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THE VOLUMES OF SIMPLICES, OR, FIND THE PENGUIN

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Abstract

Given the equations for the $n+1$ hyperplanes that bound an n-simplex in $\mathbf{R}^n$, simple formulae are derived for the contents of the $n-r$ simplices $(0 \leq r < n)$ embedded in it. For example, when $n = 3$, the formulae include the volume of the tetrahedron, the areas of its faces and the lengths of its edges.

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1. Introduction

Imagine you have fitted a radio-transmitter to a penguin. The bird is now at sea and you wish to locate it, so you position observers on the shore with tracking equipment. At a pre-arranged time the observers get a directional fix on it. Unfortunately, due to measurement errors two 'fixes' don't always meet in a 'penguin'. You therefore take three fixes and make the optimistic assumption that the bird lies in the triangle they form. The probability that this is so is at most $1/4$, under plausible assumptions (see [4]). Nevertheless, the area of the triangle gives an estimate of the sizes of the errors in the measuring process. Helping the zoologists to find the area when the lines were given, Watterson derived the following neat formula: the area of the

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triangle formed by the three lines $a_{i0} + a_{i1}x + a_{i2}y = 0$, $i = 0, 1, 2$, is

$$\frac{\pm\frac{1}{2}\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix}^2}{\begin{vmatrix} a_{01} & a_{02} \\ a_{11} & a_{12} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{01} & a_{02} \end{vmatrix}} \tag{1}$$

where we have used $\pm$ to avoid confusion between modulus signs and determinants.

This formula appeared to be new, but in correspondence Dr. Dennis De Turck has informed us that this formula and a similar one for the volume of a tetrahedron were derived by Joachimsthal [3] in 1850! In this paper we cover both cases by deriving the formula for the volume of an n-simplex. In Section 3 we generalise this formula further by deriving the volume of the $n - r$ simplices embedded in it (for example the lengths of edges and areas of faces of a tetrahedron).

2. The volume of an n-dimensional simplex

In $\mathbb{R}^n$, in general, $n + 1$ hyperplanes enclose an n-simplex. The simplex, S, is non-degenerate when it cannot be contained in an $n - 1$ dimensional subspace of $\mathbb{R}^n$.

THEOREM 1. *Let the n-simplex S be bounded by the $n + 1$ hyperplanes*

$$S_i\colon a_{i0} + \sum_{j=1}^{n} a_{ij}x_j = 0, \qquad i = 0, 1, \dots, n,$$

and let A° be the $(n + 1) \times (n + 1)$ matrix with elements a_{ij}, $0 \le i, j \le n$. Then the volume of S is given by

$$\mathrm{Vol}(S) = \frac{\pm|A^\circ|^n}{n!\prod_{i=0}^{n} A_{i0}^\circ}, \tag{2}$$

where A_{i0}° is the cofactor of a_{i0} in A°.

This is the generalisation of the formula (1) for the triangle.

PROOF. The idea is to transform simplex S into simplex S' that has its vertices located on the co-ordinate axes. The volume of such simplices is well known.

Consider the transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$T\colon \mathbf{x}' = A\mathbf{x} + \mathbf{a}_0$$

where A is the $n \times n$ matrix with elements a_{ij}, $1 \leq i, j \leq n$, and $\mathbf{a}_0^T = (a_{10}, \ldots, a_{n0})$. It can be easily seen that T transforms hyperplanes S_i into hyperplanes S_i' with equations $x_i' = 0$, $i = 1, \ldots, n$.

The simplex S is transformed into the simplex S' which is bounded by the n co-ordinate hyperplanes S_i', $i = 1, \ldots, n$ and S_0'. We have

$$\mathrm{Vol}(S) = |J| \mathrm{Vol}(S')$$

where J is the Jacobian of the transformation. $\mathrm{Vol}(S')$ is easy to determine. It is known that the volume of the simplex with vertices $(0, 0, \ldots, 0), (1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is $1/n!$ (see [2]).

Denote the intercept of S_0' with each x_i'-axis by c_i; then a simple transformation of co-ordinates $y_i = x_i'/c_i$ gives

$$\text{(3)} \qquad \mathrm{Vol}(S') = \pm \int_{S'} 1 \, dx_1' \cdots dx_n' = \pm \prod_{i=1}^{n} c_i / n!.$$

To find the c_i's, let $P_i = \bigcap_{k \neq i} S_k$, $i = 0, 1, \ldots, n$, be the vertex which lies in all the hyperplanes except S_i. If we designate the position vector of P_i as $\mathbf{x}_i$ with $\mathbf{x}_i^T = (x_{i1}, \ldots, x_{in})$, then by Cramer's rule

$$\text{(4)} \qquad x_{ij} = A_{ij}^\circ / A_{i0}^\circ,$$

where A_{ij}° is the co-factor of a_{ij} in A°. Now T maps P_i into P_i' with position vector $\mathbf{x}_i'$ and since the n hyperplanes $S_1', \ldots, S_n'$ have equations $x_i' = 0$ and since $P_i' = \bigcap_{j \neq i} S_j'$, it follows that

$$x_{ij}' = 0, \qquad j \neq i,$$

and

$$x_{ii}' = \sum_{k=1}^{n} a_{ik} x_{ik} + a_{i0}.$$

Substituting from (4) we have

$$\text{(5)} \qquad c_i = x_{ii}' = \sum_{k=1}^{n} a_{ik} \frac{A_{ik}^\circ}{A_{i0}^\circ} + a_{i0} = \frac{|A^\circ|}{A_{i0}^\circ}.$$

Because the Jacobian of the transformation T is $|A|^{-1} = (A_{00}^\circ)^{-1}$, we see from (3) and (5) that (2) holds.

