INVARIANT SUB-BUNDLES OF THE TANGENT BUNDLE OF A HOMOGENEOUS SPACE

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Let $M = G/H$ be the homogeneous space of a Lie group G and a closed subgroup H. Denote by $p : G \to G/H$ the canonical projection, $e \in G$ the identity and $x_0 = p(e)$. Let W be a subspace of the tangent space $T_{x_0}(M)$.

Definition. A lift W^* of W is a subspace of the Lie algebra $\mathfrak{G}$ of G satisfying $\mathfrak{H} \cap W^* = \{0\}$ and $p_* W^* = W$, where $p_* : \mathfrak{G} \to T_{x_0}(M)$ denotes the tangent map of p at e.

Consider a G-invariant sub-bundle $\mathfrak{W}$ of the tangent bundle of M **(4)**, i.e., a field $\mathfrak{W}$ of vector subspaces $\mathfrak{W}_x \subset T_x(M)$ for every $x \in M$ satisfying

$$(\mu_g)_{*_x} \mathfrak{W}_x \subset \mathfrak{W}_{\mu_g(x)} \qquad \text{for } g \in G, x \in M. \tag{1}$$

Here $\mu_g : M \to M$ denotes the diffeomorphism defined by $g \in G$ and $(\mu_g)_{*_x} : T_x \to T_{\mu_g(x)}$ the induced tangent map at x. As $(\mu_g)_{*_x}$ is an isomorphism for every x, we have, of course, equality in (1). We shall prove (see **8**) the following theorem.

THEOREM 1. *Let $\mathfrak{W}$ be a G-invariant sub-bundle of the tangent bundle of the homogeneous space G/H. Then $\mathfrak{W}$ is analytic. $\mathfrak{W}$ is involutive if and only if some lift $W^* \subset \mathfrak{G}$ of $W = \mathfrak{W}_{x_0}$ satisfies $p_*[W^*, W^*] \subset W$. If some lift of W satisfies this condition, then any lift does.*

Suppose G/H to be reductive with respect to a direct decomposition

$$\mathfrak{G} = \mathfrak{H} \oplus N, \tag{2}$$

i.e., N is a linear complement of $\mathfrak{H}$ satisfying $[\mathfrak{H}, N] \subset N$. Then there is a natural lift for $W \subset T_{x_0}(M)$, namely the well-defined subspace $W^* \subset N$ projecting onto W under p_*. Denote by X_N the projection of $X \in \mathfrak{G}$ in N with respect to the decomposition (2). Then Theorem 1 implies:

COROLLARY. *Let $\mathfrak{W}$ be as in Theorem* 1 *and suppose G/H to be reductive with respect to the decomposition* (2). *Then $\mathfrak{W}$ is involutive if and only if the natural lift $W^* \subset N$ of $\mathfrak{W}_{x_0}$ satisfies $[W^*, W^*]_N \subset W^*$. Moreover, if G/H is locally symmetric, i.e., $[N, N] \subset \mathfrak{H}$, then $\mathfrak{W}$ is necessarily involutive.*

We had proved these latter statements in **(6; 7)** by using the canonical connection on G/H. Theorem 1 shows that this is not essential.

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THEOREM 2. *Let* $\mathfrak{G}$ *be a Lie algebra,* $\mathfrak{H}$ *a subalgebra,* $p_* : \mathfrak{G} \to \mathfrak{G}/\mathfrak{H}$ *the canonical projection, and* $W \subset \mathfrak{G}/\mathfrak{H}$ *a subspace. Then the following conditions are equivalent:*

(i) W *is invariant under the natural action of* $\mathfrak{H}$ *on* $\mathfrak{G}/\mathfrak{H}$, *and* $p_*[W^*, W^*] \subset W$ *for some (and hence any) lift* W^* *of* W;

