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SEQUENTIAL BINARY ARRAYS AND CIRCULANT MATRICES

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Abstract

A periodic binary array on the square grid is said to be *sequential* if and only if each row and each column of the array contains a given periodic binary sequence or some cyclic shift or reversal of this sequence. Such arrays are of interest in connection with experimental layouts. This paper extends previous results by characterizing sequential arrays on sequences of the type $(1,\ldots,1,0,\ldots,0)$ and solving the problem of equivalence of such arrays (including a computation of the number of equivalence classes).

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1. Introduction

A *periodic binary sequence* $\beta_n = \{b_i\}$ of period n is a sequence of zeros and ones such that n is the smallest positive integer for which

$$b_i = b_{i+n} \quad \text{for all } i.$$

Such a sequence is clearly determined by any of its segments of length n. A *periodic binary array* $B_n = \{b_{ij}\}$ of period n on the square grid is an array each of whose rows and columns is a periodic binary sequence of period n. Such an array is called *sequential* if the same sequence (or its cyclic shifts or their reversals) occurs in every row and column. (Sequential arrays on square and also on triangular and hexagonal grids are of interest in connection with some problems in agricultural statistics; see [3, 4, 7]). An array of period n can be regarded as consisting of repetitions of an $n \times n$ matrix. In particular, if the array is sequential, then its corresponding matric has the same sequence (or its cyclic shifts or reversals) in every row and column; we shall call such a matrix *sequential*.

Two binary arrays will be called *equivalent* if one can be obtained from the other by interchanging zeros with ones, by translation, by rotation, by reflection, or by some finite sequence of these operations. Two $n \times n$ binary matrices will also be called equivalent if they generate equivalent arrays. Thus we have

1.1. DEFINITION. (a) Two $n \times n$ sequential binary matrices $A = (a_{ij})$ and $B = (b_{ij})$ are *equivalent* if one can be obtained from the other by a cyclic shift of rows S_R (downwards), a cyclic shift of columns S_C (to the right), rotation R clockwise through 90°, transposition T, complementation C, or by any finite sequence of these operations.

We record the action of these operations.

$$
\begin{aligned}
B &= S_R(A) \text{ if and only if for all } i, j, & b_{ij} &= a_{i-1,j},\\
B &= S_C(A) \text{ if and only if for all } i, j, & b_{ij} &= a_{i,j-1},\\
B &= R(A) \text{ if and only if for all } i, j, & b_{ij} &= a_{n-j+1,i},\\
B &= T(A) \text{ if and only if for all } i, j, & b_{ij} &= a_{ji},\\
B &= C(A) \text{ if and only if for all } i, j, & b_{ij} &= 1 - a_{ij}.
\end{aligned}
$$

Thus A and B are equivalent if and only if they are in the same orbit of the group

$$
\begin{aligned}
G \times Z_2 = \langle S_R, S_C, R, T \mid\, & S_R^n = S_C^n = R^4 = T^2 = 1,\ S_R S_C = S_C S_R,\\
& S_C^{-1} R = R S_R,\ S_R R = R S_C,\ RT = TR^3, TS_C = S_R T, TS_R = S_C T \rangle\\
& \times \langle C \mid C^2 = 1 \rangle
\end{aligned}
$$

acting on the set of $n \times n$ binary matrices.

For convenience we let x^k denote the sequence $xx\ldots x$ of length k. Thus $0^2 1^3$ denotes 00111.

The sequences for which the sequential arrays have been characterized are

(i) $\tau(n, k) = 10^{k-1}10^{n-k-1}$, when k is 2 or $n-2$ or when $(n, k) = 1$, in [2, Theorem 1] and [5, Theorem 2], and

(ii) $\psi(n, k) = 1^k 0^{n-k}$ for $k \leqslant 3$ (or $k \geqslant n-3$), in [2, Theorem 1] and [5, Theorem 1].

In this paper we shall characterise in Theorem 1 the sequential arrays on $\psi(n, k)$ for any k. One class of examples of these sequential arrays has the structure of a generalized circulant array, a generalization different from the x-step circulants, $(x, n) = 1$, discussed and characterized in [2, Theorem 2]. We shall show in Theorem 2 that the problem of determining equivalence of these generalized circulant arrays on $\psi(n, k)$ reduces to the problem of determining

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equivalence of certain integer sequences, and we shall compute the number of equivalence classes of these sequences in Theorem 3. Finally, we compute the number of equivalence classes of sequential arrays on $\psi(n,k)$ in Theorem 4. Before stating our results we discuss the generalized circulant matrices which arise.

1.2. DEFINITION. (a) Let g, n be positive integers such that g divides n. Let
$\mathscr{A} = \left\{\mathbf{a} = (a_1,\ldots,a_g) \in \mathbf{Z}^g \mid \sum a_i = \varepsilon g \text{ for } \varepsilon = \pm 1, \text{ and } \varepsilon a_i \geqslant 0 \text{ for all } i\right\}$.
Elements of $\mathscr{A}$ with $\varepsilon = 1$ are called *positive*, and otherwise are called *negative*.

(b) An $n \times n$ matrix $C = (c_{ij})$ is called an **a**-*circulant matrix*, where $\mathbf{a} \in \mathscr{A}$, if

(i) for $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant n$, $c_{i+1,j} = c_{i,j-a_i}$ (with subscripts to be read modulo n), and

(ii) for $i > g$ and $1 \leqslant j \leqslant n$, $c_{ij} = c_{i-g,j-\varepsilon g}$.

(c) An array of period n is called **a**-*circulant* if it consists of repetitions of an **a**-circulant matrix.

1.3. REMARKS. (a) For $1 \leqslant i \leqslant g$ and any j, row $jg+i+1$ of C is obtained from row $jg+i$ by shifting $|a_i|$ places to the right if **a** is positive or to the left if **a** is negative. If **a** is an all-1 sequence, then C is an ordinary (1-step) circulant matrix. Figure 1 shows a $(2,2,0,0)$-circulant matrix with first row $\psi(8,4)$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 1

(b) There are many periodic binary sequences ψ of period n for which an **a**-circulant array or matrix is sequential on ψ, namely if

$$\psi = 1^{m_1}0^{n_1}1^{m_2}0^{n_2}\ldots 1^{m_k}0^{m_k},$$

where $k \geqslant 1$, where $\Sigma m_i + \Sigma n_i = n$, and where g divides $(m_1, n_1, \ldots, m_k, n_k)$, then, with **a** as in Definition 1.2, the **a**-circulant matrix C with first row ψ is sequential on ψ. In fact the columns of C are cyclic shifts of ψ if **a** is negative and cyclic shifts of the reverse of ψ if **a** is positive. (This follows directly from Definition 1.2, and the details are omitted.) A different construction of sequential matrices on ψ was given in [2, Section 2], where it was noted that the Kronecker product

$$B \otimes J$$

is sequential on ψ, where J is the $g \times g$ all-1 matrix, and where B is a sequential matrix on

$$\pi = 1^{e_1}0^{f_1}1^{e_2}0^{f_2}\dots 1^{e_k}0^{f_k},$$

with $m_i = ge_i$ and $n_i = gf_i$ for $1 \leqslant i \leqslant k$. We note that some sequential matrices on ψ may be obtained by both constructions; for example, if we take the matrix B above to be the 1-step circulant matrix with first row π, then $B \otimes J$ is the $(0, 0, \dots, 0, g)$-circulant matrix with first row ψ. In Figure 2 the first matrix is not only a $(0, 2)$-circulant matrix with first row $\psi(8, 2)$, but also a Kronecker product of the 4×4 identity matrix and the 2×2 all-1 matrix J. The second matrix is also sequential on $\psi(8, 2)$; it is a Kronecker product but is not equivalent to an **a**-circulant matrix for any **a**.

$$\begin{pmatrix} 1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix} \qquad \begin{pmatrix} 1&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1 \end{pmatrix}$$

(a) (b)

FIGURE 2

(a) The $(0, 2)$-circulant matrix with first row $\psi(8, 2)$. It is also $I \otimes J$ where I is the 4×4 identity matrix and J is the 2×2 all-1 matrix.

