{K}$, the field of fractions of $\mathcal{O}$. (This result holds in particular when $\mathfrak{p}$ and $\mathfrak{q}$ are distinct nonzero prime ideals of a Dedekind domain $\mathcal{O}$.)

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6.2.3 Let $\mathfrak{p}$ be a prime ideal of a commutative ring R and let $\mathfrak{a}$ be an ideal of R not contained in $\mathfrak{p}$. Show that $R/\mathfrak{a} \otimes_R R_\mathfrak{p} = 0$ and that $\mathfrak{a}_\mathfrak{p} = R_\mathfrak{p}$.

6.2.4 Let Σ be a right Ore set in a ring R, let I be an index set and, for $i \in I$, let M_i be a right R-module. Show that

$$(\bigcap_I M_i)_\Sigma \subseteq \bigcap_I (M_i)_\Sigma.$$

For a counterexample to equality, take $R = \mathbb{Z}$, take Σ to be the odd numbers, and $M_i = 3^i\mathbb{Z}$, $i = 1, 2, \ldots$

6.2.5 Let $\mathcal{O}$ be a Dedekind domain and let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$. Using (2.3.20), calculate $M_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \otimes_\mathcal{O} M$ for an arbitrary finitely generated $\mathcal{O}$-module M. (The result should be compatible with the structure theorem for modules over the local principal ideal domain $\mathcal{O}_\mathfrak{p}$, which is trivially a Dedekind domain – see (2.3.20).)

Let M and N be finitely generated $\mathcal{O}$-modules. Show that the following statements are equivalent.

(i) $M_\mathfrak{p} \cong N_\mathfrak{p}$ for all nonzero prime ideals $\mathfrak{p}$.
(ii) $T(M) \cong T(N)$, and $M/T(M)$ and $N/T(N)$ have the same rank.

Prove further:

(iii) if $\mathcal{O}$ is a principal ideal domain and $M_\mathfrak{p} \cong N_\mathfrak{p}$ for all nonzero prime ideals $\mathfrak{p}$, then $M \cong N$;
(iv) if $\mathcal{O}$ is not a principal ideal domain, (iii) may be false.

Why does this not contradict (6.2.4)?

Remark. Let R be an $\mathcal{O}$-order. Two (finitely generated) R-modules M, N are said to be in the same *genus* if they are locally isomorphic, that is, $M_\mathfrak{p} \cong N_\mathfrak{p}$ for all nonzero prime ideals $\mathfrak{p}$ of $\mathcal{O}$. The determination of the possible genera of R-modules is an important problem – see (7.3.32)(b) and [Curtis & Reiner 1981], §31.

6.2.6 Let $\mathcal{O}$ be a Dedekind domain and let a be a nonzero element. Write $\mathcal{O}_a = \mathcal{O}_{\langle a \rangle}$, where $\langle a \rangle = \{1, a, a^2, \ldots\}$.

Describe the ring $\mathcal{O}_a$ as in Exercise 6.1.6, and use the technique of Exercise 6.2.5 to find the finitely generated $\mathcal{O}_a$-modules.

Note. $\mathcal{O}_a = \mathcal{O}_\mathbf{Q}$, where $\mathbf{Q}$ is the set of distinct prime ideals that do *not* occur in the factorization of $a\mathcal{O}$.

6.2.7 Show that a commutative polynomial ring in infinitely many variables is an example of a (right) Ore domain which is not (right) Noetherian. So the converse to (6.2.17) fails.

6.2.8 Let Σ be a right Ore set in a ring R. Show that the canonical inclusion $\iota : R \to R_\Sigma$ is an epimorphism in the category $\mathcal{R}_{ING}$.

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6.2.9 Let α be an endomorphism of a ring A, and extend α to endomorphisms (of the same name) of the skew polynomial ring $A[T,\alpha]$ and the skew Laurent polynomial ring $A[T,T^{-1},\alpha]$. Let $\alpha_\#$ be any of the extension-of-scalars functors associated with these homomorphisms (3.3.4), (3.3.20).

Show the following.

(i) There is a natural transformation $\alpha_\# \to \mathrm{Id}_{A[T,\alpha]}$.
(ii) If α is an automorphism, this transformation is a natural isomorphism.
(iii) There is a natural isomorphism $\alpha_\# \to \mathrm{Id}_{A[T,T^{-1},\alpha]}$.

6.2.10 Let $\mathcal{D}$ be a division ring and let α be an endomorphism of $\mathcal{D}$ which is either proper or has infinite inner order, that is, α is either not surjective, or no power of α is an inner automorphism ([BK: IRM] (3.2.13)). Using the fact that the twosided ideals of $\mathcal{D}[T,\alpha]$ have the form $T^i\mathcal{D}[T,\alpha]$ ([BK: IRM] (3.2.18)), show that the skew Laurent polynomial ring $\mathcal{D}[T,T^{-1},\alpha]$ is a simple domain.

Is it the same as the 'division ring of fractions' $\mathcal{D}[T,\alpha]_\Sigma$, where Σ is the set of nonzero elements of $\mathcal{D}[T,\alpha]$?

