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The Attenuated Geodesic X-ray Transform

In Definition 5.3.3 we introduced a very general attenuated X-ray transform $I_{\mathcal{A}}$ in the context of an arbitrary non-trapping manifold (M,g) with strictly convex boundary, where $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{m\times m})$ was a matrix attenuation. In this chapter we shall focus on the scalar case $m = 1$ and in this case the attenuation will be denoted by a. We shall see that under the assumption that $a \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$ and that (M,g) is a simple surface, the attenuated X-ray transform I_a is injective on $C^\infty(M)$. Along the way we will revisit the existence of holomorphic integrating factors, but first we give a brief summary of the classical situation of the Euclidean plane.

12.1 The Attenuated X-ray Transform in the Plane

We start with a smooth function $a \in C^\infty(\mathbb{R}^2)$ with compact support contained inside the unit disk $\mathbb{D}$. For $(x,v) \in S\mathbb{R}^2$ we set

$$Da(x,v) := \int_0^\infty a(x+tv)\,dt.$$

In the classical literature the function Da is called the divergent beam X-ray transform of a at x in the direction of v. Note that if M denotes the closed unit disk, then $Da|_{SM} = u^a$, where as ever u^a denotes the unique solution to the transport problem $Xu = -a$ with $u|_{\partial_- SM} = 0$. Note also that

$$Da(x+tv,v) = Da(x,v) - \int_0^t a(x+rv)\,dr. \tag{12.1}$$

The classical attenuated X-ray transform of a compactly supported function f in $\mathbb{R}^2$ is defined using $\rho := \exp(-Da)$ as weight. It is most frequently expressed in parallel-beam geometry, using the coordinates $(s,\omega) \in \mathbb{R} \times S^1$, as

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$$R_a f(s,\omega) = \int_{-\infty}^{\infty} \exp\big(-Da\big(s\omega + t\omega^{\perp}, \omega^{\perp}\big)\big) f\big(s\omega + t\omega^{\perp}\big)\, dt. \tag{12.2}$$

Note that for $a = 0$ this reduces to the Radon transform in Section 1.1. Using (12.1) we may rewrite this as

$$\begin{aligned} R_a f(s,\omega) &= \exp\big(-Da(s\omega,\omega^{\perp})\big) \\ &\quad\times \int_{-\infty}^{\infty} \exp\left[\int_0^t a(s\omega + r\omega^{\perp})\, dr\right] f\big(s\omega + t\omega^{\perp}\big)\, dt. \end{aligned} \tag{12.3}$$

Suppose now that f is supported in the closed unit disk M. We may think of f as a function in M, and consider the (Euclidean) attenuated X-ray transform in M as in Section 5.3 given by

$$I_a f(x,v) = \int_0^{\tau(x,v)} \exp\left[\int_0^t a(x+rv)\, dr\right] f(x+tv)\, dt, \quad (x,v) \in \partial_+ SM.$$

We wish to express $R_a f$ in terms of $I_a f$. If we now introduce a map $\mathbf{h}\colon SM \to [-1,1] \times S^1$ by

$$\mathbf{h}(x,v) = \big(\langle x, v_{\perp}\rangle, v_{\perp}\big)$$

as we did in Section 9.5, then we see that $\mathbf{h}^* R_a f$ is a first integral of the geodesic flow on SM. A short computation shows that its restriction to $\partial_+ SM$ gives via (12.2) (or via (12.3))

$$\mathbf{h}^* R_a f|_{\partial_+ SM} = e^{-I_0(a)} I_a(f). \tag{12.4}$$

It follows that R_a is injective if and only if I_a is injective. Moreover, as we saw in Section 5.3 there is a connection to the transport equation: one has $I_a f = u|_{\partial_+ SM}$ where u is the solution of

$$Xu + au = -f \ \text{ in } SM, \quad u|_{\partial_- SM} = 0. \tag{12.5}$$

The literature on R_a is extensive, so we limit ourselves to giving some of the highlights and discussing them from the perspective of the present monograph. One reason for the interest in R_a is that it naturally arises in single photon emission computed tomography (SPECT). This is an imaging method in nuclear medicine, where typically a radioactive tracer material is injected into the bloodstream of the patient and one measures the gamma radiation produced by the material. The function f represents the spatial density of emitters (emitting gamma photons isotropically) and a is a linear attenuation coefficient. The function $R_a f$ measures the intensity of gamma photons at the detector in the direction of a specific line.

