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ON THE 4-DIMENSIONAL POINCARÈ CONJECTURE FOR MANIFOLDS WITH 2-DIMENSIONAL SPINES

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We shall work in the piecewise-linear category, so that all manifolds and subsets thereof, as well as all maps are assumed to be piecewise-linear. If M is a manifold, denote by $\#_k M$ the k-fold connected sum of copies of M and by $2M$ the double of M, that is the manifold obtained by sewing two copies of M together by the identity map on their boundaries. $P \subset M$ is pointlike in M if $M \sim P$ is homeomorphic with the complement of a point. B^n denotes an n-ball. Our principal theorems are the following.

THEOREM. *If M^4 is a regular neighborhood of a contractible 2-complex then M^4 can be embedded in $\#_k (S^2 \times S^2)$ for some k.*

See [6] for a different proof of this theorem for differentiable manifolds.

THEOREM 2. *If M^4 is a compact contractible 4-manifold with a 2-dimensional spine then $2M^4 \sim \mathring{B}^4$ is homeomorphic with a pointlike subset of $\#_k (S^2 \times S^2)$ for some k.*

COROLLARY 1. *If the 3-dimensional Poincarè conjecture is false then a counter-example can be constructed in $\#_k (S^2 \times S^2)$.*

COROLLARY 2. *If M^4 is a compact contractible 4-manifold with a 2-dimensional spine then for some k, $\#_k (S^2 \times S^2)$ contains a countable infinity of disjoint homeomorphic images of M^4.*

1. **Transforming group presentations.** Let $X = \{x_1, x_2, \ldots, x_k\}$ and let $F(X)$ be the free group on the alphabet X. If $\{r_1, \ldots, r_m\}$ is a set of words in $F(X)$, we define a *Q-transformation* of the set $\{r_1, \ldots, r_m\}$ to be the result of a finite sequence of transformations of the following types: for any i between 1 and m $\{r_1, \ldots, r_i, \ldots, r_m\} \to \{r_1, \ldots, r_i', \ldots, r_m\}$ where $r_i' = uv$, v being a conjugate of $r_i^{\pm 1}$ and u being a consequence of the relators $\{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m\}$. Two presentations $\phi_1 = \langle X \mid R \rangle$ and $\phi_2 = \langle Y \mid S \rangle$ are Q-equivalent if there is a Q-transformation of R onto S.

The reader should note that we are interested in presentations of groups and not just the groups presented. Thus $\langle x \mid x \rangle$, $\langle x \mid x, 1 \rangle$ and $\langle x \mid xx^{-1}x \rangle$ all present the

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trivial group but are different presentations. In this sense we think of the relators in a presentation as elements of the free semigroup on $X \cup X^{-1}$.

THEOREM 3. *If* $\phi_1=\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m\rangle$ presents the trivial group then $\phi_2=\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m, 1_1, 1_2, \ldots, 1_{k-1}\rangle$ is Q-equivalent with $\phi_3=\langle x_1, x_2, \ldots, x_k \mid x_1, x_2, \ldots, x_k, 1_1, 1_2, \ldots, 1_{m-1}\rangle$ where 1_i denotes the empty word.

Proof. The theorem follows immediately from the fact that $\{r_1, \ldots, r_m, 1\}$ is Q-equivalent with $\{r_1, \ldots, r_m, w\}$ where w is any consequence of $\{r_1, \ldots, r_m\}$.

If ϕ is a finite presentation, we define a 2-dimensional cell complex K_ϕ in the usual way; that is, if $\phi=\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m\rangle$ then K_ϕ is obtained from ϕ by taking an oriented bouquet of circles $\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_k$ and attaching disks $d_1, \ldots, d_m$ to this bouquet by the formulae given by the words $r_1, \ldots, r_m$.

The crucial link between the algebraic theorem on Q-transformations and the geometry is given by the following.