6.2.11 Let $\mathcal{D}$ be a division ring and let α be an automorphism of $\mathcal{D}$. Describe all finitely generated right modules over the skew Laurent polynomial ring $\mathcal{D}[T,T^{-1},\alpha]$.

Hint. As well as (6.2.11), you may need the structure theorem for $\mathcal{D}[T][\alpha]$-modules ([BK: IRM] (3.3.6)) together with facts about total division ([BK: IRM] (3.2.20), (3.2.21)).

6.2.12 **Invariant basis number lost** ([Cohn 1985], Exercise 0.8.17)

Let $\mathcal{K}[T]$ be the commutative polynomial ring over a field $\mathcal{K}$ and let $S = \mathcal{K}[T]\langle A_{hi}, B_{jk}\rangle$ be the free associative $\mathcal{K}[T]$-algebra in $2mn$ variables

$$A_{hi},\ h = 1,\dots,m,\ i = 1,\dots,n \ \text{ and } \ B_{jk},\ j = 1,\dots,n,\ k = 1,\dots,m.$$

Let $A = (A_{hi})$ and $B = (B_{jk})$ be the matrices with these variables as entries, and let $\mathfrak{a}$ be the twosided ideal of S generated by the entries of the matrices

$$AB - T\cdot I_m \ \text{ and } \ BA - T\cdot I_n.$$

Put $R = S/\mathfrak{a}$.

(a) Show that there is a ring homomorphism from R to $\mathcal{K}$ and hence that R has invariant basis number.

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(b) Show that the localization $R_{\langle T\rangle}$ does not have invariant basis number if $m \neq n$.

6.2.13 **Localization with zerodivisors**

Suppose that we allow zerodivisors in Σ, as in Exercise 6.1.8. Define the localization M_Σ of a right R-module M to be $M \otimes_R R_\Sigma$, with natural R-module homomorphism $\tau : M \longrightarrow M_\Sigma$. As in (6.2.1), define the Σ-*torsion submodule* $T_\Sigma(M)$ of M to be the kernel of τ, and say that M is a Σ-*torsion* module if $M = T_\Sigma(M)$, and that M is Σ-*torsion-free* if $T_\Sigma(M) = 0$.

Verify that assertions (i), (ii) and (iii) of (6.2.2) remain valid in this more general setting.

Deduce that, if $M \longrightarrow N$ is an injective homomorphism of R-modules, then the induced homomorphism $M_\Sigma \longrightarrow N_\Sigma$ is also injective.

Conclude that R_Σ is a flat left R-module.

(Detailed accounts of this more general construction, together with its applications to module theory, can be found in [Cohn 1979] §12.1, 12.2, [Rowen 1988], Chapter 3, and [McConnell & Robson 1987], Chapter 2.)

6.3 CATEGORICAL LOCALIZATION

Finally in this chapter, we outline a far-reaching generalization of localization. In this approach, we specify a suitable subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$, and construct a quotient category $\mathcal{A}/\mathcal{C}$ in which the objects of $\mathcal{C}$ become isomorphic to the zero object 0. Thus, any morphism in $\mathcal{A}$ which has both kernel and cokernel in $\mathcal{C}$ becomes an isomorphism in $\mathcal{A}/\mathcal{C}$. For a right Ore set Σ in a ring R, we recover the previous notion of localization by taking the subcategory $\mathcal{T}\textsc{or}_{\Sigma,R}$ of $\mathcal{M}\textsc{od}_R$ – see Exercise 6.3.2 below.

The quotient category construction has been used recently by [Quillen] in a new treatment of module theory for nonunital rings. We indicate some of his results in a series of exercises below (6.3.4 to 6.3.7).

The arguments require many detailed verifications, which are left to the reader. The burden can be eased by assuming that we work with subcategories of $\mathcal{M}\textsc{od}_R$ for some ring R. In view of the Embedding Theorems (2.3.22) and our intended applications, this hypothesis causes no real loss. Alternative expositions of the construction can be found in [Gabriel 1962], [Swan 1968], [Faith 1973] or [Popescu 1973].

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For technical reasons, we assume that our categories are small, or at least, have a small skeleton (1.3.15).

6.3.1 Serre subcategories

Suppose then that $\mathcal{A}$ is a (small) abelian category. A *Serre subcategory* (also known as a thick or dense subcategory) of $\mathcal{A}$ is a full additive subcategory $\mathcal{C}$ with the following property.

SSub: Given a short exact sequence in $\mathcal{A}$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

M is in $\mathcal{C}$ if and only if both M' and M'' are in $\mathcal{C}$.

Notice, in particular, that $\mathcal{C}$ is closed under isomorphisms, that is, if $A \in \mathcal{A}$ and $A \cong C$ with $C \in \mathcal{C}$, then $A \in \mathcal{C}$ already.

6.3.2 Examples

When Σ is a right Ore set of non-zerodivisors in a ring R, then $\mathcal{T}\mathit{or}_{\Sigma,R}$ is a Serre subcategory of $\mathcal{M}\mathit{od}_R$, by part (i) of (6.2.6). If R is right Noetherian, then the category $\mathcal{M}_R$ of finitely generated right R-modules is an abelian category by (2.3.6), and a Serre subcategory of $\mathcal{M}\mathit{od}_R$ by the familiar result that, in an exact sequence as above, M is finitely generated if and only if both M' and M'' are finitely generated (see [BK: IRM] (3.1.2)). Further, the category $\mathcal{T}_{\Sigma,R}$ of finitely generated Σ-torsion modules is then a Serre subcategory of both $\mathcal{M}\mathit{od}_R$ and $\mathcal{M}_R$.