In our discussion we shall assume that a is known and the objective is to recover f from $R_a f$. Remarkably, even in the Euclidean plane the full resolution of the injectivity question for R_a is relatively recent and is due to Arbuzov et al. (1998). A couple of years later, Novikov (2002b) gave an explicit inversion formula based on complexifying the transport problem (12.5) and solving a scalar Riemann–Hilbert problem. Shortly after, Boman and Strömberg (2004) produced an inversion formula that applied to a larger class of attenuations, namely $a \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. An inversion formula in fan-beam coordinates for the unit disk is provided in Kazantsev and Bukhgeim (2007). For an exposition of some these developments we refer to Finch (2003). We remark that in dimensions $n \geq 3$ the problem of recovering f from its Euclidean attenuated X-ray transform is formally overdetermined and can be reduced to inversion on small two-dimensional slices, see e.g. Markoe and Quinto (1985); Ilmavirta (2016).

In the two-dimensional results above, holomorphic integrating factors for the attenuation a play a prominent role. As we explained in Section 9.5.3 these are easy to come by in the Euclidean case, but for an arbitrary simple surface one needs to deploy some microlocal tools. In Proposition 10.1.2 we have already produced holomorphic and antiholomorphic integrating factors for any attenuation $a \in \Omega_{-1} \oplus \Omega_1$ on a simple surface. Below we shall extend this result to attenuations $a \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. This result will allow us to invert the attenuated geodesic X-ray transform.

12.2 Injectivity Results for Scalar Attenuations

We begin with definitions. Let (M, g) be a compact non-trapping surface with strictly convex boundary, and let $\mathcal{A} \in C^\infty(SM)$ be a general attenuation. In this chapter, the attenuation $\mathcal{A}$ will always be scalar and we will write $\mathcal{A} = a$ to emphasize this. Recall that in Section 5.3 we introduced the attenuated X-ray transform of $f \in C^\infty(SM)$ as

$$I_a f = u^f|_{\partial_+ SM},$$

where u^f is the solution of

$$Xu + au = -f \text{ in } SM, \qquad u|_{\partial_- SM} = 0.$$

Noting that $X(e^{-u^a}) = ae^{-u^a}$, we see that the previous equation is equivalent with

$$X\big(e^{-u^a} u\big) = -e^{-u^a} f \text{ in } SM, \qquad e^{-u^a} u|_{\partial_- SM} = 0.$$

A short computation shows that in the scalar case $I_a f$ has the explicit formula

$$I_a f(x, v) = \int_0^{\tau(x,v)} \exp\left[\int_0^t a(\varphi_s(x, v))\, ds\right] f(\varphi_t(x, v))\, dt$$

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for $(x, v) \in \partial_+ SM$.

We will mostly be interested in the case where $f \in C^\infty(M)$ (i.e. f is a 0-tensor).

Definition 12.2.1 If $a \in C^\infty(SM)$, the attenuated geodesic X-ray transform on 0-tensors is defined by

$$I_{a,0} \colon C^\infty(M) \to C^\infty(\partial_+ SM), \quad I_{a,0} f := I_a(\ell_0 f).$$

As discussed in Section 12.4, there are counterexamples showing that $I_{a,0}$ is not injective when $a \in C^\infty(SM)$ is arbitrary. However, injectivity will hold in the important special case where $a \in C^\infty(M)$, or more generally when a has the special form

$$a(x, v) = h(x) + \theta_x(v),$$

where $h \in C^\infty(M, \mathbb{C})$ is a function and θ is a smooth complex-valued 1-form, which we identify with the function $\theta_x(v)$ on SM. Since we are working in two dimensions, we may equivalently say that we will consider attenuations of the form

$$a = a_{-1} + a_0 + a_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1.$$

We first consider the case $a_0 = 0$ (i.e. a is purely a 1-form). In this setting we can prove a fairly general result.