(ii) $p_*^{-1}W \subset \mathfrak{G}$ *is a subalgebra.*

Combining Theorems 1 and 2 we obtain

THEOREM 3. *Let* G/H *be a homogeneous space. Assign to every* G*-invariant sub-bundle* $\mathfrak{W}$ *of the tangent bundle of* G/H *the subspace*

$$\alpha(\mathfrak{W}) = p_*^{-1}\mathfrak{W}_{x_0} \subset \mathfrak{G}.$$

α *is an injection into the set of subspaces of* $\mathfrak{G}$ *containing* $\mathfrak{H}$. $\mathfrak{W}$ *is involutive if and only if* $\alpha(\mathfrak{W})$ *is a subalgebra. If* H *is connected, then* α *is a bijection of the set of involutive* $\mathfrak{W}$*'s onto the set of subalgebras of* $\mathfrak{G}$ *containing* $\mathfrak{H}$.

Remark. If $H = \{e\}$, α reduces to the correspondence between left-invariant fields of vector subspaces on G and subspaces of $\mathfrak{G}$.

COROLLARY. *Let* G/H *be a homogeneous space such that* $\mathfrak{H}$ *is a maximal subalgebra of* $\mathfrak{G}$. *Then there is no involutive* G*-invariant sub-bundle of the tangent bundle* $T(G/H)$ *of* G/H *different from* $T(G/H)$.

As an application we consider a G-invariant almost complex structure on $M = G/H$. It is defined by a linear map

$$J_{x_0} = J : T_{x_0}(M) \to T_{x_0}(M)$$

satisfying $J^2 = -1$ and commuting with the action of the linear isotropy group on $T_{x_0}(M)$; see (**2**, p. 83). The eigenspaces of $J_x : T_x(M) \to T_x(M)$ with respect to the eigenvalues i and $-i$ define a direct decomposition of the complexified tangent space $T_x^{\mathbf{C}} = V_x \oplus W_x$, where V_x and W_x are complex conjugate subspaces. The arising (complex) sub-bundles $\mathfrak{V}$ and $\mathfrak{W}$ of the complexified tangent bundle $T^{\mathbf{C}}(M)$ of M are G-invariant, i.e., invariant under the natural action of G on $T^{\mathbf{C}}(M)$. By (**2**, p. 78), the almost complex structure is complex if and only if one (and hence both) of the sub-bundles $\mathfrak{V}$ and $\mathfrak{W}$ is involutive. The preceding theorem—more precisely modified versions for sub-bundles of $T^{\mathbf{C}}(G/H)$—shows therefore

THEOREM 4. *Let* G/H *be a homogeneous space with a* G*-invariant almost complex structure. Consider the eigenspaces* V *and* W *of* $J : T_{x_0}^{\mathbf{C}} \to T_{x_0}^{\mathbf{C}}$ *with respect to the eigenvalues* i *and* $-i$. *Then the almost complex structure is complex if and only if one of the following two equivalent conditions is satisfied:*

(i) $p_*[V^*, V^*] \subset V$ *for some (and hence any) lift* V^* *of* V *in the complexified algebra* $\mathfrak{G}^{\mathbf{C}}$,

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(ii) *$p_*^{-1} V \subset \mathfrak{G}^{\mathbf{C}}$ is a subalgebra.*

These conditions for V can be equivalently replaced by the corresponding conditions for W.

This theorem should be compared with the result of Frölicher (**2**, p. 93), which is also an easy consequence of our considerations. If G/H is reductive, the condition (i) can be put in a more convenient form and one obtains the

COROLLARY. *Let G/H be a homogeneous space, reductive with respect to the decomposition $\mathfrak{G} = \mathfrak{H} \oplus N$, and admitting a G-invariant almost complex structure. Consider the eigenspaces V and W of $J : T_{x_0}^{\mathbf{C}} \to T_{x_0}^{\mathbf{C}}$ with respect to the eigenvalues i and $-i$. The almost complex structure is complex if and only if the natural lift V^* of V in $N^{\mathbf{C}}$ (the lift W^* of W) satisfies the condition $[V^*, V^*]_{N^{\mathbf{C}}} \subset V^*$ (the condition $[W^*, W^*]_{N^{\mathbf{C}}} \subset W^*$). Moreover, if G/H is locally symmetric, then any G-invariant almost complex structure on G/H is complex.*

The last statement is known for symmetric spaces (**3**, p. 302).