Since an abelian group is the same as a $\mathbb{Z}$-module, we obtain in particular the original examples [Serre 1953] of Serre subcategories of the category of abelian groups: all finitely generated abelian groups; all torsion abelian groups; all finite abelian groups; and all abelian p-groups for a prime p. These examples extend in the expected way when $\mathbb{Z}$ is replaced by a Dedekind domain $\mathcal{O}$ and p by a prime ideal $\mathfrak{p}$ of $\mathcal{O}$.

6.3.3 $\mathcal{C}$-isomorphisms

Let $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. We say that a morphism in $\mathcal{A}$ is a $\mathcal{C}$-*isomorphism* if both its kernel and cokernel are objects of $\mathcal{C}$. For any pair of morphisms $\alpha : L \to M$ and $\beta : M \to N$ in $\mathcal{A}$ there is an induced exact sequence

$$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta\alpha \to \operatorname{Ker}\beta \to \operatorname{Cok}\alpha \to \operatorname{Cok}\beta\alpha \to \operatorname{Cok}\beta \to 0$$

(by routine checking in a module category, or by the universal properties of kernels and cokernels in the abstract), from which we see that if any two of α, β, $\beta\alpha$ are $\mathcal{C}$-isomorphisms, then so is the third.

Many arguments with $\mathcal{C}$-isomorphisms involve the construction of a pull-back square

$$\begin{array}{ccc} M \times_{M''} L'' & \xrightarrow{\overline{\beta}} & L'' \\ \downarrow{\scriptstyle \overline{\theta}} & & \downarrow{\scriptstyle \theta} \\ M & \xrightarrow{\beta} & M'' \end{array}$$

or a push-out square

$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \overline{\phi}} \\ M' & \xrightarrow{\overline{\mu}} & M' \oplus_{L'} L \end{array}$$

in which the original morphisms β, θ or ϕ, μ are known to be $\mathcal{C}$-isomorphisms. (The definitions are reviewed in Exercises 1.4.10 ff.) Since there are induced isomorphisms

$$\operatorname{Ker}\beta \cong \operatorname{Ker}\overline{\beta} \quad \text{and} \quad \operatorname{Cok}\phi \cong \operatorname{Cok}\overline{\phi},$$

together with an injection $\operatorname{Cok}\overline{\theta} \rightarrowtail \operatorname{Cok}\theta$ and a surjection $\operatorname{Ker}\mu \twoheadrightarrow \operatorname{Ker}\overline{\mu}$, we see that the morphisms $\overline{\beta}$, $\overline{\theta}$, $\overline{\phi}$ and $\overline{\mu}$ will again be $\mathcal{C}$-isomorphisms, as also will be the diagonal morphisms $\beta\overline{\theta}$ and $\overline{\mu}\phi$.

6.3.4 Denominator sets

We commence the construction of the quotient category $\mathcal{A}/\mathcal{C}$, in which the $\mathcal{C}$-isomorphisms are to become invertible. This procedure was introduced in [Grothendieck 1957], 'systematizing and making more supple the 'language modulo $\mathcal{C}$' of [Serre 1953]' (our translation). It was then treated at greater length in [Gabriel 1962]. Our technique is based on [Heller 1965], but modified so as to emphasize the link with the definition of the ring of fractions in (6.1.7) above. In that construction, where elements of the Ore set Σ are inverted, the first step is to introduce a directed set structure on Σ.

Here, we begin by attaching a directed set $\mathrm{D}(M, N)$ to each pair M, N of objects of $\mathcal{A}$. The motive underlying the choice of the members of $\mathrm{D}(M, N)$

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is that they are pairs of $\mathcal{C}$-isomorphisms which become denominators of homomorphisms from M to N in the quotient category. Such subsets are our present counterparts of Σ; more precisely, in the case of the Ore set, it suffices to consider only the pair R, R and to fix the second $\mathcal{C}$-isomorphism as the identity (see Exercise 6.3.2 for more details of the relation between the two constructions).

Given a pair M, N of objects of $\mathcal{A}$, we define a directed denominator set (5.1.1) $\mathrm{D}(M, N)$ as follows. The members of $\mathrm{D}(M, N)$ are to be all the pairs of $\mathcal{C}$-isomorphisms

$$\sigma : M' \longrightarrow M, \quad \tau : N \longrightarrow N'$$

in $\mathcal{A}$. Such a pair is written $\tau\backslash/\sigma$, and can be conveniently visualized as a diagram

$$\tau\backslash/\sigma : \qquad M \xleftarrow{\ \sigma\ } M' \qquad\qquad N' \xleftarrow{\ \tau\ } N.$$

At this point we must invoke the hypothesis that $\mathcal{A}$ is small, or that it has been replaced by some equivalent small category, to ensure that $\mathrm{D}(M, N)$ is a set.