(b) The Kronecker product $B \otimes J$ where B is a 4×4 permutation matrix and J is the 2×2 all-1 matrix. It is not equivalent to a generalized circulant matrix.

We shall show that when $\psi = \psi(n, k) = 1^k 0^{n-k}$, then all sequential matrices on ψ are equivalent to a matrix obtained by one of these two constructions.

THEOREM 1. *Any sequential binary matrix on the sequence*

$$\psi(n, k) = 1^k 0^{n-k},$$

where $1 \leqslant k \leqslant n/2$, *is equivalent to either*

(a) *the* **a**-*circulant matrix with* $\psi(n, k)$ *as first row, for some* $\mathbf{a} = (a_1, \dots, a_g) \in \mathscr{A}$, *where* $g = (n, k)$, *or*

(b) *the Kronecker product* $P \otimes J$ *of a permutation matrix* P *of dimension* n/k *and the* $k \times k$ *all*-1 *matrix* J, *where here* k *divides* n.

Moreover, any $n \times n$ *matrix satisfying* (a) *or* (b) *is sequential on* $\psi(n, k)$.

1.4. REMARKS. (a) From Remark 1.3(b) above, the matrices occurring in (a) or (b) are all sequential on $\psi(n,k)$.

(b) Since the sequences $\psi(n,k)$ and $\psi(n,n-k)$ are equivalent under complementation (that is, under interchanging zeros with ones), Theorem 1 also characterizes the sequential matrices on $\psi(n,k)$ for $n/2 < k < n$.

The problem now arises of determining conditions for the equivalence of two generalized circulant sequential matrices on a given periodic binary sequence, in particular on the sequence $\psi(n,k)$. We consider this problem for sequences of the type introduced in Remark 1.3(b). We do not solve the equivalence problem completely for all sequences of this type, but we do solve it for a subclass of sequences containing all the $\psi(n,k)$; see Theorem 2 and Remarks 1.6. First we need the following definitions.

1.5. DEFINITIONS. (a) The *shift operator* σ, the *reversal operator* ρ, and the *negation operator* η acting on sequences of integers of finite length are defined by

$$\sigma(a_1,\ldots,a_m) = (a_m, a_1,\ldots,a_{m-1}),$$
$$\rho(a_1,\ldots,a_m) = (a_m, a_{m-1},\ldots,a_1),$$
$$\eta(a_1,\ldots,a_m) = (-a_1,-a_2,\ldots,-a_m).$$

(b) An element $\mathbf{a}$ of $\mathscr{A}$ (as defined in 1.2) is said to be in *normal form* if $\mathbf{a}$ is positive and if its last entry is non-zero, that is, if $\mathbf{a}$ has the form

$$\mathbf{a} = (0^{c_1}, b_1+1,\ldots,0^{c_l}, b_l+1),$$

where $l \geqslant 1$, where the b_i and c_i are non-negative integers, and where $\Sigma b_i = \Sigma c_i = g - l$. The operator τ is defined on the subset $\mathscr{A}^*$ of elements of $\mathscr{A}$ in normal form by

$$\tau(0^{c_1}, b_1+1,\ldots,0^{c_l}, b_l+1) = (0^{b_1}, c_1+1,\ldots,0^{b_l}, c_l+1).$$

(c) Two elements $\mathbf{a}$, $\mathbf{a}'$ of $\mathscr{A}$ are said to be *equivalent*, written $\mathbf{a} \sim \mathbf{a}'$, if and only if $\mathbf{a}'$ can be obtained from $\mathbf{a}$ by a finite sequence of the operations η, ρ, σ, τ (we recall that τ may only be applied to sequences in normal form). Clearly this relation $\sim$ is an equivalence relation on $\mathscr{A}$. We shall sometimes consider the restriction of $\sim$ to $\mathscr{A}^*$ and still use the notation $\sim$.

(d) For a binary sequence ψ of length n, where n is a multiple of g, and for $\mathbf{a} \in \mathscr{A}$, the $\mathbf{a}$-circulant matrix with first row ψ is denoted by $A(\mathbf{a},\psi)$.

Now with ψ as in Remark 1.3(b) (so that $A(\mathbf{a},\psi)$ is sequential on ψ), it is clear that any $\mathbf{a}$-circulant matrix which is sequential on ψ is equivalent to either $A(\mathbf{a},\psi)$ or $A(\mathbf{a},\rho\psi)$ (by permuting columns). So the problem of equivalence of generalized circulant matrices which are sequential on ψ is reduced to the problem of the equivalence of $A(\mathbf{a},\psi)$ with $A(\mathbf{a}',\psi)$ or with $A(\mathbf{a}',\rho\psi)$, for $\mathbf{a}, \mathbf{a}' \in \mathscr{A}$.

THEOREM 2. (a) *Let* $\psi = (1^{m_1}0^{n_1}\dots 1^{m_k}0^{n_k})$, *where* $\sum m_i + \sum n_i = n$, *and let* g *be a divisor of* $(m_1, n_1, \dots, m_i, n_k)$. *Then for* $\mathbf{a}, \mathbf{a}' \in \mathscr{A}$, *we have* $\mathbf{a} \sim \mathbf{a}'$ *if and only if* $A(\mathbf{a}, \psi)$ *is equivalent to at least one of* $A(\mathbf{a}', \psi)$ *and* $A(\mathbf{a}', \rho\psi)$.

(b) *Suppose that either* $\rho\psi = \sigma^u\psi$ *for some* u, *or* $\rho\psi = \sigma^u\bar{\psi}$ *for some* u, *where* $\bar{\psi} = (0^{m_1}, 1^{n_1}, \dots, 0^{m_k}, 1^{n_k})$ *is the complement of* ψ. *Then for* $\mathbf{a}, \mathbf{a}' \in \mathscr{A}$, *we have that*

(i) $A(\mathbf{a}, \psi)$, $A(\mathbf{a}, \rho\psi)$ *are equivalent, and that*

(ii) $\mathbf{a} \sim \mathbf{a}'$ *if and only if* $A(\mathbf{a}, \psi)$ *and* $A(\mathbf{a}', \psi)$ *are equivalent.*

1.6. REMARKS. (a) In Theorem 2(b) we have each $\mathbf{a}$-circulant matrix which is sequential on ψ equivalent to $A(\mathbf{a}, \psi)$, and $A(\mathbf{a}, \psi)$ equivalent to $A(\mathbf{a}', \psi)$ if and only if $\mathbf{a} \sim \mathbf{a}'$.

(b) The sequences $\psi(n, k) = 1^k0^{n-k}$, $1^k0^l1^k0^m$, and $1^l0^k1^m0^k$, where g divides (k, l, m), all satisfy the conditions of Theorem 2(b).