Proof of Theorem 1. The involutivity of $\mathfrak{W}$ means that the bracket of two sections of the sub-bundle $\mathfrak{W}$ is again a section of $\mathfrak{W}$. The evaluation of the bracket is very simple for a particular choice of the sections that we proceed to explain (**5**, p. 42).

Consider an arbitrary linear complement N of $\mathfrak{H}$ in $\mathfrak{G}$. There exists an open neighbourhood U of $O \in N$, such that $\exp : \mathfrak{G} \to G$ maps U homeomorphically onto $Q^* = \exp(U)$, and such that p maps Q^* homeomorphically onto an open neighbourhood Q of $x_0 \in M = G/H$; (**3**, Chapter II, Lemma 4.1).

Let $X \in T_{x_0}(M)$. For every $x \in Q$ there is a unique $g \in Q^*$ satisfying $\mu_g(x_0) = x$, namely the g projecting on x. Therefore

$$\tilde{X}_{\mu_g(x_0)} = (\mu_g)_* X \tag{3}$$

defines a vector field $\tilde{X}$ on Q. Here $(\mu_g)_* : T_{x_0} \to T_{\mu_g(x)}$. Consider the lift $X^* \in N$ of X. Denote by $\tilde{X}^*$ the left-invariant vector field on G defined by X^*. We claim that

$$(\tilde{X}^*(p^*f))(g) = (p^*(\tilde{X}f))(g) \tag{4}$$

for any $f \in C^\infty(M)$ and any $g \in Q^*$: the left-hand side is

$$(\tilde{X}^*(p^*f))(g) = ((L_g)_* X^*)p^*f$$

where $(L_g)_* : \mathfrak{G} \to T_g(G)$ denotes the tangent map at e of the left translation by g, whereas the right-hand side is

$$\begin{aligned}(p^*(\tilde{X}f))(g) &= (\tilde{X}f)(p(g)) = ((\mu_g)_* X)f \\ &= (p_{*_g}(L_g)_* X^*)f = ((L_g)_* X^*)p^*f\end{aligned}$$

in view of $\mu_g \circ p = p \circ L_g$ and $X = p_* X^*$.

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Now let X, $Y \in T_{x_0}(M)$ and X^*, Y^* be their lifts with respect to N. Then for $f \in C^\infty(M)$

$$Y(\tilde{X}f) = Y^*(p^*(\tilde{X}f)) = Y^*(\tilde{X}^*(p^*f))$$

because of (4) and the fact that $Y^*(p^*(\tilde{X}f))$ depends only on the restriction of $p^*(\tilde{X}f)$ to $\exp(\mathbf{R}Y^*) \subset Q^*$. This shows (**5**, p. 43) immediately that

$$[\tilde{Y}, \tilde{X}]_{x_0} = p_*[Y^*, X^*]. \tag{5}$$

Now let $\mathfrak{W}$ be a G-invariant sub-bundle of the tangent bundle of M. The subspace $\mathfrak{W}_{x_0} = W \subset T_{x_0}(M)$ is invariant under the action of the isotropy group H. Let W^* be an arbitrary lift of W and choose a linear complement N of $\mathfrak{H}$ in $\mathfrak{G}$ containing W^*. By the G-invariance of $\mathfrak{W}$ it is clear that for any $X \in W$ the vector field $\tilde{X}$ on Q defined by (3) will be a local section of the sub-bundle $\mathfrak{W}$, i.e., $\tilde{X}_x \in \mathfrak{W}_x$ for $x \in Q$. Hence if we choose a basis $X_1, \ldots, X_p$ of W, $p = \dim W$, and construct the corresponding vector fields according to (3), we obtain p analytic vector fields $\tilde{X}_1, \ldots, \tilde{X}_p$ on Q such that the vectors $\tilde{X}_1(x), \ldots, \tilde{X}_p(x)$ form a basis of $\mathfrak{W}_x$ for every $x \in Q$. This shows that $\mathfrak{W}$ is an analytic field of vector subspaces (**1**, p. 87).