The partial ordering $\leq$ on $\mathrm{D}(M, N)$ is defined by writing

$$\tau_0\backslash/\sigma_0 \leq \tau_1\backslash/\sigma_1$$

if in $\mathcal{A}$ there are commuting triangles

$$\begin{array}{ccccccc} M & \xleftarrow{\ \sigma_0\ } & M'_0 & \qquad & N'_0 & \xleftarrow{\ \tau_0\ } & N \\ & {}^{\sigma_1}\nwarrow & \uparrow\sigma & & \tau\downarrow & \swarrow{}^{\tau_1} & \\ & & M'_1 & & N'_1 & & \end{array}$$

Notice that both σ and τ are $\mathcal{C}$-isomorphisms, since σ_0, σ_1, τ_0 and τ_1 are all $\mathcal{C}$-isomorphisms.

The definition can be reformulated as stipulating that

$$\tau_0\backslash/\sigma_0 \leq \tau\tau_0\backslash/\sigma_0\sigma$$

for any pair of $\mathcal{C}$-isomorphisms

$$\sigma : M'_1 \longrightarrow M'_0, \quad \tau : N'_0 \longrightarrow N'_1.$$

It is clear that this partial ordering is transitive, and we can exhibit an upper bound of two members $\tau_1\backslash\alpha_1/\sigma_1$ and $\tau_2\backslash\alpha_2/\sigma_2$ of $\mathrm{D}(M, N)$ as the middle 'row' of the diagram

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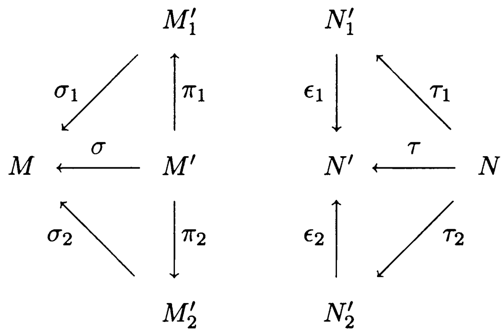

in which $M' = M_1' \times_M M_2'$ is the pull-back of M_1' and M_2' over M and $N' = N_1' \oplus_N N_2'$ is the push-out of N_1' and N_2' over N, and the morphisms are those arising in the pull-back and push-out squares. (Essentially, we have just shown how the existence of pull-backs and push-outs in a category $\mathcal{A}$ affords a directed set structure on the product of left- and right-fibre categories $\mathrm{Id}_{\mathcal{A}}/M \times N\backslash \mathrm{Id}_{\mathcal{A}}$.)

6.3.5 Some direct systems of morphisms

Next, we associate with $\mathrm{D}(M,N)$ a direct system $\mathrm{M}(M,N)$ of abelian groups by attaching to a pair $\tau\backslash/\sigma$ as above the abelian group $\mathrm{Mor}_{\mathcal{A}}(M',N')$. We write $\tau\backslash\alpha/\sigma$ to indicate that a morphism $\alpha : M' \to N'$ belongs to the copy of $\mathrm{Mor}_{\mathcal{A}}(M',N')$ which is labelled by the pair $\tau\backslash/\sigma$, and we visualise this through the diagram

$$\tau\backslash\alpha/\sigma : \qquad M \xleftarrow{\ \sigma\ } M' \xrightarrow{\ \alpha\ } N' \xleftarrow{\ \tau\ } N.$$

Notice that when σ and τ are already invertible in $\mathcal{A}$, this diagram does represent a morphism $\tau^{-1}\alpha\sigma^{-1}$ from M to N.

Suppose that $\mathrm{Mor}_{\mathcal{A}}(M_0',N_0')$ is labelled by $\tau_0\backslash/\sigma_0$, that $\mathrm{Mor}_{\mathcal{A}}(M_1',N_1')$ is labelled by $\tau_1\backslash/\sigma_1$ and that $\tau_0\backslash/\sigma_0 \leq \tau_1\backslash/\sigma_1 = \tau\tau_0\backslash/\sigma_0\sigma$. Then we define a homomorphism

$$\phi_{10} : \mathrm{Mor}_{\mathcal{A}}(M_0',N_0') \longrightarrow \mathrm{Mor}_{\mathcal{A}}(M_1',N_1')$$

by

$$\phi_{10}(\tau_0\backslash\alpha_0/\sigma_0) = \tau\tau_0\backslash\tau\alpha_0\sigma/\sigma_0\sigma = \tau_1\backslash\alpha_1/\sigma_1,$$

where $\alpha_1 = \tau\alpha_0\sigma$. That is, ϕ_{10} is defined so that in commutative diagrams of the form

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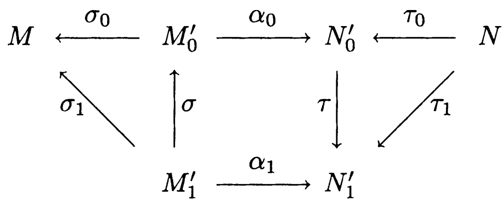


the bottom is the image of the top. It is easy to verify that $\mathrm{M}(M, N)$ is a direct system of abelian groups (5.1.3).

