SOME ALGEBRAIC PROPERTIES OF $F(X)$ AND $K(X)$

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Introduction

Throughout we consider operators on a reflexive Banach space X. We consider certain algebraic properties of $F(X)$, $K(X)$ and $B(X)$ with the general aim of examining their dependence on the possession by X of the approximation property. $B(X)$ (resp. $K(X)$) denotes the algebra of all bounded (resp. compact) operators on X and $F(X)$ denotes the closure in $B(X)$ of its finite rank operators. The two questions we consider are:

(1) Is $K(X)$ equal to the set of all operators in $B(X)$ whose right and left multiplication operators on $F(X)$ (or on $B(X)$) are weakly compact?

(2) Is $F(X)$ a dual algebra?

The answer to both questions is in the affirmative if X has the approximation property. In Sections 2 and 3 we discuss the general cases and show for example that every element of $K(X)$ does act weakly compactly on $F(X)$. However, completely satisfactory answers are only obtained when X is taken to be a closed subspace of l_p $(1<p<\infty)$. That this is not quite as restrictive as might at first appear is illustrated by (1) and (4). For such X it is shown in Section 2 that the answer to (1) is always " yes " and in Section 4 that the answer to (2) is " yes " if and only if X has the approximation property.

Other algebras of operators on X which we shall consider are $N(X)$ the nuclear operators on X (with the nuclear norm τ) and $N'(X) = (N(X))'$ (with the operator norm). We denote by γ the greatest cross-norm on $X \otimes X'$.

If two Banach spaces E and F are isomorphic then

$$d(E, F) = \inf\{\| T \| \, . \, \| T^{-1} \| \colon T \text{ an isomorphism of } E \text{ onto } F\}.$$

If S is a subset of a normed linear space then lin S denotes the linear hull of S and $\overline{\text{lin}}\, S$ its closure. If $k>0$ and E is a normed linear space then $(E)_k$ is the closed ball in E of radius k—that is $\{x \in E\colon \| x \| \leqq k\}$. The identity operator on a normed linear space E is denoted by I_E or simply by I.

We shall use a characterisation of compact sets in Banach spaces. For a specific reference to its proof we quote (3), Lemma 2.

Lemma 0. *Let K be a compact subset of a Banach space E. Then K is contained in the closed convex hull of a sequence in E converging to* 0.

1. Some general properties of l_p spaces

In this section we collect together some basic results on l_p spaces that will be used subsequently. The most important of these is due to Pelczynski (**12**):

Theorem 1.1. *Let X be an infinite dimensional subspace of $l_p(1 \leqq p<\infty)$. Then X contains an infinite dimensional subspace Y which is complemented in l_p and isomorphic to l_p.*

We shall also require a variant of this result.

Theorem 1.2. *Let X be a subspace of $l_p(1 \leqq p<\infty)$ and let $\{x_n\}$ be a sequence in X which converges weakly to 0 and satisfies $\| x_n \| = 1$ (n = 1, 2, ...). Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is a basis for a subspace Y of X which is isomorphic to l_p and complemented in l_p.*

Proof. Let $\{e_i\}$ denote the usual basis in l_p, and let $\{e_i^*\}$ denote the biorthogonal functionals. By hypothesis there are increasing sequences $\{p_n\}$ and $\{q_n\}$ of positive integers with $q_1 = 1$ and

$$4\left\| \sum_{i=1}^{q_n} e_i^*(x_{p_n})e_i \right\| \leqq \frac{1}{2^n}$$

$$4\left\| \sum_{i=q_{n+1}+1}^{\infty} e_i^*(x_{p_n})e_i \right\| \leqq \frac{1}{2^n}$$

Let $z_n = \sum_{i=q_n+1}^{q_{n+1}} e_i^*(x_{p_n})e_i$. Then $\{z_n\}$ is a block basic sequence with respect to $\{e_i\}$ and so, by Lemma 1 in (**12**); $\overline{\text{lin}}\,\{z_n\}$ is isometrically isomorphic to l_p and there is a projection P of norm 1 of l_p onto $\overline{\text{lin}}\,\{z_n\}$.