Suppose now that $\mathfrak{W}$ is involutive. Then certainly

$$[\tilde{X}_i, \tilde{X}_j]_{x_0} \in W \qquad \text{for } 1 \leqslant i, j \leqslant p. \tag{6}$$

By (5) this can be expressed equivalently as

$$p_*[X_i^*, X_j^*] \in W \qquad \text{for } i \leqslant i, j \leqslant p, \tag{7}$$

where $X_i^* \in W^*$ is the lift of $X_i \in W$. The vectors X_i^* form a basis of W^* and hence

$$p_*[W^*, W^*] \subset W. \tag{8}$$

Suppose conversely that (8) is satisfied. This implies (6). In view of the homogeneity of M, for the involutivity of $\mathfrak{W}$ it is sufficient to see that we have

$$[A, B]_{x_0} \in W \tag{9}$$

for any pair of vector fields A, B on Q with $A_x \in \mathfrak{W}_x$, $B_x \in \mathfrak{W}_x$ for all $x \in Q$. By linearity, it is sufficient to consider the case where $A = a\tilde{X}_i$, $B = b\tilde{X}_j$ with $a, b \in C^\infty(Q)$. In this case we obtain immediately (cf. **1**, p. 88)

$$\begin{aligned}[A, B] &= a\tilde{X}_i \cdot b\tilde{X}_j - b\tilde{X}_j \cdot a\tilde{X}_i \\ &= (a \cdot b)[\tilde{X}_i, \tilde{X}_j] + (a \cdot \tilde{X}_j b)\tilde{X}_j - (b \cdot \tilde{X}_j a)\tilde{X}_i.\end{aligned}$$

This together with (6) shows (9) and hence the involutivity of $\mathfrak{W}$.

The last statement of Theorem 1 is finally a consequence of our previous considerations, as the lift W^* of W was chosen arbitrarily. A direct proof is also given by Theorem 2, as W is invariant under the natural action of $\mathfrak{H}$ on $T_{x_0}(M)$ (see the proof of Theorem 3 for this fact).

Proof of Theorem 2. The natural action of $\mathfrak{H}$ on $\mathfrak{G}/\mathfrak{H}$ can be described as follows. Let $K \in \mathfrak{H}$ and consider the commutative diagram

$$\text{(10)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{H} & \xrightarrow{\ i\ } & \mathfrak{G} & \xrightarrow{\ p_*\ } & \mathfrak{G}/\mathfrak{H} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{ad}\, K} & & \uparrow{\scriptstyle \mathrm{ad}\, K} & & \uparrow{\scriptstyle \gamma_K} & & \\ 0 & \longrightarrow & \mathfrak{H} & \xrightarrow{\ i\ } & \mathfrak{G} & \xrightarrow{\ p_*\ } & \mathfrak{G}/\mathfrak{H} & \longrightarrow & 0 \end{array}$$

with exact lines, where ad denotes the adjoint representation of $\mathfrak{G}$, i the inclusion $\mathfrak{H} \to \mathfrak{G}$, and γ_K the unique map filling in. For $X \in \mathfrak{G}/\mathfrak{H}$, the element $\gamma_K(X) \in \mathfrak{G}/\mathfrak{H}$ is defined by

$$\text{(11)}\qquad \gamma_K(X) = p_*[K, X^*] \qquad \text{for } X^* \in p_*^{-1}(X).$$

It is independent of the choice of X^* in view of the exactness of the lines in the diagram.

(ii) $\Rightarrow$ (i). Suppose $p_*^{-1}W$ to be a subalgebra. Then certainly

$$p_*[\mathfrak{H}, p_*^{-1}W] \subset W;$$

hence by (11), $\gamma_K(W) \subset W$ for any $K \in \mathfrak{H}$. Let W^* be some lift of W. Then $W^* \subset p_*^{-1}W$, and

$$p_*[W^*, W^*] \subset p_*[p_*^{-1}W, p_*^{-1}W] \subset p_*(p_*^{-1}W) = W$$

proves the second condition of (i) (for any lift W^* of W).