Theorem 12.2.2 *Let (M, g) be a compact non-trapping surface with strictly convex boundary and I_0^* surjective. Let θ be any smooth complex-valued 1-form. Then $I_{\theta,0}$ is injective.*

Proof Suppose that $f \in C^\infty(M)$ and $I_{\theta,0} f = 0$. By Theorem 5.3.6 there is a smooth function u such that $Xu + \theta u = -f$ and $u|_{\partial SM} = 0$. Since $X + \theta$ maps even (odd) functions to odd (even) and $f \in \Omega_0$ we may assume without loss of generality that u is odd.

Using Proposition 10.1.2 we know that there exists w holomorphic and even with $Xw = \theta$. Thus we have

$$X\big(e^w u\big) = e^w((Xw)u + Xu) = -e^w f. \tag{12.6}$$

Note that $e^w u$ is odd and consider

$$q := \sum_{-\infty}^{-1} \big(e^w u\big)_k.$$

Since $e^w f$ is holomorphic, (12.6) gives

$$Xq = \eta_+ q_{-1} \in \Omega_0.$$

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But $q|_{\partial SM} = 0$ since $u|_{\partial SM} = 0$, hence injectivity of I_0 gives $q = 0$ (see Lemma 10.2.2). This means that $e^w u$ is holomorphic and thus u is holomorphic. Using Proposition 10.1.2 again but with $\tilde{w}$ antiholomorphic, we deduce that u is also antiholomorphic. Since we assumed u odd we must have $u = 0$ and thus $f = 0$ as claimed. □

This result has the following important corollary on the existence of solutions of transport equations with prescribed zeroth Fourier mode (the case $\theta = 0$ was proved in Theorem 8.2.2).

Corollary 12.2.3 *Let (M,g) be a simple surface and let θ be a smooth complex-valued 1-form. Then, given $f \in C^\infty(M,\mathbb{C})$ there exists $u \in C^\infty(SM,\mathbb{C})$ such that*

$$\begin{cases} Xu + \theta u = 0, \\ u_0 = f. \end{cases}$$

Proof Consider any smooth function $\mathbb{W}\colon SM \to \mathbb{C} \setminus \{0\}$ such that $X\mathbb{W} - \theta\mathbb{W} = 0$. Then by Lemma 5.4.6 injectivity of $I_{\theta,0}$ is equivalent to injectivity of $I_{\mathbb{W},0}$. Combining Theorem 12.2.2 with Corollary 8.4.6 we deduce the existence of u when θ is replaced by $-\bar{\theta}$. Since θ was an arbitrary complex 1-form, this proves the result. □

The next theorem may be seen as the dual statement at the level of the transport equation to the injectivity of the geodesic X-ray transform on the spaces Ω_k.

Theorem 12.2.4 *Let (M,g) be a simple surface. Given $f \in \Omega_k$ there exists $u \in C^\infty(SM)$ such that*

$$\begin{cases} Xu = 0, \\ u_k = f. \end{cases}$$

Proof Let $r := e^{ik\theta} \in \Omega_k$. Then $\theta := r^{-1}X(r) \in \Omega_{-1} \oplus \Omega_1$ is a 1-form. By Corollary 12.2.3, there exists a smooth u such that $Xu + \theta u = 0$ and $u_0 = r^{-1}f \in \Omega_0$. Now observe that

$$X(ru) = r(Xu + \theta u) = 0.$$

Since $(ru)_k = ru_0 = f \in \Omega_k$, the theorem is proved. □

Armed with this theorem we can now prove the existence of holomorphic integrating factors for $a \in C^\infty(M,\mathbb{C})$.