6.3.6 The quotient category

We can now define the *quotient category* $\mathcal{A}/\mathcal{C}$. The objects of $\mathcal{A}/\mathcal{C}$ are the same as those of $\mathcal{A}$, while the morphisms of $\mathcal{A}/\mathcal{C}$ are given by

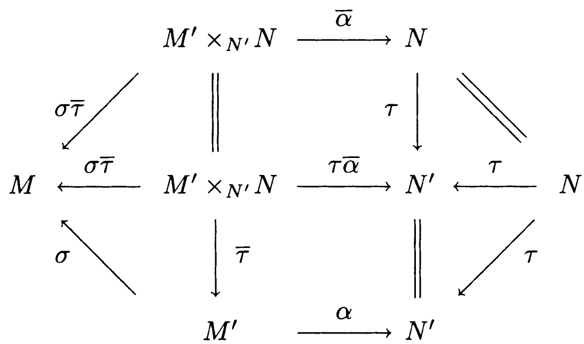
$$\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M, N) = \operatorname{dir}\lim M(M, N).$$

By construction, $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M, N)$ is an abelian group (5.1.4), whose elements will be written $[\tau\backslash\alpha/\sigma]$. As a preliminary to the definition of the product of morphisms in $\mathcal{A}/\mathcal{C}$, we need to find some canonical forms for the morphisms.

Given $(\tau\backslash\alpha/\sigma)$, we use the pull-back

$$\begin{array}{ccc} M' \times_{N'} N & \xrightarrow{\overline{\alpha}} & N \\ \downarrow{\scriptstyle \overline{\tau}} & & \downarrow{\scriptstyle \tau} \\ M' & \xrightarrow{\alpha} & N' \end{array}$$

to obtain the commutative diagram



which (read upside-down) shows that

$$[\tau\backslash\alpha/\sigma] = [\tau\backslash\tau\overline{\alpha}/\sigma\overline{\tau}] = [id_N\backslash\overline{\alpha}/\sigma\overline{\tau}],$$

that is, every member of $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M,N)$ can be represented by an element of the form $[id_N\backslash\beta/\pi]$. We also note that this representation is additive, in that if we have

$$\alpha_1 + \alpha_2 : M' \to N',$$

then

$$\overline{\alpha_1 + \alpha_2} = \overline{\alpha}_1 + \overline{\alpha}_2.$$

Similarly, using the push-out, we can show that $[\tau\backslash\alpha/\sigma] = [\rho\backslash\gamma/id_M]$ for some ρ, γ, the representation again being additive.

We can now define the product of morphisms. Given

$$\lambda \in \mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(L,M) \ \text{ and } \ \mu \in \mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M,N),$$

choose representatives

$$\lambda = [id_M\backslash\beta/\pi] \ \text{ and } \ \mu = [\rho\backslash\gamma/id_M]$$

and put

$$\mu\lambda = [\rho\backslash\gamma\beta/\pi].$$

We leave to the reader the verifications of the facts that the product is well defined and biadditive with respect to the abelian group structures in $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(L,M)$ and $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M,N)$, so that $\mathcal{A}/\mathcal{C}$ is an additive category as in (2.2.1).

6.3.7 The quotient functor

The *quotient functor* $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is defined by sending M in $\mathcal{A}$ to its namesake in $\mathcal{A}/\mathcal{C}$ and $\alpha : M \to N$ in $\mathcal{A}$ to $[id_N\backslash\alpha/id_M]$ in $\mathcal{A}/\mathcal{C}$. It is clear that T is an additive functor.

Now suppose that $\alpha : M \to N$ is a $\mathcal{C}$-isomorphism. Then $[id_M\backslash id_M/\alpha]$ is an element of $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(N,M)$, and

$$[id_N\backslash\alpha/id_M]\cdot[id_M\backslash id_M/\alpha] = [id_N\backslash\alpha/\alpha],$$

which is easily seen to be the identity $[id_N\backslash id_N/id_N]$ in $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(N,N)$. Likewise, $[id_N\backslash\alpha/id_M]$ has left inverse $[\alpha\backslash id_N/id_N]$, so we find that

$$[id_M\backslash id_M/\alpha] = (T(\alpha))^{-1} = [\alpha\backslash id_N/id_N],$$

which shows that every $\mathcal{C}$-isomorphism in $\mathcal{A}$ has acquired an inverse in $\mathcal{A}/\mathcal{C}$. Hence we have, in $\mathcal{A}/\mathcal{C}$,

$$[\tau\backslash\alpha/\sigma] = (T\tau)^{-1}(T\alpha)(T\sigma)^{-1}.$$

It is straightforward to verify that any zero object 0 in $\mathcal{A}$ becomes a zero object $T0$ in $\mathcal{A}/\mathcal{C}$. Further, if C is an object in $\mathcal{C}$, then the zero morphism $0 : 0 \to \mathcal{C}$ is a $\mathcal{C}$-isomorphism, so that TC is isomorphic to 0 in $\mathcal{A}/\mathcal{C}$. One can also verify that if $A \in \mathcal{A}$ has $TA \cong 0$ in $\mathcal{A}/\mathcal{C}$, then A is in $\mathcal{C}$ already, and that if $T\alpha$ is an isomorphism in $\mathcal{A}/\mathcal{C}$, then α is a $\mathcal{C}$-isomorphism in $\mathcal{A}$.