Then for $n \geqq 1$

$$\begin{aligned} 1 = \| x_{p_n} \| &= \left\| \sum_{i=1}^{\infty} e_i^*(x_{p_n})e_i \right\| \\ &\leqq \left\| \sum_{i=1}^{q_n} e_i^*(x_{p_n})e_i \right\| + \| z_n \| + \left\| \sum_{i=q_{n+1}+1}^{\infty} e_i^*(x_{p_n})e_i \right\| \\ &\leqq \tfrac{1}{4} + \| z_n \|. \end{aligned}$$

So $\| z_n \| \geqq \frac{3}{4}$. Further

$$\begin{aligned} \sum_{n=1}^{\infty} \| x_{p_n} - z_n \| &\leqq \sum_{n=1}^{\infty} \left[\left\| \sum_{i=1}^{q_n} e_i^*(x_{p_n})e_i \right\| + \left\| \sum_{i=q_{n+1}+1}^{\infty} e_i^*(x_{p_n})e_i \right\| \right] \\ &\leqq \tfrac{1}{2}. \end{aligned}$$

Choose bi-orthogonal functionals $\{z_n^*\}$ corresponding to the basis $\{z_n\}$ in $\overline{\text{lin}}\,\{z_n\}$. Then $\| z_n^* \| = \dfrac{1}{\| z_n \|} \leqq \frac{4}{3}$. So

$$\sum_{n=1}^{\infty} \| x_{p_n} - z_n \| \, \| z_n^* \| \leqq \tfrac{4}{3} \cdot \tfrac{1}{2} = \tfrac{2}{3} < 1$$

and

$$\| P \| \sum_{n=1}^{\infty} \| x_{p_n} - z_n \| \, \| z_n^* \| < 1.$$

So by Theorems 3 and 4 pp. 63-64 in (9), $\{x_{p_n}\}$ is a basic sequence which is equivalent to $\{z_n\}$ and which has $\overline{\text{lin}}\,\{x_{p_n}\}$ complemented in l_p. Since $\overline{\text{lin}}\,\{z_n\}$ is isometrically isomorphic to l_p it follows that $Y = \overline{\text{lin}}\,\{x_{p_n}\}$ is isomorphic to l_p.

Finally we require a result which is certainly well known; for completeness we include a proof.

Theorem 1.3. *Let X be a finite dimensional subspace of $l_p (1 \leqq p < \infty)$. Then for any $\varepsilon > 0$ there is a finite (say r-) dimensional subspace Y of l_p with $X \subseteq Y$ and $d(Y, l_p^{(r)}) < \dfrac{1+\varepsilon}{1-\varepsilon}$.*

Proof. Choose a basis $\{x_1, \ldots, x_n\}$ of unit vectors in X and bi-orthogonal functionals $\{x_1^*, \ldots, x_n^*\}$ in X^*. Let P be a projection of l_p onto X. Let c be a constant such that $\sum_{i=1}^{n} | \alpha_i | \leqq c \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|$ for any scalars $\alpha_1, \ldots, \alpha_n$.

Let $\{e_i\}$ be the usual basis in l_p. Choose r so that if $y_i = \sum_{j=1}^{r} e_j^*(x_i) e_j$ then

$$\| P \| \cdot \sum_{i=1}^{n} \| x_i - y_i \| \, \| x_i^* \| < \varepsilon.$$

Define T: $l_p \to l_p$ by

$$Tx = x - Px + \sum_{i=1}^{n} x_i^*(Px) \cdot y_i \quad (x \in l_p).$$

Then as in the proof of Theorem 4, p. 65 in (9), T is an isomorphism of l_p onto itself, $1 - \varepsilon \leqq \| Tx \| \leqq 1 + \varepsilon$ ($x \in l_p$, $\| x \| = 1$) and $Tx_i = y_i$ $1 \leqq i \leqq n$. Let S be the restriction of T^{-1} to $\text{lin}\,\{e_1, \ldots, e_r\}$, and let Y be the range of S. Then $X \subseteq Y$ and $d(l_p^{(r)}, Y) \leqq \| S \| \, \| S^{-1} \| \leqq \| T \| \, \| T^{-1} \| \leqq \dfrac{1+\varepsilon}{1-\varepsilon}$.

2. The bidual of $F(X)$ and its w.c.c. elements

It is well known (see for example (6), Theorem 5.3) that the dual of $F(X)$ is $N(X) = X \otimes_\tau X'$ and it is easy to verify that $N'(X)$ is the bounded weak operator closure of $F(X)$ in $B(X)$. Also well known (see Schatten (13) Theorem 3.2) is the fact that $X \otimes_\gamma X'$ has dual $B(X)$.