(c) To refine Theorem 2(b) and Remark 1.6(a) a little, we observe that every element of $\mathscr{A}$ is equivalent to a sequence in normal form, so that each generalized circulant matrix which is sequential on ψ is equivalent to $A(\mathbf{a}, \psi)$ for some $\mathbf{a}$ in the subset $\mathscr{A}^*$ of sequences in normal form. Thus the number of equivalence classes of generalized circulant matrices sequential on ψ, with ψ as in Theorem 2(b), is equal to the number of equivalence classes in $\mathscr{A}$; this in turn is equal to the number of equivalence classes in $\mathscr{A}^*$ (of the restriction of $\sim$ to $\mathscr{A}^*$). Finally we note that equivalent sequences in $\mathscr{A}$ have the same number of nonzero entries.

THEOREM 3. *The number* $N(g, l)$ *of equivalence classes of the equivalence relation* $\sim$ *in* $\mathscr{A}$ *(or* $\mathscr{A}^*$*) which have exactly* l *nonzero entries is*

$$N(g, 1) = 1,$$

and if $l \geqslant 2$, *then*

$$N(g,l) = \frac{1}{4l}\sum_{d\mid(g,l)} \phi(d)\binom{(g/d)-1}{(l/d)-1}^2 + \frac{1}{4l}\sum_{d\mid(g,v)} \phi(d)\binom{(g/d)-1}{(l/d)-1}$$
$$+ \frac{1}{4l}\delta_1 \sum_{d\mid(g,l/2)} \phi(2d)\binom{(g/d)-1}{(l/d)-1} + \frac{1}{4}\delta_2\binom{[(g-1)/2]}{[(l-1)/2]}^2 + \frac{1}{4}\delta_3,$$

where (i) $l = 2^u v$ *with* $u \geqslant 0$ *and* v *odd*,

(ii) $$\delta_1 = \begin{cases} 1 & \text{if } l \text{ is even}, \\ 0 & \text{if } l \text{ is odd}, \end{cases}$$

(iii) $$\delta_2 = \begin{cases} 1 & \text{if } l \text{ is odd}, \\ 0 & \text{if } l \text{ is even and } g \text{ is odd}, \\ 2(g/l) - 1 & \text{if } l \text{ and } g \text{ are both even}, \end{cases}$$

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(iv)
$$\delta_3 = \begin{cases} 1 & \textit{if } l \textit{ divides } g, \\ 0 & \textit{otherwise}. \end{cases}$$

Finally, to complete the enumeration of the equivalence classes of sequential arrays on $\psi(n,k)$, we have

THEOREM 4. *The number of equivalence classes of binary sequential arrays on* $\psi(n,k) = 1^k 0^{n-k}$, *where* $1 \leqslant k \leqslant n/2$, *is*
$$\sum_{1+\delta \leqslant l \leqslant g} N(g,l) + \delta T(n/k),$$
where
$$\delta = \begin{cases} 1 & \textit{if } k \textit{ divides } n, \\ 0 & \textit{otherwise}, \end{cases}$$
$N(g,l)$ *is given in Theorem* 3, *and* $T(m)$ *is the number of equivalence classes of* $m \times m$ *permutation matrices as given in* [6, *Theorem* 1].

This theorem depends essentially on the following proposition, which reduces the problem of equivalence of sequential matrices of the form $B \otimes J$ defined in Remark 1.3(b) to the equivalence of the smaller matrices B. We note that the generalized circulant matrix $A(\mathbf{a}, \psi)$ is equivalent to a Kronecker product $B \otimes J$ if and only if $g = k$ divides n and $\mathbf{a} \sim (k, 0^{k-1})$. Thus the summation in Theorem 4 is over $1 + \delta \leqslant l \leqslant g$.

PROPOSITION 5. *Let* $\psi = (1^{m_1}0^{n_1}\ldots 1^{m_k}0^{n_k})$, *where* $\sum m_i + \sum n_i = n$, *and let* g *be a divisor of* $(m_1, n_1, \ldots, m_k, n_k)$. *Further, let* $m_i = e_i g$ *and* $n_i = f_i g$ *for* $1 \leqslant i \leqslant k$, *and let* $\pi = (1^{e_1}0^{f_1}\ldots 1^{e_k}0^{f_k})$. *If* B *and* D *are sequential matrices on* π, *then* B *and* D *are equivalent if and only if* $B \otimes J$ *and* $D \otimes J$ (*sequential matrices on* ψ) *are equivalent*, *where* J *is the* $g \times g$ *all*-1 *matrix*.

The rest of the paper is organized so that Theorem N is proved in section $N + 1$, for $N = 1, 2, 3, 4$.

2. Classification of sequential matrices on $\psi(n,k)$

We prove Theorem 1 in this section. Let $D = (d_{ij})$ be a sequential binary matrix on $\psi(n,k) = 1^k 0^{n-k}$, where $k \leqslant n/2$. As the result has already been proved for $k = 1, 2$ [2, Theorem 1] and $k = 3$ [5, Theorem 1], we shall assume that $k \geqslant 4$. We shall denote the ith row of D by R_i and the ith column of D by C_i for $1 \leqslant i \leqslant n$, and we set $g = (n,k)$. Finally, we shall say that D is $\mathbf{b}$-*circulant*

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on rows $1, \dots, p$ where $\mathbf{b} = (b_1, \dots, b_q)$ has non negative integer entries (and $p > 0, q > 0$), if R_{jq+i+1} is obtained from R_{jq+i} by shifting b_i places to the right for all $1 \leqslant jq + i < p$ and all $1 \leqslant i \leqslant q$.

We may assume that $R_1 = \psi(n,k) \neq R_n$ (by applying $S_C^u S_R^v$ for some u, v if necessary). Suppose first that $R_1 = \cdots = R_k$. Since R_{k+1} contains $\psi(n,k)$, there is an integer h with $k \leqslant h \leqslant n-k$ such that $R_{k+1} = (0^h 1^k 0^{n-k-h})$. Then as C_{h+i} contains $\psi(n,k)$ for each $i = 1, \dots, k$, it follows that

$$C_{h+i} = (0^k 1^k 0^{n-2k}) \quad \text{and} \quad R_{k+i} = R_{k+1} \quad \text{for } i = 1, \dots, k$$

(see Figure 3). Continuing this argument, we see that the rows and columns of D are partitioned into sets of size k, where two rows or columns are in the same set if and only if they are equal. The sets consist of k consecutive rows or columns, and part (a) of Theorem 1 is true.