Putting all this together, we find that a morphism $(T\tau)^{-1}(T\alpha)(T\sigma)^{-1}$ has a kernel, which can be taken to be $T(\operatorname{Ker}\alpha)$, and a cokernel, $T(\operatorname{Cok}\alpha)$. More checking confirms that $\mathcal{A}/\mathcal{C}$ inherits the structure of an abelian category (2.3.1) from $\mathcal{A}$.

Notice that a sequence in $\mathcal{A}/\mathcal{C}$ is a short exact sequence if and only if it is isomorphic to the image under T of a short exact sequence of $\mathcal{A}$. Thus the functor T is exact (2.4.6), where we view $\mathcal{A}$ and $\mathcal{A}/\mathcal{C}$ as repletely exact categories.

6.3.8 The universal property

Finally, we record the universal property of the quotient functor. Suppose that $\mathcal{B}$ is an abelian category, which we regard as a repletely exact category, and that $G : \mathcal{A} \to \mathcal{B}$ is any exact functor which maps the objects of $\mathcal{C}$ to zero. Then there is a unique exact functor H which makes the following triangle commute:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;G\;} & \mathcal{B} \\ {\scriptstyle T}\downarrow & \nearrow{\scriptstyle H} & \\ \mathcal{A}/\mathcal{C} & & \end{array}$$

Conversely, whenever there is a commuting triangle of additive functors as above, G is exact if and only if H is exact.

6.3.9 Some comments on localization in general

1. In section 6.1, we constructed a ring of fractions with respect to a suitable subset Σ of a ring R. Another approach is to consider a set of matrices over R, of varying sizes, which ought to become invertible over some ring

of fractions, rather than a set of elements of R itself. This approach is discussed in [Cohn 1985], §7.

2. As we remarked in (6.1.17), a noncommutative ring has a maximal ring of fractions with respect to an Ore set of non-zerodivisors, but this ring may be no bigger than the original ring. The analysis of the circumstances in which it is possible to construct a useful (that is, Artinian semisimple) ring of fractions for a noncommutative ring is rather delicate. The main result is Goldie's Theorem, which is discussed in [McConnell & Robson 1987], Chapter 2.

3. A more general method of localization for rings is prompted by the quotient category construction. The point of view here is that the main interest of localization theory lies in the distinction between torsion and torsion-free modules. Thus, starting with a Serre subcategory $\mathcal{T}$ of $\mathcal{M}od_R$, the 'torsion' modules, the aim is to construct a localization ring S such that $\mathcal{M}od_S \simeq \mathcal{M}od_R/\mathcal{T}$. Then $\mathcal{T}$ comprises precisely those modules X with $X \otimes_R S = 0$. If this method is to work, one must have $S \cong T(R)$, T being the quotient functor. More details of the relationship between this type of localization and our previous notion of localization with respect to an Ore set are sketched in Exercise 6.3.2 below. General discussions are given by [Golan 1975], [Popescu 1973], and [Stenström 1975], among others.

4. Another variant reflects the fact that in recent years there has been much interest in defining localizations on non-abelian (for example, topological) categories. The starting point is the premise (of [Adams 1975]), that localization is an idempotent monad functor L. This means that L is a functor from a category $\mathcal{C}$ to itself and there exists a natural transformation η from $Id_{\mathcal{C}}$ to L, with L naturally isomorphic to L^2 by $L\eta = \eta_L$, as in the following commutative diagram (for each object X of $\mathcal{C}$).

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & LX \\ {\scriptstyle \eta_X}\downarrow & /\!\!/ & \downarrow{\scriptstyle \eta_{LX}} \\ LX & \xrightarrow[L\eta_X]{} & L^2X \end{array}$$

The same point of view is helpful with general localization theories for modules. Exercise 6.3.1 below shows how an idempotent monad functor can be defined by localization of rings as considered here. Further developments can be found in [Casacuberta 1994] and [Berrick 1999].

Exercises

6.3.1 Show that the composite functor $L : \mathcal{M}od_R \to \mathcal{M}od_R$, $L(M) = \iota^{\#}(M \otimes_R R_\Sigma)$, is idempotent. (Here, $\iota : R \to R_\Sigma$ is the canonical inclusion and $\iota^{\#}$ is the restriction functor.)

Show also that L is an exact functor. (Some authors, but not topologists, require that this be true of generalized localizations.)

6.3.2 **Localization of rings performed categorically**

Let Σ be a right Ore set in a ring R.

(i) Show that $\mathcal{T}or_{\Sigma,R}$ is a Serre subcategory of $\mathcal{M}od_R$.

(ii) If R is not Noetherian, show that $\mathcal{T}_{\Sigma,R}$ need not be a Serre subcategory of $\mathcal{M}od_R$ even though it is a sub-Q-exact category.

(iii) Let $\mathcal{A}/\mathcal{C}$ be the quotient category $\mathcal{M}od_R/\mathcal{T}or_{\Sigma,R}$, and let $T : \mathcal{M}od_R \to \mathcal{A}/\mathcal{C}$ be the quotient functor. Verify that there is a ring isomorphism $\mathrm{End}_{\mathcal{A}/\mathcal{C}}(T(R)) \cong R_\Sigma$.