THEOREM 4. *If ϕ_1 and ϕ_2 are Q-equivalent and K_{ϕ_1} is a spine of a compact 5-manifold, M^5 then K_{ϕ_2} is also a spine of M^5.*

Proof. Let N be a regular neighborhood of the 1-skeleton of K_{ϕ_1} in M^5. M^5 is obtained from N by attaching 2-handles along simple closed curves on ∂N. The homotopy class of each of these curves is determined by the homotopy class obtained by collapsing N onto the 1-skeleton of K_{ϕ_1}. Now homotopic simple closed curves in a 4-manifold are ambient isotopic. It follows that if r is a relator of ϕ_1 and N' is a regular neighborhood of K_{ϕ_3}, where ϕ_3 is ϕ_1 with the relator r omitted, then a simple closed curve in $\partial N'$ representing r is homotopic in $\partial N'$ with a simple closed curve in $\partial N'$ representing uv, where u is any consequence of the relators of ϕ_3 and v is a conjugate of $r^{\pm 1}$. It follows that a regular neighborhood of K_{ϕ_1} is also a regular neighborhood of K_{ϕ_2} if ϕ_2 is obtained from ϕ_1 by replacing the relator r of ϕ_1 by the relator wr. But these are exactly the moves that can be accomplished by Q-transformations.

Andrews and Curtis have proved Theorem 4 implicitly in their proof of Theorem 2 of [1].

2. **Sums of manifolds with boundary.** Let M_1 and M_2 be n-manifolds with non-empty boundary and let B_1^{n-1} and B_2^{n-1} be $(n-1)$-balls in the boundaries of M_1 and M_2 respectively. We define the sum $M_1 \triangle M_2$ to be $M_1 \cup_f M_2$ where $f\colon B_1^{n-1} \to B_2^{n-1}$ is a homeomorphism. If ∂M_1 and ∂M_2 are connected, this sum is independent of the choice of B_1^{n-1}, B_2^{n-1} and depends only on the orientation class of f.

LEMMA 1. *If K_i is a spine of M_i $(i=1, 2)$* then $K_1 \vee K_2$ is a spine of $M_1 \triangle M_2$.

NOTATION. We denote by $\vee_k S^2$ the wedge of k 2-spheres.

The following lemma is an easy consequence of uniqueness of regular neighborhoods.

LEMMA 2. *If* $f: \vee_k S^2 \to E^n$ *is a p.l. embedding and* $N^n(f(\vee_k S^2))$ *is a regular* neighborhood of $f(\vee_k S^2)$ *in* E^n, $n \geq 5$, *then* $N^n(f(\vee_k S^2))$ *is homeomorphic with* $\triangle_k(S^2 \times B^{n-2})$. ($\triangle_k (S^2 \times B^{n-2})$ *denotes the k-fold* $\triangle$ *sum.*)

3. **Proof of Theorems 1 and 2 and their corollaries.** Let M^4 be a compact contractible 4-manifold with a 2-dimensional spine K_{ϕ_1}. Let $Q = M^4 \times I$, then $M^4 \subset \partial Q$. Suppose $\phi_1 = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle$ presents the trivial group. Then by Theorem 3, $\phi_2 = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m, 1_1, \ldots, 1_{k-1} \rangle$ is Q-equivalent with $\phi_3 = \langle x_1, \ldots, x_k \mid x_1, \ldots, x_k, 1_1, \ldots, 1_{m-1} \rangle$. Now note that K_{ϕ_2} is a spine of $Q \triangle N^5(\vee_{k-1} S^2)$. It follows from Theorem 4 that K_{ϕ_3} is a spine of $Q \triangle N^5(\vee_{k-1} S^2)$. Collapsing disks, we see that $Q \triangle N^5(\vee_{k-1} S^2)$ is a regular neighborhood of some embedding $f: \vee_{k-1} S^2 \to \mathrm{Int}(Q \triangle N^5(\vee_{k-1} S^2))$. However, since Q is contractible, $Q \triangle N^5(\vee_{k-1} S^2)$. can be embedded in E^5. From this it follows that $Q \triangle N^5(\vee_{k-1} S^2)$ is a homeomorphic with a regular neighborhood of $f(\vee_{k-1} S^2)$ in $N^5(\vee_{k-1} S^2) \subset E^5$. By Lemma 2, $Q \triangle N^5(\vee_{k-1} S^2)$ is homeomorphic with $N^5(\vee_{k-1} S^2)$.