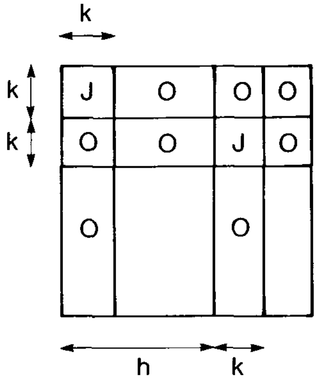

FIGURE 3

Thus we may suppose that no k consecutive rows or columns of D are equal. Then there exists an integer c_1 satisfying $0 \leqslant c_1 \leqslant k-2$ and such that $R_1 = \cdots = R_{c_1+1} \neq R_{c_1+2}$. Further, we may assume that the first entry of R_{c_1+2} is 0 (if necessary by applying $S_C^u S_R^v R$ to D for some u, v). Thus there exists an integer b_1 with $0 \leqslant b_1 \leqslant n-k-1$ such that

$$(1) \qquad R_{c_1+2} = (0^{b_1+1} 1^k 0^{n-k-b_1-1}).$$

Since $C_1, \dots, C_{b_1+1}$ contain $\psi(n,k)$, it follows that the first x entries of R_n are all equal to 1, where $x = \min(k, b_1+1)$. If $b_1 + 1 \geqslant k$, then $R_n = R_1$, contrary to our assumptions. Thus $0 \leqslant b_1 \leqslant k-2$, and $R_n = (1^u 0^{n-k} 1^{k-u})$ for some u with $b_1 + 1 \leqslant u \leqslant k-1$. In particular

$$(2) \qquad d_{nk} = d_{n,k+1} = 0 \quad \text{and} \quad d_{nn} = 1.$$

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Hence, using (1) and (2), we obtain

(3) $$C_k = (1^k 0^{n-k}) \quad \text{and} \quad C_{k+1} = (0^{c_1+1} 1^k 0^{n-k-c_1-1}).$$

2.1. LEMMA. *There exist non-negative integers $b_1, \ldots, b_l$ and $c_1, \ldots, c_l$ for some $l \geqslant 2$ such that*

(a) $\Sigma_{1 \leqslant i \leqslant l}(b_i + 1) = \Sigma_{1 \leqslant i \leqslant l}(c_i + 1) = k$

and

(b) *D is* **a**-*circulant on rows* $1, \ldots, k+1$, *where*

$$\mathbf{a} = (0^{c_1}, b_1 + 1, \ldots, 0^{c_l}, b_l + 1).$$

PROOF. We have already defined integers b_1, c_1 satisfying $0 \leqslant b_1 \leqslant k-2$, $0 \leqslant c_1 \leqslant k-2$ such that D is $(0^{c_1}, b_1+1)$-circulant on rows $1, \ldots, c_1+2$. We shall define the other b_i, c_i recursively as follows.

Suppose that for some positive integer x we have defined non-negative integers $b_1, \ldots, b_x$ and $c_1, \ldots, c_x$ with

$$\sum_{1 \leqslant i \leqslant x} (b_i + 1) \leqslant k, \qquad \sum_{1 \leqslant i \leqslant x} (c_i + 1) \leqslant k$$

and such that D is $(0^{c_1}, b_1 + 1, \ldots, 0^{c_x}, b_x + 1)$-circulant on rows $1, \ldots, \Sigma_{1 \leqslant i \leqslant x}(c_i + 1) + 1$ (represented diagrammatically in Figure 4).

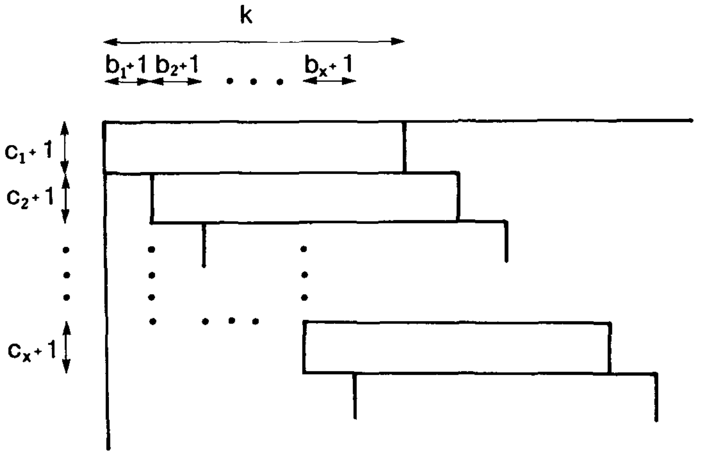

FIGURE 4

First we show that $\Sigma_{1 \leqslant i \leqslant x}(b_i + 1) = k$ if and only if $\Sigma_{1 \leqslant i \leqslant x}(c_i + 1) = k$. Assume that $\Sigma(b_i + 1) = k$. Then the kth entry of row $\Sigma(c_i + 1) + 1$ is zero, and $\Sigma(c_i + 1) + 1 \leqslant k + 1$. However, by (3) the first k entries in C_k are 1, and hence $\Sigma(c_i + 1) = k$. Conversely, assume that $\Sigma(c_i + 1) = k$. If $\Sigma(b_i + 1) \leqslant k - 1$, then the kth entry in R_{k+1} would be 1, whereas by (3) it is 0. Thus $\Sigma(b_i + 1) = k$.

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Next, if one, and hence both, of $\Sigma_{1 \leqslant i \leqslant x}(b_i + 1)$ and $\Sigma_{1 \leqslant i \leqslant x}(c_i + 1)$ are less than k, we have to define b_{x+1} and c_{x+1}: assume that $b = \Sigma_{1 \leqslant i \leqslant x}(b_i + 1) < k$ and that $c = \Sigma_{1 \leqslant i \leqslant x}(c_i + 1) < k$. Then there exists a non-negative integer c_{x+1} such that $R_{c+1} = \cdots = R_{c+c_{x+1}+1} \neq R_{c+c_{x+1}+2}$. The kth entry in R_{c+1}, and hence in $R_{c+c_{x+1}+1}$, is 1 (since $b < k$), and by (3) it follows that $c + c_{x+1} + 1 = \Sigma_{1 \leqslant i \leqslant x+1}(c_i + 1) \leqslant k$. Set $e = c + c_{x+1} + 2$. There exists a non-negative integer b_{x+1} such that, setting $f = b + b_{x+1} + 1$, we have $R_e = (0^f 1^k 0^{n-k-f})$. Since $c_1 + 1 \leqslant e \leqslant k + 1$, it follows from (3) that the $(k + 1)st$ entry in R_e is 1. Thus

$$b + b_{x+1} + 1 = \sum_{1 \leqslant i \leqslant x+1} (b_i + 1) \leqslant k.$$

By the definition of b_{x+1} and c_{x+1}, D is $(0^{c_1}, b_1 + 1, \ldots, 0^{c_{x+1}}, b_{x+1} + 1)$-circulant on rows $1, \ldots, e$.

It follows that there exists an integer $l \geqslant 2$ and nonnegative integers b_i, c_i, $1 \leqslant i \leqslant l$ such that (a) and (b) are true. This proves Lemma 2.1.

By the division algorithm there are unique integers $q \geqslant 2$ and $r \in [0, k - 1]$ such that

$$n = qk + r. \tag{4}$$

Also there exists an integer m, $0 \leqslant m \leqslant l - 1$, such that

$$\sum_{1 \leqslant i \leqslant m} (b_i + 1) \leqslant r < \sum_{1 \leqslant i \leqslant m+1} (b_i + 1) \tag{5}$$

(where the expression on the left is zero if $m = 0$), since $r < k = \Sigma(b_i + 1)$. By continuing our above argument we deduce that D is $\mathbf{a}$-circulant (with $\mathbf{a}$ as given in Lemma 2.1), on rows $1, \ldots, c$, where $c = (q - 1)k + \Sigma_{1 \leqslant i \leqslant m+1}(c_i + 1)$. (After this set of rows we must worry about 1's occurring in column 1.) We may now easily complete the proof of Theorem 1 in the case where $g = (n, k) = k$.