Proposition 12.2.5 (Holomorphic integrating factors, part II) *Let (M,g) be a simple surface. Given $a \in \Omega_0$, there exists $w \in C^\infty(SM)$ such that w is*

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holomorphic and $Xw = a$. Similarly, there exists $\tilde{w} \in C^\infty(SM)$ such that $\tilde{w}$ is antiholomorphic and $X\tilde{w} = a$.

Proof We do the proof for w holomorphic; the proof for $\tilde{w}$ antiholomorphic is analogous.

First we note, as in the proof of Proposition 10.1.2, that the equation $\eta_- f_1 = a$ can always be solved. Indeed this is the case since it is equivalent to solving a $\bar{\partial}$-equation on a disk: by Lemma 6.1.8

$$\eta_- f_1 = e^{-2\lambda}\bar{\partial}\big(f e^{\lambda}\big),$$

where $f_1 = f e^{i\theta}$. Hence we just need to solve $\bar{\partial}(f e^{\lambda}) = e^{2\lambda} a$, which is always possible, e.g. by extending a as a smooth compactly supported function outside the disk and applying the Cauchy transform.

Next, using Theorem 12.2.4 there is a smooth function u such that $Xu = 0$ and $u_1 = f_1$. Now take $w = u_1 + u_3 + u_5 + \cdots$. Then $Xw = \eta_- u_1 = a$ and w is the desired holomorphic integrating factor. □

We now state the final version on the existence of holomorphic integrating factors.

Proposition 12.2.6 (Holomorphic integrating factors, final version) *Let (M, g) be a simple surface. Given $a = a_{-1} + a_0 + a_{-1} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$, there exists $w \in C^\infty(SM)$ such that w is holomorphic and $Xw = a$. Similarly, there exists $\tilde{w} \in C^\infty(SM)$ such that $\tilde{w}$ is antiholomorphic and $X\tilde{w} = a$.*

Proof This is a direct consequence of Propositions 10.1.2 and 12.2.5. □

We can now prove the main result of this section. For $a = a_0$ this was first proved in Salo and Uhlmann (2011).

Theorem 12.2.7 *Let (M, g) be a simple surface, and assume that $a = a_{-1} + a_0 + a_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. Then $I_{a,0}$ is injective.*

Proof This proof is very similar in spirit to that of Theorems 12.2.2 and 10.2.3. Suppose that $f \in C^\infty(M)$ satisfies $I_{a,0} f = 0$. By Theorem 5.3.6 there is a smooth function u such that $Xu + au = -f$ and $u|_{\partial SM} = 0$.

Using Proposition 12.2.6 we know that there exists w holomorphic with $Xw = a$. Thus we may write

$$X\big(e^w u\big) = e^w((Xw)u + Xu) = -e^w f. \tag{12.7}$$

Consider

$$q := \sum_{-\infty}^{-1} \big(e^w u\big)_k.$$

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Since $e^w f$ is holomorphic, (12.7) gives

$$Xq = \eta_+ q_{-2} + \eta_+ q_{-1} \in \Omega_{-1} \oplus \Omega_0.$$

But $q|_{\partial SM} = 0$, hence splitting into even and odd degrees, Theorem 10.2.3 gives that $q = 0$. This means that $e^w u$ is holomorphic and thus u is holomorphic. Using Proposition 12.2.6 again but with $\tilde{w}$ antiholomorphic we deduce that u is also antiholomorphic. Hence $u = u_0$. To complete the proof we need to show that u_0 also vanishes (and hence $f = 0$ as well).