(iv) Show that there is an exact functor $H : \mathcal{M}od_R/\mathcal{T}or_{\Sigma,R} \to \mathcal{M}od_{R_\Sigma}$ such that there is a commuting triangle

$$\begin{array}{ccc} \mathcal{M}od_R & \xrightarrow{-\otimes_R R_\Sigma} & \mathcal{M}od_{R_\Sigma} \\ {\scriptstyle T}\downarrow & \nearrow{\scriptstyle H} & \\ \mathcal{M}od_R/\mathcal{T}or_{\Sigma,R} & & \end{array}$$

(v) Show that the morphism $M \to M \otimes_R R_\Sigma$ in $\mathcal{M}od_R$ is a $\mathcal{T}or_{\Sigma,R}$-isomorphism.

(vi) Conclude that H is an equivalence of categories.

6.3.3 **The morphisms of a quotient category**

Let $\mathcal{A}$ be a (small) abelian category, let $\mathcal{C}$ be a Serre subcategory, and take objects M, N of $\mathcal{A}$. Our use of the directed set $\mathrm{D}(M,N)$ (6.3.4) in the construction of the morphisms $\mathrm{Mor}_{\mathcal{A}/\mathcal{C}}(M,N)$ of the quotient category is extravagant, in that we can use smaller directed sets ([Grothendieck 1957]).

Given $\sigma : M \to M'$, factor it as $M \xrightarrow{\epsilon} \mathrm{Im}\,\sigma \xrightarrow{\mu} M'$ with ϵ an

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epimorphism and μ a monomorphism. Show that the push-out

$$\begin{array}{ccc} M' & \xrightarrow{\alpha} & N \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon'} \\ \operatorname{Im}\sigma & \xrightarrow{\alpha'} & Q \end{array}$$

leads to the equalities

$$[id_N \backslash \alpha/\sigma] = [\epsilon' \backslash \epsilon'\alpha/\sigma] = [\epsilon' \backslash \alpha'\epsilon/\mu\epsilon] = [\epsilon' \backslash \alpha'/\mu]$$

in $\operatorname{Mor}_{\mathcal{A}/\mathcal{C}}(M, N)$.

Combining this observation with those made in (6.3.6), deduce that $\operatorname{Mor}_{\mathcal{A}/\mathcal{C}}(M, N)$ can be constructed as the direct limit associated with any of the following directed subsets of $\mathrm{D}(M, N)$.

(a) $(\epsilon \backslash / \mu)$ with $\mu : M' \to M$ a monomorphism having cokernel in $\mathcal{C}$ and $\epsilon : N \to N'$ an epimorphism with kernel in $\mathcal{C}$.
(b) $(\epsilon \backslash / id_M)$ with $\epsilon : N \to N'$ an epimorphism and $\operatorname{Ker}\epsilon \in \mathcal{C}$.
(c) $(id_N \backslash / \mu)$ with $\mu : M' \to M$ a monomorphism and $\operatorname{Cok}\mu \in \mathcal{C}$.

When $\mathcal{A}$ is a module category, we can further contract these directed sets by requiring that M' is a submodule of M and that N' is a quotient N/L' for a submodule L' of N.

6.3.4 Nilmodules

Let $\mathfrak{a}$ be a (twosided) ideal of a ring R.

(a) A right R-module is called a *nilmodule* for the pair $(R, \mathfrak{a})$ if there is a natural number n such that M is $\mathfrak{a}^n$-torsion (that is, $M\mathfrak{a}^n = 0$). Show that the nilmodules for $(R, \mathfrak{a})$ form the objects of a Serre subcategory, denoted $\mathcal{N}il_{R,\mathfrak{a}}$, of $\mathcal{M}od_R$. Write the quotient category as $\mathcal{M}od_{R,\mathfrak{a}}$.
(b) Check the following extreme cases.
    (i) If $\mathfrak{a} = R$, then $\mathcal{N}il_{R,\mathfrak{a}} = \{0\}$, making $\mathcal{M}od_{R,\mathfrak{a}}$ isomorphic to $\mathcal{M}od_R$.
    (ii) If $\mathfrak{a}$ is actually a unital ring and R is its unitalization $\mathfrak{a} \times \mathbb{Z}$ (see (1.3.2)(iv)), then $\mathcal{M}od_R$ is equivalent to $\mathcal{M}od_{\mathfrak{a}} \times \mathcal{M}od_{\mathbb{Z}}$ and $\mathcal{N}il_{R,\mathfrak{a}}$ corresponds to $\mathcal{M}od_{\mathbb{Z}}$. This leaves $\mathcal{M}od_{R,\mathfrak{a}}$ equivalent to $\mathcal{M}od_{\mathfrak{a}}$.
    (iii) If $\mathfrak{a}$ is nilpotent, then $\mathcal{M}od_{R,\mathfrak{a}}$ is the zero abelian category.
    (iv) More generally, suppose that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ have both $\mathfrak{a}^n \subseteq \mathfrak{b}$ and $\mathfrak{b}^n \subseteq \mathfrak{a}$ for some natural number n. Then $\mathcal{M}od_{R,\mathfrak{a}}$ is isomorphic to $\mathcal{M}od_{R,\mathfrak{b}}$.