2.2. Lemma. *If k divides n, then D is $\mathbf{a}$-circulant, with $\mathbf{a}$ as in Lemma 2.1, and Theorem 1 is true.*

Proof. Let $n = qk$. Then $c = (q - 1)k + c_1 + 1$ (the parameter m of (5) is 0 here), and $R_c = (0^{n-k}1^k)$. Since

$$C_1 = \cdots = C_{b_1+1} = (1^{c_1+1}0^{n-k}1^{k-c_1-1}) \neq C_{b_1+2} = (1^{c_1+c_2+2}0^{n-k}1^{k-c_1-c_2-2}),$$

it follows that $R_{c+1} = \cdots = R_{c+c_2+1} = (1^{b_1+1}0^{n-k}1^{k-b_1-1})$. Also columns $b_1 + 2, \ldots, b_1 + b_2 + 3$ determine rows $c + c_2 + 2, \ldots, c + c_2 + c_3 + 2$ in the same way as columns $1, \ldots, b_1 + 1$ determined rows $c + 1, \ldots, c + c_2 + 1$. Continuing in this fashion we see that $D$is $\mathbf{a}$-circulant.

To complete the proof of Theorem 1 we assume that $r > 0$ (see (4)) and refine the proof of Lemma 2.2. Set

$$b = \sum_{1 \leqslant i \leqslant m+1} (b_i + 1), \qquad c = (q - 1)k + \sum_{1 \leqslant i \leqslant m+1} (c_i + 1).$$

We have $R_c = (0^{(q-1)k+b-(b_{m+1}+1)}1^k 0^{r-b+b_{m+1}+1})$. First we determine the entries of R_{c+1}: the $(c+1)$st entries in $C_{(q-1)k+b}$ and $C_{(q-1)k+b+1}$ are 0 and 1, respectively, and it follows that

(6) $$d_{c+1,j} = 1 \quad \text{for } 1 \leqslant j \leqslant b-r, \quad \text{and} \quad d_{c+1,b-r+1} = 0.$$

Next we consider $C_1 = (1^{c_1+1}0^{n-k}1^{k-c_1+1})$; since the cth entry is 0 and the $(c+1)$st entry is 1, it follows that $c+1 = n-k+c_1+2$, that is,

(7) $$r = \sum_{2 \leqslant i \leqslant m+1} (c_i + 1).$$

(In particular $m \geqslant 1$.) Also since the $(c+1)$st entry in C_{b_1+1} is 1 and in C_{b_1+2} is 0, it follows from (6) that $b_1 + 1 = b - r$, that is,

(8) $$r = \sum_{2 \leqslant i \leqslant m+1} (b_i + 1).$$

Suppose now that $r > \sum_{1 \leqslant i \leqslant m}(b_i + 1)$. Then the last entry in R_c is 0, and the last entry in R_{c+1} is 1. Thus, by considering C_n, it follows that $n - c \geqslant k$, that is, $r \geqslant \sum_{1 \leqslant i \leqslant m+1}(c_i + 1)$, which contradicts (7). Thus

$$r = \sum_{1 \leqslant i \leqslant m} (b_i + 1),$$

which, together with (8), yields $b_1 = b_{m+1}$. It also implies that $R_c = (0^{n-k}1^k)$, so that the first entry of C_n equal to 1 is entry $c - c_{m+1}$. By (2) the last entry in C_n is 1, and so it follows that $c_{m+1} = c_1$.

If we consider in turn rows $c + \sum_{m+2 \leqslant i \leqslant j}(c_i + 1) + 1$ for $m + 2 \leqslant j \leqslant l$ and rows $qk + \sum_{1 \leqslant i \leqslant j}(c_i + 1) + 1$ for $1 \leqslant j \leqslant m$, we will find as above that

$$b_i = b_{m+i} \quad \text{and} \quad c_i = c_{m+i} \quad \text{for all } i \geqslant 1,$$

where the subscripts are to be read modulo l. Thus, if $s = (m, l)$, it follows that

$$b_i = b_{s+i} \quad \text{and} \quad c_i = c_{s+i} \quad \text{for all } i.$$

Hence $\mathbf{a} = (\mathbf{a}')^{l/s}$, where $\mathbf{a}' = (0^{c_1}, b_1 + 1, \ldots, 0^{c_s}, b_s + 1) \in \mathbb{Z}^t$, and where $t = \sum_{1 \leqslant i \leqslant s}(c_i + 1) = k/(l/s) = ks/l = \sum_{1 \leqslant i \leqslant s}(b_i + 1)$. It follows that t divides k, and, as $r = \sum_{1 \leqslant i \leqslant m}(b_i + 1) = t(m/s)$, t also divides r, and hence t divides $(n, k) = g$. Let $g = tu$. It follows from our arguments that D is $\mathbf{a}$-circulant and hence is $(\mathbf{a}')^u$-circulant. Finally, $(\mathbf{a}')^u = (0^{c_1}, b_1 + 1, \ldots, 0^{c_{su}}, b_{su} + 1) \in \mathbb{Z}^g$ with $\sum_{1 \leqslant i \leqslant su}(b_i + 1) = \sum_{1 \leqslant i \leqslant su}(c_i + 1) = g$. This completes the proof of Theorem 1.

3. Equivalence of generalized circulant sequential matrices

In this section we prove Theorem 2. Let $\psi = (1^{m_1}0^{n_1} \cdots 1^{m_k}0^{n_k})$ with $\sum m_i + \sum n_i = n$ and let g divide $(m_1, n_1, \ldots, m_k, n_k)$. We shall use the operations on matrices defined in 1.1 and the operations on sequences defined in 1.5. Let

$\mathbf{a} = (a_1, \dots, a_g) \in \mathscr{A}$ and let $\psi' = \sigma^u \rho^v(\psi)$, where $0 \leqslant u < n$, and where v is 0 or 1. It is straightforward to check the following, and the details are omitted.

$$A(\sigma\mathbf{a}, \psi') = S_C^{a_g} S_R^{-1} A(\mathbf{a}, \psi'),$$
$$A(\rho\mathbf{a}, \rho\psi') = S_R^{-1} R^2 A(\mathbf{a}, \psi'),$$
$$A(\eta\mathbf{a}, \rho\psi') = RTA(\mathbf{a}, \psi').$$

(We note that $A(\mathbf{a}, \sigma\psi') = S_C A(\mathbf{a}, \psi')$.) Also if $\mathbf{a} = (0^{c_1}, b_1 + 1, \dots, 0^{c_l}, b_l + 1) \in \mathscr{A}^*$ (so that column 1 of $A(\mathbf{a}, \psi')$ is then $\sigma^{c_1+1}\rho\psi'$), then

$$A(\tau\mathbf{a}, \rho\psi') = S_C^{-(c_1+1)} TA(\mathbf{a}, \psi'),$$

and

$$A(\eta\tau\mathbf{a}, \psi') = S_C^{c_1+1} RA(\mathbf{a}, \psi').$$

It follows from these equations that if $\mathbf{a}' \in \mathscr{A}$ can be obtained from $\mathbf{a} \in \mathscr{A}$ by a finite sequence of the operations σ, ρ, η, τ, then $A(\mathbf{a}, \psi)$ is equivalent to $A(\mathbf{a}', \psi)$ or $A(\mathbf{a}', \rho\psi)$.