If $B \in B(X)$ and L_B, R_B denote respectively left and right multiplication by B on $F(X)$ then $(L_B)^{**}$ and $(R_B)^{**}$ are respectively left and right operator multiplication on $N'(X)$; in particular $N'(X)$ is an ideal of $B(X)$.

The equivalences between the following are either well known or simple to verify (see Grothendieck (7)):

(1) X has $a.p$;

(2) $\tau = \gamma$;

(3) $B(X) = N'(X)$;

(4) $I \in N'(X)$.

Theorem 2.1. (Olubummo (11).) *If $F \in F(X)$ then L_F and R_F are w.c.c.*

The following theorem is probably known. See (10) for the particular case when X is a Hilbert space.

Theorem 2.2. *If X has a.p. and $B \in B(X)$ has either L_B or R_B w.c.c. then $B \in F(X)$.*

Proof. By (5), Theorem VI 4.2, L_B is w.c.c. if and only if $(L_B)^{**}(B(X)) \subseteq F(X)$ and so, in particular, if L_B is w.c.c. then $B = B.I \in F(X)$. The proof that $B \in F(X)$ if R_B is w.c.c. is similar.

Note. Thus if X has a.p. $F(X)$ $(= K(X))$ can be characterised as those operators on X by which left and right multiplication on $F(X)$ is w.c.c. If X lacks a.p. then *a priori* either $F(X)$ or $K(X)$ might be characterised in this way. In one direction we have the following strengthening of Theorem 2.1.

Theorem 2.3. *If $K \in K(X)$ then L_K and R_K are w.c.c.*

Proof. It is sufficient to give the proof for R_K. R_K^* is left multiplication by K on $N(X)$, so by (5), Theorem VI 4.8 it is sufficient to prove that this is w.c.c. Suppose the contrary. Then by Theorem 8.1 in (8), $(R_K)^*$ is a factor of the summing operator σ: $l_1 \to l_\infty$, i.e.

$$\exists U\colon N(X) \to l_\infty,\ S\colon l_1 \to N(X) \text{ such that } U(R_K)^*S = \sigma.$$

Suppose without loss of generality that $\| S \| = 1$. Let $\{e_i\}$ be the usual basis in l_1 and let $t_i = Se_i$. Then $U(R_K)^*t_i = (0, 0, \ldots, 0, 1, 1, \ldots)$ where there are precisely $i-1$ zeros.

Since $K \in K(X)$, $K(X_1) \subseteq \overline{\text{co}}\,(k_n)$ where $\{k_n\}$ is a sequence in X that converges to 0. Let t_i have a representation $\sum_{n=1}^{\infty} x_n^{(i)} \otimes x_n^{*(i)}$ where

$$\| x_n^{(i)} \| \leqq 1,\ \sum_{n=1}^{\infty} \| x_n^{*(i)} \| \leqq 2,\ \| x_n^{*(i)} \| \leqq 1.$$

Then $Kt_i = \sum_{n=1}^{\infty} Kx_n^{(i)} \otimes x_n^{*(i)}$. Choose a sequence of positive numbers $\{\varepsilon_n\}$ so that $\sum_{n=1}^{\infty} \varepsilon_n < \frac{1}{6\|U\|}$ and then choose for each $i \in N$, $n \in N$ a sequence of non-negative numbers $\{\lambda_m^{(n,i)}\}_{m=1}^{\infty}$ with only a finite number non-zero so that

$$\| Kx_n^{(i)} - \sum_m \lambda_m^{(n,i)} k_m \| < \varepsilon_n \quad i, n \in N \text{ and } \sum_m \lambda_m^{(n,i)} = 1.$$

Let $u_{n,i} = Kx_n^{(i)} - \sum_m \lambda_m^{(n,i)} k_m$. Then

$$\begin{aligned} Kt_i &= \sum_n \left(\sum_m k_m \otimes \lambda_m^{(n,i)} x_n^{*(i)} + u_{n,i} \otimes x_n^{*(i)}\right) \\ &= \sum_m \sum_n k_m \otimes \lambda_m^{(n,i)} x_n^{*(i)} + \sum_n u_{n,i} \otimes x_n^{*(i)} \\ &= \sum_m k_m \otimes y_m^{*(i)} + Ni \end{aligned} \tag{1}$$