We see that M_4 can be embedded in $\partial N^5(\vee_{k-1} S^2)$. Now note that $N^5(\vee_{m-1} S^2) = N^5(\vee_{m-2} S^2) \triangle N^5(\vee_1 S^2)$ so that $\partial N^5(\vee_{m-1} S^2) = \#_{m-1} (S^2 \times S^2)$. To prove Theorem 2 note that if M^4 is contractible and K_{ϕ_1} is its spine, then ϕ_1 must have the same number of generators as relators (if there were more relators than generators then $H_2(K_{\phi_1}) \neq 0$). It follows that $Q \triangle N^5(\vee_{m-1} S^2) = N^5(\vee_{m-1} S^2)$; thus $2M^4 \# (\#_{m-1}(S^2 \times S^2)) = \#_{m-1} (S^2 \times S^2)$. This shows that the complement of $2M^4 \sim \mathring{B}^4$ in $2M^4 \# (\#_{m-1} (S^2 \times S^2))$ is homeomorphic with the complement of a ball in $\#_{m-1} (S^2 \times S^2)$ and so $2M^4 \sim \mathring{B}^4$ is pointlike.

Proof of Corollary 1. Let P^3 be a contractible 3-manifold with 2-sphere boundary. Then $P^3 \times I$ is a contractible 4-manifold with a 2-dimensional spine, thus $2(P^3 \times I) \sim \mathring{B}^4$ can be embedded in $\#_k (S^2 \times S^2)$. Now $P^3 \subset \mathrm{Bd}(P^3 \times I) \subset 2(P^3 \times I) \sim \mathring{B}^4 \subset \#_k (S^2 \times S^2)$.

Proof of Corollary 2. This follows from Theorem 2 and Lemma 8 of [3].

4. **General comments.** We would have a cross category version of the Poincarè conjecture for 4-manifolds with 2-dimensional spines if we knew the following

CONJECTURE 1. A pointlike subset of $\#_k (S^2 \times S^2)$ is cellular for every k.

From this conjecture would follow:

1. If M^4 is a homotopy 4-sphere with a 2-dimensional spine then M^4 is 4-sphere.
2. A homotopy 3-ball can be embedded in E^4.
3. If the 3-dimensional Poincarè conjecture is false then a counterexample can be constructed in E^4.
4. A regular neighborhood of a contractible 2-complex in a 5-manifold is a topological 5-ball.

The author has patterned this paper after the paper of Andrews and Curtis [1] that was his original source on the subject of Q-transformations. Andrews and

Curtis prove consequences 1-4 of Conjecture 1 based on the conjecture that every presentation of the trivial group is Q-equivalent with an obviously trivial presentation of the form $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n, 1, \ldots, 1\rangle$. This conjecture seems quite difficult to handle and may well be false. Although the author and B. Levinger have devoted much effort to it they have not been able to show that the presentation $\langle a, b \mid a^{-3}b^{-1}a^2b, b^{-3}a^{-1}b^2a\rangle$ is Q-equivalent with the trivial presentation $\langle a, b \mid a, b\rangle$.

Implicit in Andrews' and Curtis' paper is the following theorem.

THEOREM 5. *If ϕ_1 is Q-equivalent with the obviously trivial presentation and K_{ϕ_1} is the spine of a contractible 5-manifold N^5 then N^5 is a combinatorial 5-ball.*

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