Conversely, suppose that $\mathbf{a}$, $\mathbf{a}' \in \mathscr{A}$ and that $A(\mathbf{a}, \psi)$ and $A(\mathbf{a}', \psi')$ are equivalent, where now ψ' is either ψ or $\rho\psi$, that is, $A(\mathbf{a}', \psi')$ can be obtained from $A(\mathbf{a}, \psi)$ by applying an element of the group $G \times Z_2$ defined in 1.1. Each element of this group has a unique representation as $S_R^a S_C^b R^c T^d C^e$, where the integers a, $b \in [0, n-1]$, $c \in [0, 3]$ and d, $e \in [0, 1]$. Thus we have

$$A(\mathbf{a}, \psi) = S_R^a S_C^b R^c T^d C^e A(\mathbf{a}', \psi').$$

If $e = 1$, then $A(\mathbf{a}, \psi)$ is sequential both on ψ and on its complement $\bar{\psi}$, so that $\bar{\psi}' = \sigma^u \rho^v \psi'$ for some u, v, and we have $CA(\mathbf{a}', \psi') = A(\mathbf{a}', \bar{\psi}') = S_C^u A(\mathbf{a}', \rho^v \psi')$. Thus we may assume that $e = 0$. Also if $d = 1$, then $TA(\mathbf{a}', \psi') = R^{-1} A(\eta\mathbf{a}', \rho\psi')$, so that we may assume that $d = 0$. Clearly there is an integer u such that $\sigma^u \mathbf{a}'$ (if $\mathbf{a}'$ is positive) or $\sigma^u \eta\mathbf{a}'$ (if $\mathbf{a}'$ is negative) is in normal form, that is, has the form $(0^{c_1}, b_1 + 1, \dots, 0^{c_l}, b_l + 1)$. Since $A(\mathbf{a}', \psi') = S_R^v S_C^w A(\sigma^u \mathbf{a}', \psi')$ for some v, w, and since $(S_R^a S_C^b R^c)(S_R^v S_C^w) = S_R^x S_C^y R^c$ for some x, y, it follows that we may assume that $\mathbf{a}'$ or $\eta\mathbf{a}'$ is in normal form as above. If $\mathbf{a}'$ is in normal form, then $RA(\mathbf{a}', \psi') = S_C^{-(c_1+1)} A(\eta\tau\mathbf{a}', \psi')$, while if $\eta\mathbf{a}'$ is in normal form, then

$$RA(\mathbf{a}', \psi') = R^2 T(RTA(\mathbf{a}', \psi')) = R^2 TA(\eta\mathbf{a}', \rho\psi') = R^2 S_C^{c_1+1} A(\tau\eta\mathbf{a}', \psi')$$
$$= S_C^{-(c_1+1)} R^2 A(\tau\eta\mathbf{a}', \psi') = S_C^{-(c_1+1)} S_R A(\rho\tau\eta\mathbf{a}', \rho\psi').$$

Thus we may assume that $c = 0$. Finally,

$$S_R^a S_C^b A(\mathbf{a}', \psi') = S_C^x A(\sigma^{-a}\mathbf{a}', \psi') = A(\sigma^{-a}\mathbf{a}', \sigma^x \psi')$$

for some x and we conclude that $\mathbf{a} \sim \mathbf{a}'$. Thus part (a) of Theorem 2 is proved.

If $\rho\psi = \sigma^u \psi$ for some u, then $A(\mathbf{a}, \rho\psi) = A(\mathbf{a}, \sigma^u \psi) = S_C^u A(\mathbf{a}, \psi)$, and therefore $A(\mathbf{a}, \psi)$ and $A(\mathbf{a}, \rho\psi)$ are equivalent. Similarly, if $\rho\psi = \sigma^u \bar{\psi}$, then $A(\mathbf{a}, \rho\psi) = S_C^u A(\mathbf{a}, \bar{\psi}) = S_C^u CA(\mathbf{a}, \psi)$, and again $A(\mathbf{a}, \psi)$ and $A(\mathbf{a}, \rho\psi)$ are equivalent. Thus, if either of these two conditions is satisfied, then it follows from part (a) that $\mathbf{a} \sim \mathbf{a}'$ if and only if $A(\mathbf{a}, \psi)$ is equivalent to $A(\mathbf{a}', \psi)$. This completes the proof of Theorem 2.

4. Enumeration of the equivalence classes of $\mathscr{A}$

In Remark 1.6(c), we observed that equivalent sequences in $\mathscr{A}$ had the same number l of nonzero entries for some $1 \leqslant l \leqslant g$. We also observed that every element of $\mathscr{A}$ is equivalent to an element in normal form. Hence we shall calculate the number $N(g, l)$ of equivalence classes of the restriction of $\sim$ to the set $\mathscr{A}_l^*$ of elements of $\mathscr{A}^*$ with exactly l nonzero entries. If $l = 1$, then $N(g, 1) = |\mathscr{A}_1^*| = 1$, so we shall assume that $2 \leqslant l \leqslant g$. First we describe the equivalence relation on $\mathscr{A}_l^*$ in a different way which will allow us to use the technique of "Burnside counting" (see [1], p. 191) to compute $N(g, l)$.

We define operations σ^*, ρ^*, τ^* on $\mathscr{A}_l^*$ which are analogous to σ, ρ, τ as follows: for

$$\mathbf{a} = \left(0^{c_1}, b_1 + 1, \ldots, 0^{c_l}, b_l + 1\right) \in \mathscr{A}_l^*,$$

let

$$\begin{aligned} \sigma^*\mathbf{a} &= \left(0^{c_l}, b_l + 1, 0^{c_1}, b_1 + 1, \ldots, 0^{c_{l-1}}, b_{l-1} + 1\right), \\ \rho^*\mathbf{a} &= \left(0^{c_l}, b_{l-1} + 1, \ldots, b_1 + 1, 0^{c_1}, b_l + 1\right), \text{ and} \\ \tau^*\mathbf{a} &= \left(0^{b_1}, c_1 + 1, \ldots, 0^{b_l}, c_l + 1\right). \end{aligned}$$

4.1. Lemma. *For $\mathbf{a}, \mathbf{a}' \in \mathscr{A}_l^*$, $\mathbf{a}$ is equivalent to $\mathbf{a}'$ if and only if $\mathbf{a}'$ can be obtained from $\mathbf{a}$ by a finite sequence of the operations σ^*, ρ^*, τ^*.*

Proof. Since, applied to $\mathbf{a} = (0^{c_1}, b_1 + 1, \ldots, 0^{c_l}, b_l + 1)$, we have $\sigma^* = \sigma^{c_l + 1}$, $\rho^* = \sigma^{-1}\rho$, and $\tau^* = \tau$, it follows that if $\mathbf{a}'$ can be obtained from $\mathbf{a}$ by a finite sequence of σ^*, ρ^*, τ^*, then $\mathbf{a}' \sim \mathbf{a}$. Conversely, suppose that $\mathbf{a} \sim \mathbf{a}'$, that is, $\mathbf{a}'$ can be obtained form $\mathbf{a}$ by a finite sequence of σ, ρ, τ, η. As η commutes with the other three operations, and as $\mathbf{a}$ and $\mathbf{a}'$ are both positive, we may assume that η does not occur in this sequence. Also, as $\rho = \sigma\rho^*$ and $\tau = \tau^*$, we have

$$\mathbf{a}' = \sigma^{u_1}\rho^{*v_1}\tau^{*w_1} \cdots \sigma^{u_t}\rho^{*v_t}\tau^{*w_t}\mathbf{a}$$

for some non-negative integers u_i, v_i, w_i with at least one of u_i, v_i, w_i positive for each i, and some $t \geqslant 1$. Since ρ^* and τ^* are only defined on elements of $\mathscr{A}_l^*$, and since $\mathbf{a}'$ and $\mathbf{a}$ are in $\mathscr{A}_l^*$ it follows that for each i, σ^{u_i} is a possibly trivial power of σ^*. This proves the lemma.