where $N_i = \sum_n u_{n,i} \otimes x_n^{*(i)}$, $\tau(N_i) \leqq \sum_n \| u_{n,i} \| \| x_n^{*(i)} \| \leqq \sum_n \varepsilon_n < \frac{1}{6\|U\|}$ and $y_m^{*(i)} = \sum_n \lambda_m^{(n,i)} x_n^{*(i)}$; the change of order of the summation is justified by observing that

$$\begin{aligned} \sum_n \sum_m \| k_m \otimes \lambda_m^{(n,i)} x_n^{*(i)} \| &= \sum_n \left(\| x_n^{*(i)} \| \sum_m \| k_m \| \, | \lambda_m^{(n,i)} |\right) \\ &\leqq 2 \sup_m \| k_m \| < \infty. \end{aligned}$$

Similarly

$$\begin{aligned} \sum_m \| y_m^{*(i)} \| &= \sum_m \left(\| \sum_n \lambda_m^{(n,i)} x_n^{*(i)} \|\right) \leqq \sum_m \sum_n \| \lambda_m^{(n,i)} x_n^{*(i)} \| \\ &= \sum_n \left(\sum_m | \lambda_m^{(n,i)} | \, \| x_n^{*(i)} \|\right) = \sum_n \| x_n^{*(i)} \| \leqq 2 \end{aligned} \tag{2}$$

Since $k_m \to 0$ as $m \to \infty$ it follows from (2) that there is a constant M (independent of i) such that

$$\sum_{m=M+1}^{\infty} \| k_m \| \| y_m^{*(i)} \| \leqq \frac{1}{6\|U\|}$$

and hence $\tau\left(\sum_{m=M+1}^{\infty} k_m \otimes y_m^{*(i)}\right) \leqq \frac{1}{6\|U\|}$. Thus from (1)

$$Kt_i = \sum_{m=1}^{M} k_m \otimes y_m^{*(i)} + G_i \tag{3}$$

where $\tau(G_i) \leqq \frac{1}{3\|U\|}$. It follows that $U\left(\sum_{m=1}^{M} k_m \otimes y_m^{*(i)}\right)$ has its jth component $\leqq \frac{1}{3}$ in modulus for $j < i$ and $\geqq \frac{2}{3}$ in modulus for $j \geqq i$.

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Now U is norm and hence weakly continuous. $(X^*)_1 \oplus \ldots \oplus (X^*)_1$ (M terms) is compact in the Cartesian product of the weak topologies. So

$$\mathscr{C} = \left\{ \sum_{m=1}^{M} k_m \otimes y_m^* : y_m^* \in X_1^* \right\}$$

is weakly compact in $N(X)$. Therefore $U\mathscr{C}$ is weakly compact in l_∞. But $\left\{ U\left(\sum_{m=1}^{M} k_m \otimes y_m^{*(i)} \right) \right\}_{i=1}^{\infty}$ is a sequence in $U\mathscr{C}$ that has no weakly convergent subsequence. This contradiction proves the theorem.

Note that the above proof could be modified with $X \otimes_\gamma X'$ replacing $N(X)$. Thus if $K \in K(X)$ then left and right multiplication by K on $B(X)$ is w.c.c. The theorem of Olubummo (quoted here as Theorem 2.1) gives this same result for elements of $F(X)$.

Theorem 2.4. *Let X be a subspace of l_p $(1 < p < \infty)$. Then an operator B on X is compact if and only if L_B and R_B are weakly compact.*

Proof. The only implication to be proved is that if L_B and R_B are w.c.c. then $B \in K(X)$. Suppose the contrary—L_B is w.c.c. and $B \in B(X) - K(X)$. Then there is a sequence $\{x_n\}$ in X which converges weakly to 0 but has $\| Bx_n \| \not\to 0$. Replacing by a subsequence and multiplying by a constant if necessary we may suppose that $\| x_n \| = 1$, $\| Bx_n \| \geqq c > 0$ $\forall n \in N$.

By Theorem 1.2 and again replacing $\{x_n\}$ by a subsequence if necessary we may suppose that $\{x_n\}$ is a basis for a subspace Y of X which is complemented in l_p and hence also in X. Let P: $X \to Y$ be a projection and let P_n be the projection of Y onto lin$\{x_1, \ldots, x_n\}$. Then $P_n \to I_F$ in the strong operator topology on Y and so $P_n P \to P$ in the bounded weak operator topology on X. So $P \in (F(X))''$ and it follows from (5) Theorem VI 4.2 that $BP \in F(X)$. Since $x_n \to 0$ weakly as $n \to \infty$ and $BPx_n = Bx_n \not\to 0$ in norm as $n \to \infty$ this is a contradiction. This establishes the theorem.