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(c) Generalizing (b)(ii) above, we consider the situation where $\mathfrak{a}$ is itself a unital ring (under the multiplication of R), but R is arbitrary. Then it is easy to see ([BK: IRM], Exercise 2.6.13) that $\mathfrak{a} = Re$ for some nonzero central idempotent e of R, and the ring decomposition $R = \mathfrak{a} \times R(1-e)$ gives a canonical splitting of each R-module M as $Me \oplus M(1-e)$, and so an equivalence of categories

$$\mathcal{M}\mathit{od}_R \simeq \mathcal{M}\mathit{od}_{\mathfrak{a}} \times \mathcal{N}\mathit{il}_{R,\mathfrak{a}},$$

whence $\mathcal{M}\mathit{od}_{R,\mathfrak{a}} \simeq \mathcal{M}\mathit{od}_{\mathfrak{a}}$.

6.3.5 Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of repletely exact abelian categories. Show that the full subcategory of $\mathcal{A}$ with objects M having FM zero in $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.

(From (6.3.8), every Serre subcategory arises in this way.)

6.3.6 **Relative Morita equivalence** ([Quillen])

Let R and S be rings, and let $\mathfrak{a}$ be a (twosided) ideal of R and $\mathfrak{b}$ an ideal of S. Suppose that we have a Morita context

$$\begin{pmatrix} R & W \\ V & S \end{pmatrix}, \quad \sigma : W \otimes_S V \to R, \ \tau : V \otimes_R W \to S.$$

Write $\sigma(W\mathfrak{b} \otimes_S V) = W\mathfrak{b}V$, an ideal of R, and define the ideal $V\mathfrak{a}W$ of S similarly. Suppose that for some natural number n, both $\mathfrak{a}^n \subseteq W\mathfrak{b}V$ and $\mathfrak{b}^n \subseteq V\mathfrak{a}W$.

(a) Suppose that $\alpha : L \to M$ is an R-module homomorphism and that $\operatorname{Ker}\alpha$ is $\mathfrak{a}^k$-torsion. Show that $\operatorname{Ker}(\alpha \otimes id_W)$ is $V\mathfrak{a}^kW$-torsion and thence $\mathfrak{b}^{kn}$-torsion. Do likewise for $\operatorname{Cok}\alpha$.

Deduce that the functor

$$\mathcal{M}\mathit{od}_R \longrightarrow \mathcal{M}\mathit{od}_S : \qquad M \longmapsto M \otimes_R W$$

sends $\mathcal{N}\mathit{il}_{R,\mathfrak{a}}$-isomorphisms to $\mathcal{N}\mathit{il}_{S,\mathfrak{b}}$-isomorphisms (in the notation of Exercise 6.3.4), and hence induces a functor from the quotient category $\mathcal{M}\mathit{od}_{R,\mathfrak{a}}$ to $\mathcal{M}\mathit{od}_{S,\mathfrak{b}}$.

(b) Deduce from Exercise 4.2.3(b) that the induced functor is an equivalence of categories, and that it is naturally isomorphic to the functor from $\mathcal{M}\mathit{od}_{R,\mathfrak{a}}$ to $\mathcal{M}\mathit{od}_{S,\mathfrak{b}}$ induced by the functor $M \mapsto \operatorname{Hom}(V_R, M)$ on $\mathcal{M}\mathit{od}_{R,\mathfrak{a}}$.

6.3.7 (a) Let R be a ring with (twosided) ideal $\mathfrak{a}$. Since $\mathfrak{a}$ is a nonunital ring, we may embed it as an ideal of its unitalization $\overline{\mathfrak{a}}$ (as in

(1.3.2)(iv)). Show that there is then a Morita context of the form

$$\begin{pmatrix} R & \mathfrak{a} \\ R & \bar{\mathfrak{a}} \end{pmatrix},$$

which by the previous exercise gives rise to an equivalence from $\mathcal{M}\text{OD}_{R,\mathfrak{a}}$ to $\mathcal{M}\text{OD}_{\bar{\mathfrak{a}},\mathfrak{a}}$. It follows that the category $\mathcal{M}\text{OD}_{R,\mathfrak{a}}$ can be thought of as independent of the choice of ring R in which a given nonunital ring $\mathfrak{a}$ is embedded. When taken in conjunction with Exercise 6.3.4, this makes a case for $\mathcal{M}\text{OD}_{R,\mathfrak{a}}$ to be regarded as 'the' generalization to nonunital rings $\mathfrak{a}$ of the category $\mathcal{M}\text{OD}_R$. (See (4.2.4).)

(b) Show that an idempotent e in a ring R gives rise to a Morita context

$$\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix},$$

and thence to an equivalence from $\mathcal{M}\text{OD}_{R,ReR}$ to $\mathcal{M}\text{OD}_{eRe}$.

(c) In (b), replace R by its *cone*, CR, which is the ring of all infinite matrices over R that have only a finite number of nonzero entries in each row and column, and take for the idempotent the matrix

$$\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Obtain an equivalence from $\mathcal{M}\text{OD}_{CR,mR}$ to $\mathcal{M}\text{OD}_R$, where mR denotes the ideal of CR comprising those matrices with only finitely many nonzero entries.