(i) $\Rightarrow$ (ii). Let W^* be some lift of W and write $A = p_*^{-1}W = \mathfrak{H} \oplus W^*$. Then

$$p_*[A, A] = p_*\{[\mathfrak{H}, \mathfrak{H}] + [\mathfrak{H}, W^*] + [W^*, W^*]\} \subset p_*[\mathfrak{H}, W^*] + W.$$

But the $\mathfrak{H}$-invariance of W implies by (11) that $p_*[\mathfrak{H}, W^*] \subset W$ and hence $p_*[A, A] \subset W = p_* A$. Therefore

$$[A, A] \subset p_*^{-1}(p_*[A, A]) \subset p_*^{-1}(p_* A).$$

But $A \supset \mathfrak{H} = \ker p_*$ and hence $p_*^{-1}(p_* A) = A$. This shows that A is a subalgebra.

Now it is also clear that the condition $p_*[W^*, W^*] \subset W$ for some lift of W implies the same condition for any lift of W, because it implies (ii), and (ii) implies this condition for any lift of W.

Proof of Theorem 3. Consider a G-invariant sub-bundle W of the tangent bundle of $M = G/H$ and $W = \mathfrak{W}_{x_0} \subset T_{x_0}(M)$. W is invariant under the natural representation $\sigma : H \to GL(T_{x_0})$ of the isotropy group H on T_{x_0}, i.e.,

$$\sigma(h)W = (\mu_h)_* W \subset W \qquad \text{for } h \in H.$$

We show that W is invariant under the natural action of $\mathfrak{H}$ on T_{x_0} as described in the proof of Theorem 2. For this, it is sufficient to see that this action is the induced representation $\bar{\sigma} : \mathfrak{H} \to \mathrm{End}(T_{x_0})$ (see lemma below). For $h \in H$, the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{H} & \xrightarrow{i} & \mathfrak{G} & \xrightarrow{p_*} & T_{x_0} & \longrightarrow & 0 \\
 & & \uparrow \mathrm{Ad}(h) & & \uparrow \mathrm{Ad}(h) & & \uparrow \sigma(h) & & \\
0 & \longrightarrow & \mathfrak{H} & \xrightarrow{i} & \mathfrak{G} & \xrightarrow{p_*} & T_{x_0} & \longrightarrow & 0
\end{array} \tag{12}
$$

where Ad denotes the adjoint representation of G. The lines being exact, $\sigma(h)$ is uniquely determined by this diagram. Let $K \in \mathfrak{H}$ and consider the 1-parameter subgroup $t \to \exp tK$ of H. For any t we have a commutative diagram (12_t) with $h_t = \exp tK$. Now

$$\mathrm{ad}\, K = \frac{d}{dt} \mathrm{Ad}(\exp tK) \Big|_{t=0};$$

comparing with diagram (10) we see that

$$\gamma_K = \frac{d}{dt} \sigma(\exp tK) \Big|_{t=0}.$$

But

$$\bar{\sigma}(K) = \frac{d}{dt} \sigma(\exp tK) \Big|_{t=0}$$

and this shows that $\gamma_K = \bar{\sigma}(K)$. The rest of the proof follows from Theorems 1 and 2 by the following standard observation.

LEMMA. *Let $\sigma : H \to GL(E)$ be a representation of the Lie group H in the finite-dimensional* **R**-*vector space E. Let $\bar{\sigma} : \mathfrak{H} \to \mathrm{End}(E)$ be the induced representation. Suppose H is connected. Then a subspace $W \subset E$ is H-invariant, i.e., $\sigma(h)W \subset W$ for every $h \in H$ if and only if W is $\mathfrak{H}$-invariant, i.e., $\bar{\sigma}(K)W \subset W$ for every $K \in \mathfrak{H}$.*

Added in proof: The results of this paper were announced in **(8)**. G. Legrand informed me that he obtained the same results in C.R. Acad. Sci. Paris, *258* (1964), 4648–4650.

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