3. The duality of $F(X)$

It is well known and follows easily from (2) that $F(X)$ is an annihilator algebra and is dual if X has the approximation property. On the other hand Davie (3) has shown that if X lacks the approximation property then there is a reflexive space Y such that $F(X \oplus Y)$ is not dual.

Conjecture. X has a.p. if and only if $F(X)$ is dual.

We shall show that this conjecture is true for certain particular cases. In the meantime we discuss the general case. Our starting point is the following result of Bonsall and Goldie, (2):

Theorem 3.1. *Let A be an annihilator algebra. Then A is dual if and only if $a \in \overline{Aa} \cap \overline{aA}$ for each a in A.*

Two corollaries follow immediately:

Corollary 3.2. *If $F(X)$ is dual and if for any compact subset K of X there exists an operator F in $F(X)$ with $\overline{F(X)_1} \supseteq K$ then X has a.p.*

Corollary 3.3. *$F(X)$ is dual if and only if the following cannot occur: there is a γ-Cauchy sequence $\{t_n\}$ in $X \otimes X'$ and an operator F in $F(X)$ such that either* $\operatorname{tr}(t_n FG) \to 0$ *and* $\operatorname{tr}(t_n F) \not\to 0$ $\forall G \in F(X)$ *or* $\operatorname{tr}(t_n GF) \to 0$ *and* $\operatorname{tr}(t_n F) \not\to 0$ $\forall G \in F(X)$.

Compare this with a characterisation of the approximation property (from (2)$\Leftrightarrow$(1) at beginning of Section 2). X has a.p. if and only if the following cannot occur:

there is a γ-Cauchy sequence $\{t_n\}$ in $X \otimes X'$ such that $\operatorname{tr}(t_n G) \to 0$ and $\operatorname{tr}(t_n) \not\to 0$ $\forall G \in F(X)$.

From Theorem 3.1 we see that if $F(X)$ is dual and $F \in F(X)$ then $F \in \overline{F.F(X)}$ and $F \in \overline{F(X).F}$. It is not apparent that this implies that $F \in \overline{(F(X))_k.F}$ or $F \in \overline{F.(F(X))_k}$ for any $k > 0$. If either of these holds with k independent of F, then X has a.p.

Theorem 3.4. *Suppose that there is a positive real number k such that for each F in $F(X)$ $F \in \overline{(F(X))_k.F}$ then X has a.p.*

Proof. First let F be a fixed element of $F(X)$. Choose a sequence $\{G_n\}$ in $(F(X))_k$ that satisfies $G_n F \to F$ as $n \to \infty$. Let I_F be a w^*-cluster point of $\{G_n\}$ in $N'(X)$. Then I_F restricted to the range of F is the identity. Now let A be the net of all finite sequences in $(X)_1$. For each $\alpha = \{x_1, \ldots, x_n\} \in A$, choose $F_\alpha \in F(X)$ with $F_\alpha(X)_1 \supseteq \operatorname{lin}\{x_1, \ldots, x_n\}$. Let I be a w^*-cluster point of $\{I_{F_\alpha}\}$ in $N'(X)$. Then I is the identity operator on X. So $I \in N'(X)$ and X has a.p.

4. The situation for subspaces of l_p

Let p be a fixed number $1 < p < \infty$ and let p' satisfy $1/p + 1/p' = 1$. Let X be a subspace of l_p. Then by Theorem 1.1 X has a subspace which is isomorphic to l_p, and complemented in l_p and hence also complemented in X. So X^* has a complemented subspace Y which is isomorphic to $l_{p'}$. Let P be the projection $X^* \to Y$ and let θ be the isomorphism $Y \to l_{p'}$. Since X is a subspace of l_p, X^* is a quotient of $l_{p'}$; let Q denote the quotient mapping $l_{p'} \to X^*$.