4.2. Lemma. *The operations σ^*, ρ^*, τ^* generate a group*

$$H = \langle \sigma^*, \rho^* \,|\, (\sigma^*)^l = (\rho^*)^2 = 1,\ \rho^*\sigma^* = \sigma^{*-1}\rho^* \rangle \times \langle \tau^* \,|\, (\tau^*)^2 = 1 \rangle$$

isomorphic to the direct product of a dihedral group of order $2l$ and a cyclic group of order 2. Each element of H has a unique expression of the form

$$h = (\sigma^*)^u (\rho^*)^v (\tau^*)^w,$$

where $1 \leqslant u \leqslant l$, and where v and w are 0 or 1. Elements of $\mathscr{A}_l^*$ are equivalent if and only if they lie in the same orbit of H acting on $\mathscr{A}_l^*$, as defined above.

The proof of Lemma 4.2 is straightforward and is omitted. We may now apply the theorem of Burnside [1, p. 191] to obtain

$$N(g, l) = \frac{1}{4l} \sum_{h \in H} F(h),$$

where $F(h)$ is the number of elements of $\mathscr{A}_l^*$ fixed by h. To complete the proof of Theorem 3 we compute $F(h)$ for each $h \in H$. To help in this computation we note that elements of H which are conjugate to each other fix the same number of elements of $\mathscr{A}_l^*$, so we divide our work into the following cases.

1. $h = \sigma^{*m}$, $1 \leqslant m \leqslant l$,

2. $h = \sigma^{*m}\tau^*$, $1 \leqslant m \leqslant l$,

3(a). $h = \rho^*$

3(b). $h = \rho^*\tau^*$

(If l is odd, then H contains l elements conjugate to ρ^*, namely $\sigma^{*m}\rho^*$ for $1 \leqslant m \leqslant l$, and l elements conjugate to $\rho^*\tau^*$, namely $\sigma^{*m}\rho^*\tau^*$ for $1 \leqslant m \leqslant l$. So these cases are sufficient to complete the computation for l odd. If l is even, then H contains $l/2$ elements conjugate to each of ρ^*, $\rho^*\tau^*$, $\sigma^*\tau^*$, and $\sigma^*\rho^*\tau^*$, so we add Case 4.)

4(a). $h = \sigma^*\rho^*$,

4(b). $h = \sigma^*\rho^*\tau^*$.

Finally, for use in the computation we note that the number $c(m, n)$ of compositions of a positive integer n into m nonzero parts is

$$c(m, n) = \binom{n-1}{m-1}. \tag{9}$$

We shall always write $\mathbf{a} = (0^{c_1}, b_1 + 1, \ldots, 0^{c_l}, b_l + 1) \in \mathscr{A}_l^*$.

Case 1. $h = \sigma^{*m}$, where $1 \leqslant m \leqslant l$. Here $\sigma^{*m}\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_{l-m+i} = b_i \quad \text{and} \quad c_{l-m+i} = c_i,$$

where subscripts are to be read modulo l. Setting $e = (l, m)$, we have integers u, v such that $u(l-m) + vl = e$. Hence $b_i = b_{(l-m)+i} = \cdots = b_{u(l-m)+i} = b_{e+i}$ (reading the subscripts modulo l), and similarly $c_i = c_{e+i}$ for all i. Thus $\mathbf{a} = (0^{c_1}, b_1 + 1, \ldots, 0^{c_e}, b_{e+1})^d$, where $l = ed$, and where $g = (\Sigma_{1 \leqslant i \leqslant e}(c_i + 1))d$, so that d divides g also (or no such $\mathbf{a}$ exists). The number of choices of $c_1, \ldots, c_e$ is $c(l/d, g/d)$, and these determine the rest of the c_i. Similarly, the number of choices of $b_1, \ldots, b_e$ is $c(l/d, g/d)$. For this given value m, d is $l/(l, m)$. Further, for any divisor d of (g, l) there are (by definition of ϕ) exactly $\phi(d)$ integers u such that $1 \leqslant u \leqslant d$ and $(u, d) = 1$, and hence there are exactly $\phi(d)$ integers $m(= lu/d)$ such that $1 \leqslant m \leqslant l$ and $(m, l) = l/d$. Hence, using (9), we obtain

$$\sum_{1 \leqslant m \leqslant l} F(\sigma^{*m}) = \sum_{d \mid (g,l)} \phi(d) \binom{(g/d) - 1}{(l/d) - 1}^2. \tag{10}$$

Case 2. $h = \sigma^{*m}\tau^*$, where $1 \leqslant m \leqslant l$. Here $\sigma^{*m}\tau^*\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_{l-m+i} = c_i \quad \text{and} \quad c_{l-m+i} = b_i.$$

As in case 1, setting $e = (l, m)$ and $l = de$, we find that this is true if and only if, for all i, $b_{e+i} = c_i$ and $c_{e+i} = b_i$. Thus the b_i completely determine the c_i, and, for all i, $b_{2e+i} = b_i$. If d is odd, then a similar argument yields $b_{e+i} = b_i$ for all i, while if d is even, then we do not have this extra restriction. Thus, if d is odd, then $F(h)$ is $c(l/d, g/d)$ if d divides g and is zero otherwise (by a similar argument to Case 1). If d is even, then $F(h)$ is the number of choices of $b_1, \ldots, b_{2e}$ with $\Sigma_{1 \leqslant i \leqslant 2e}(b_i + 1) = 2g/d$, namely, $c(2l/d, 2g/d)$ if $d/2$ divides g and zero otherwise. So that we can isolate the even and odd divisors of l, we set $l = 2^u v$, where $u \geqslant 0$ and where v is odd. Then we have

$$\sum_{1 \leqslant m \leqslant l} F(\sigma^{*m}\tau^*) = \sum_{d \mid (g,v)} \phi(d) \binom{(g/d) - 1}{(l/d) - 1} + \delta_1 \sum_{d \mid (g, l/2)} \phi(2d) \binom{(g/d) - 1}{(l/d) - 1}, \tag{11}$$

where δ_1 is 0 if l is odd and is 1 if l is even.