Going back to the transport equation $Xu + au = -f$ we see that if we focus on degree -1 we have $\eta_- u_0 + a_{-1} u_0 = 0$ with $u_0|_{\partial M} = 0$. Choose some $b \in \Omega_0$ satisfying $\eta_- b = a_{-1}$. Then

$$\eta_-\big(e^b u_0\big) = 0,$$

and $e^b u_0$ is a holomorphic function on M that vanishes on the boundary, so it must be zero everywhere. □

Exercise 12.2.8 Let (M, g) be a simple surface and let $a = a_{-1} + a_0 + a_1 \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. Establish the following tensor tomography result with attenuation a: let $u \in C^\infty(SM)$ be such that

$$Xu + au = f, \qquad u|_{\partial SM} = 0.$$

Suppose $f_k = 0$ for $|k| \geq m + 1$ for some $m \geq 0$. Then $u_k = 0$ for $|k| \geq m$ (when $m = 0$, this means $u = f = 0$).

12.3 Surjectivity of $I_\perp^*$

There is another application of Theorem 12.2.4 that was already used for the characterization of the range of I_0 in the case of simple surfaces in Theorem 9.6.2.

Theorem 12.3.1 *Let (M, g) be a simple surface. Then the operator*

$$I_\perp^* \colon C_\alpha^\infty(\partial_+ SM) \to C^\infty(M)$$

is surjective.

Proof Let us recall that $I_\perp^* h = -2\pi (X_\perp h^\sharp)_0$ for $h \in C_\alpha^\infty(\partial_+ SM)$ (cf. (9.14)). Given $f \in C^\infty(M)$, consider functions $w_{\pm 1} \in \Omega_{\pm 1}$ solving (as we have done in the proof of Proposition 12.2.5):

$$\eta_- w_1 = -f/4\pi i, \quad \eta_+ w_{-1} = f/4\pi i. \tag{12.8}$$

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By Theorem 12.2.4 there are odd functions $p, q \in C^\infty(SM)$ such that $Xp = Xq = 0$ and $p_{-1} = w_{-1}$, $q_1 = w_1$. Consider the function

$$w := \sum_{-\infty}^{-1} p_k + \sum_{1}^{\infty} q_k.$$

By (12.8) we have $Xw = 0$. Let $h := w|_{\partial_+ SM} \in C_\alpha^\infty(\partial_+ SM)$. We claim that $I_\perp^* h = f$. Indeed using (12.8) again,

$$I_\perp^* h = -2\pi(X_\perp w)_0 = -2\pi i(\eta_- w_1 - \eta_+ w_{-1}) = f/2 + f/2 = f$$

as desired. □

12.4 Discussion on General Weights

Theorem 12.2.7 prompts a natural question: is it possible to prove injectivity of $I_{a,0}$ for a more general a? What would happen if we just took an arbitrary $a \in C^\infty(SM)$?

It turns out that for an arbitrary attenuation $a \in C^\infty(SM)$, injectivity of $I_{a,0}$ is no longer true even in the Euclidean case. Recall that by Lemma 5.4.6 the injectivity of $I_{a,0}$, where $a \in C^\infty(SM)$ is a general attenuation, is equivalent to the injectivity of the weighted X-ray transform $I_{\rho,0}$ for any smooth weight $\rho\colon SM \to \mathbb{C} \setminus \{0\}$ satisfying $X\rho - a\rho = 0$.

In Boman (1993), an example is given of $\rho \in C^\infty(S\mathbb{R}^2)$ with $\rho > 0$ and f with compact support in $\mathbb{R}^2$ such that $I_\rho(f) = 0$. If the weight ρ is real analytic, injectivity is known, cf. Boman and Quinto (1987). However, as of today there is no complete characterization of the set of weights for which injectivity of I_ρ holds. Novikov (2014) considers weights ρ that have a finite vertical Fourier expansion, namely $\rho \in \oplus_{-N}^{N} \Omega_k$, and shows injectivity of I_ρ on compactly supported functions in the plane under additional assumptions on ρ.

With this in mind we can now state the following open problem for simple surfaces.

Open problem. Let (M, g) be a simple surface and let $a \in \oplus_{-N}^{N} \Omega_k$ be an attenuation with finite vertical Fourier expansion. Is it true that $I_{a,0}$ is injective?