Lemma 4.1. *Let q be a real number, $1 \leqq q < \infty$. Let K be any compact subset of l_q. Then there is an element T of $F(l_q)$ that satisfies*

$$T(l_q)_1 \supseteq K.$$

Proof. We may suppose by Lemma 0 that $K = \{0\} \cup \{x_n\}_{n=1}^\infty$ where $x_n \to 0$ as $n \to \infty$. Choose an increasing sequence $\{m_n\}_{n=1}^\infty$ of positive integers such that $\|x_r\| \leqq 1/2^n$ for all $r \geqq m_n$ $(n = 1, 2, \ldots)$. Let $m_0 = 1$. Then for $n = 0, 1, 2, \ldots, \operatorname{lin}\{x_r : m_n \leqq r < m_{n+1}\}$ is a finite dimensional subspace of

l_q and so by Theorem 1.3 is contained in a finite (say r_n-) dimensional subspace E_n of l_q with $d(l_q^{(r_n)}, E_n) < 2$—i.e. there is an isomorphism $T_n : E_n \to l_q^{(r_n)}$ such that $\| T_n \| = 1$, $\| T_n^{-1} \| < 2$.

Let $\{e_i\}_{i=1}^{\infty}$ be the usual basis in l_q. Then we can identify

$$F_0 = \text{lin}\{e_1, \ldots, e_{r_0}\} \text{ with } l_q^{(r_0)}$$
$$\vdots$$
$$F_n = \text{lin}\{e_{r_0+\ldots+r_{n-1}+1}, \ldots e_{r_0+\ldots+r_n}\} \text{ with } l_q^{(r_n)}$$
$$\vdots$$

Define T on l_q by $T|_{F_n} = 2^{-n} \cdot T_n^{-1}$. Then it is clear that $T \in F(l_q)$ and $x_n \in T(l_q)_1$ for each $n \in N$.

Theorem 4.2. *If X is a subspace of l_p $(1 < p < \infty)$ then $F(X)$ is dual if and only if X has a.p.*

Proof. We use the notation of the first paragraph of Section 4. The only implication to be proved is that if $F(X)$ is dual then X has a.p. Since X is reflexive X has a.p. if and only if X^* has a.p. and $F(X)$ is dual if and only if $F(X^*)$ is dual. So it is sufficient to prove that if $F(X^*)$ is dual then X^* has a.p.

Let K be any compact set in X^*. Choose a compact set $\tilde{K}$ in $l_{p'}$ such that $Q(\tilde{K}) \supseteq K$. By Lemma 4.1 there is an element T of $F(l_p')$ such that $T(l_{p'})_1 \supseteq \tilde{K}$. Then $F = QT\theta P$ is in $F(X^*)$ and for some real number $r \geqq 1$ $(r = \| \theta^{-1} \|)$ $F(X^*)_r \supseteq K$. The theorem follows from Corollary 3.2.

REFERENCES

(1) F. E. Alexander, Compact and finite rank operators on subspaces of l_p, *Bull. London Math. Soc.* (to appear).

(2) F. F. Bonsall and A. W. Goldie, Annihilator algebras, *Proc. London Math. Soc.* **4** (1954), 154-167.

(3) A. M. Davie, A counter-example on dual Banach algebras, *Bull. London Math. Soc.* **5** (1973), 79-81.

(4) A. M. Davie, The approximation problem for Banach spaces, *Bull. London Math. Soc.* **5** (1973), 261-266.

(5) N. Dunford and J. Schwartz, *Linear operators*, Vol. I (Interscience, New York, 1958).

(6) J. Gil de Lamadrid, Measures and tensors, *Trans. Amer. Math. Soc.* **114** (1965), 98-121.

(7) A. Grothendieck, *Produits tensorielles topologiques et espaces nucléaires* (Memoirs of the American Math. Soc., No. 16 1955).

(8) J. Lindenstrauss and A. Pelczynski, Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications, *Studia Math.* **29** (1968), 275-326.

(9) J. T. Marti, *Introduction to the theory of bases* (Springer-Verlag, Berlin, 1969).

(10) T. Ogasawara, Finite dimensionality of certain Banach algebras, *J. Sci. Hiroshima Univ. Ser.A* **17** (1951), 539-363.

(11) A. Olubummo, Operators of finite rank in a reflexive Banach space, *Pacific J. Math.* **12** (1962), 1023-1027.

(12) A. Pelczynski, Projections in certain Banach spaces, *Studia Math.* **19** (1960), 209-228.

(13) R. Schatten, *A theory of cross-spaces* (Ann. Of Math. Studies 26, Princeton, 1960).

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