Case 3(a). $h = \rho^*$. Here $\rho^*\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_i = b_{l-i} \quad \text{and} \quad c_i = c_{l+1-i}.$$

Suppose first that l is odd and set $x = b_l$ and $y = c_{(l+1)/2}$. Then $b_1, \ldots, b_{(l-1)/2}$ and b_l determine all the b_i and, since $g = x + 1 + 2\Sigma_{1 \leqslant i \leqslant (l-1)/2}(b_i + 1)$, they can be chosen, for a given x, in $c((l-1)/2, (g - x - 1)/2)$ ways; we require $g - x - 1$ to be even. Similarly $c_1, \ldots, c_{(l-1)/2}$ and $c_{(l+1)/2}$ determine all the c_i and can be chosen, for a given y, in $c((l-1)/2, (g - y - 1)/2)$ ways; we require $g - y - 1$ to be even. Thus x, $y \in [0, g - l]$, and both have opposite parity to g, hence the same parity as $g - l$. Thus, even when $l = 3$, we have

$$F(\rho^*) = \sum \binom{(g - x - 3)/2}{(l-3)/2} \binom{(g - y - 3)/2}{(l-3)/2},$$

where the summation is over all $x,\ y \in [0, g-l]$ of the same parity as $g-l$, that is,

$$F(\rho^*) = \left\{\sum \binom{(g-x-3)/2}{(l-3)/2}\right\}^2 = \left\{\sum_{\frac{l-3}{2} \leqslant q \leqslant [\frac{g-3}{2}]} \binom{q}{(l-3)/2}\right\}^2 = \binom{[(g-1)/2]}{(l-1)/2}^2,$$

since

(12) $$\sum_{p \leqslant q \leqslant r} \binom{q}{p} = \binom{r+1}{p+1}.$$

Similarly, in the case where l is even, we set $x = b_l$ and $y = b_{l/2}$. If $l = 2$, then $c_1 = c_2 = (g-2)/2$, so that g must be even (or $F(h) = 0$), and $x = g - 2 - y \in [0, g-2]$. So we have

$$F(\rho^*) = \begin{cases} 0 & \text{if } g \text{ is odd, } l = 2, \\ (g-1) & \text{if } g \text{ is even, } l = 2. \end{cases}$$

For $l \geqslant 4$ and even, $b_1, \ldots, b_{l/2-1}$, x and y determine all the b_i and, for given x and y, can be chosen in $c(l/2 - 1, (g-x-y-2)/2)$ ways; so we require $g - x - y$ to be even. Similarly $c_1, \ldots, c_{l/2}$ determine all the c_i and can be chosen in $c(l/2, g/2)$ ways, so that g must be even. So if g is odd, we have $F(\rho^*) = 0$. If g is even, then

$$F(\rho^*) = \sum \binom{(g-x-y-4)/2}{(l-4)/2}\binom{(g-2)/2}{(l-2)/2},$$

where the summation is taken over all non-negative integers $x,\ y$ of the same parity such that $x + y \leqslant g - l$. Then, upon setting $x + y = 2z$, we see that each pair $x,\ y$ determines a unique $z \in [0, (g-l)/2]$, and each such z determines $2z + 1$ pairs $x,\ y$. Thus

$$\begin{aligned} F(\rho^*) &= \binom{(g-2)/2}{(l-2)/2} \sum_{0 \leqslant z \leqslant (g-l)/2} (2z+1) \binom{(g-4)/2 - z}{(l-4)/2} \\ &= \binom{(g-2)/2}{(l-2)/2} \left\{ (g-1) \sum \binom{(g-4)/2 - z}{(l-4)/2} \right. \\ &\qquad \left. - (l-2) \sum \binom{(g-2)/2 - z}{(l-2)/2} \right\}. \end{aligned}$$

since

$$(q+1)\binom{q}{p} = (p+1)\binom{q+1}{p+1}, \tag{13}$$

both summations ranging over z in $[0, (g-l)/2]$. Then, using (12) again, we obtain

$$\begin{aligned} F(\rho^*) &= \binom{(g-2)/2}{(l-2)/2}\left\{(g-1)\binom{(g-2)/2}{(l-2)/2} - (l-2)\binom{g/2}{l/2}\right\} \\ &= \binom{(g-2)/2}{(l-2)/2}^2 (2g/l-1). \end{aligned}$$

Putting all these results together, we have

$$F(\rho^*) = \delta_2\binom{[(g-1)/2]}{[(l-1)/2]}^2, \tag{14}$$

$$\text{where } \delta_2 = \begin{cases} 1 & \text{for } l \text{ odd,} \\ 0 & \text{for } l \text{ even and } g \text{ odd,} \\ 2g/l-1 & \text{for } l \text{ and } g \text{ both even.} \end{cases}$$

Case 3(b): $h = \rho^*\tau^*$. Here $\rho^*\tau^*\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_i = c_{l-i} \quad \text{and} \quad c_{i+1} = b_{l-i},$$

and this is true if and only if, for all i, $b_i = c_i = (g/l) - 1$, so that g must be divisible by l. Thus

$$F(\rho^*\tau^*) = \delta_3, \quad \text{where } \delta_3 = \begin{cases} 1 & \text{if } l \text{ divides } g, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Case 4(a): $h = \sigma^*\rho^*$. Here $\sigma^*\rho^*\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_{i+1} = b_{l-i} \quad \text{and} \quad c_{i+2} = c_{l-i}.$$

If l is odd, then the b_i, $i \neq (l+1)/2$, and the c_i, $i \neq 1$, occur in pairs. If l is even, then the b_i, and the c_i, $i \neq 1, (l+2)/2$, occur in pairs, and we obtain, in all cases,

$$F(\sigma^*\rho^*) = F(\rho^*). \tag{16}$$

Case 4(b): $h = \sigma^*\rho^*\tau^*$. Here $\sigma^*\rho^*\tau^*\mathbf{a} = \mathbf{a}$ if and only if, for all i,

$$b_{i+1} = c_{l-i}, \qquad c_{i+2} = b_{l-i},$$

and this is true if and only if, for all i, $b_i = c_i = (g/l) - 1$; in particular, g must be divisible by l. Thus

$$F(\sigma^*\rho^*\tau^*) = F(\rho^*\tau^*). \tag{17}$$

Putting all these results together, we find (from equations (10), (11), (14), (15), (16), (17)) that

$$\sum_{h\in H} F(h) = \sum_{d\mid(g,l)} \phi(d)\binom{(g/d)-1}{(l/d)-1}^2 + \sum_{d\mid(g,v)} \phi(d)\binom{(g/d)-1}{(l/d)-1} + \delta_1 \sum_{d\mid(g,l/2)} \phi(2d)\binom{(g/d)-1}{(l/d)-1} + l\delta_2\binom{[(g-1)/2]}{[(l-1)/2]}^2 + l\delta_3,$$

where u, v, δ_1, δ_2, δ_3 are as defined above. This completes the proof of Theorem 3.

5. Enumeration of the equivalence classes of sequential arays on $\psi(n,k)$

In the final section we complete the proof of Theorem 4 by proving Proposition 5. Theorem 4 follows immediately from this on taking $\psi = \psi(n,k)$.

PROOF OF PROPOSITION 5. As in Section 3, B and D (respectively, $B \otimes J$ and $D \otimes J$) are equivalent if and only if there exist integers a, b, c, d, e (respectively, v, w, x, y, z) such that

$$B = S_R^a S_C^b R^c T^d C^e(D), \quad \left(B \otimes J = S_R^v S_C^w R^x T^y C^z(D \otimes J)\right). \tag{18}$$

We note that

$$C(D\otimes J) = C(D)\otimes J,$$
$$T(D\otimes J) = T(D)\otimes J,$$
$$R(D\otimes J) = R(D)\otimes J,$$
$$S_R^{ag} S_C^{bg}(D\otimes J) = S_R^a S_C^b(D)\otimes J.$$

It follows immediately that the equivalence of B and D implies the equivalence of $B \otimes J$ and $D \otimes J$. Conversely, if $B \otimes J$ and $D \otimes J$ are equivalent, so that (18) holds, then, from the equations above, we may assume that $x = y = z = 0$. Hence, if we consider the form of the two matrices as Kronecker products, we see that (18) can hold only if v and w are both multiples of g, whence B and D are equivalent.

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