3. Volume of an $n - r$ simplex in n dimensions

We are now in a position to determine the volume of any $n - r$ simplex formed by the intersection of r of the hyperplanes with S.

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THEOREM 2. *Consider the* $n+1$ *hyperplanes*

$$S_i\colon a_{i0} + \sum_{j=1}^{n} a_{ij}x_j = 0, \qquad i = 0, 1, \dots, n.$$

The volume of the $n-r$ *simplex, lying in* $\bigcap_{j=1}^{r} S_j$ *and bounded by* $S_0, S_{r+1}, \dots, S_n$, *is*

$$\frac{\pm\sqrt{|B_{n-r}^T B_{n-r}|}}{(n-r)!\prod_{i=r+1}^{n} b_{0i}^{\circ}}. \tag{6}$$

where B_{n-r} *is the matrix consisting of the last* $n-r$ *columns of* B, *for* $B = A^{-1}$.

PROOF. The idea is similar to that in the proof of Theorem 1. Denote by $S_{12\dots r}$ the $n-r$ simplex lying in $\bigcap_{i=1}^{r} S_i$ and bounded by $S_0, S_{r+1}, \dots, S_n$. Under the inverse transformation T^{-1}, $S_{12\dots r}$ is the image of the simplex $S'_{12\dots r}$ which has its vertices on the co-ordinate axes. Therefore we know that its volume is given by

$$\operatorname{Vol}(S'_{12\dots r}) = \prod_{i=r+1}^{n} c_i/(n-r)!. \tag{7}$$

The volume of $S_{12\dots r}$ is given by the surface integral

$$\int_{S_{12\dots r}} 1\,dS = \pm\int_{S'_{12\dots r}} W\,dx'_{r+1}\dots dx'_n, \tag{8}$$

where W^2 is the Gram determinant (see Courant and John [1, page 645]),

$$W^2 = \left|\sum_{k=1}^{n} \frac{\partial x_k}{\partial x'_i}\frac{\partial x_k}{\partial x'_j}\right|_{i,j=1,\dots,n-r}.$$

To evaluate W, we look at the inverse transformation of T. If $B = A^{-1}$, then T^{-1} is given by

$$T^{-1}\colon \mathbf{x} = B\mathbf{x}' - B\mathbf{a}_0.$$

But for $\mathbf{x}' \in S'_{12\dots r}$, $x'_i = 0$, $i = 1, 2, \dots, r$, so the inverse transformation is

$$\mathbf{x} = B_{n-r}\mathbf{x}'_{n-r} - B\mathbf{a}_0, \tag{9}$$

and $\mathbf{x}'_{n-r}$ is the last $n-r$ rows of $\mathbf{x}'$. Equation (9) represents $x_1, \dots, x_n$ as linear functions of $x'_{r+1}, \dots, x'_n$. It is easily seen from (9) that

$$\frac{\partial x_i}{\partial x'_j} = b_{ij}, \qquad i = 1, \dots, n, \qquad j = r+1, \dots, n,$$

so that

$$W^2 = |B_{n-r}^T B_{n-r}|$$

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and is a constant. Substituting this into (8) and using (5) and (7), we obtain (6).

REMARK 1. The above formula can be written in terms of the matrices A° and A as

$$\frac{\pm|A_r^T A_r|^{1/2}|A^\circ|^{n-r}}{(n-r)!|A|\prod_{i=r+1}^{n} A_{i0}^\circ}.$$

This follows from the identity

$$|B_{n-r}^T B_{n-r}| = |A_r^T A_r|/|A|^2$$

where A_r is the matrix consisting of the first r columns of A. If $r < n/2$, the second form is probably easier to calculate.

REMARK 2. Formula (1) has various elementary proofs which use elementary linear algebra and geometry. Formulae (2) and (6) do not seem to have a simple elementary proof just by using linear algebra. However, as we have seen, there are quite simple proofs of these if one is prepared to use some basic calculus of multiple integration. A reflection on the proofs shows that they are simple due to the interplay of multiple integration and elementary linear algebra.

For completeness we give a result that allows us to calculate the volume of a simplex when its vertices are given.

THEOREM 3. *If* $\mathbf{y}_0, \ldots, \mathbf{y}_r$ *are* $r+1$ *points in* $\mathbf{R}^n$ *that form an r-dimensional simplex S then*

$$\text{(10)} \qquad \operatorname{Vol}(S) = \frac{1}{r!}\left(\sum_{j_1,\ldots,j_r}\begin{vmatrix} y_{0j_1} & \cdots\, y_{0j_r} & 1 \\ & \vdots & \vdots \\ y_{rj_1} & \cdots\, y_{rj_r} & 1\end{vmatrix}^2\right)^{1/2}$$

where y_{ij} *stands for the jth co-ordinate of* $\mathbf{y}_i$ *and the sum is taken over all* $\binom{n}{r}$ *possible choices of co-ordinates* $j_1, \ldots, j_r$ *out of n.*

PROOF. A short and simple proof of (10) is obtained by using our basic idea of considering S as the image under linear transformation of the r-dimensional simplex S' formed by unit vectors in $\mathbf{R}^r$. This transformation is given by

$$V^T : \mathbf{y} = V^T\mathbf{u} + \mathbf{y}_0, \qquad \mathbf{u} \in \mathbf{R}^r,\ \mathbf{y} \in \mathbf{R}^n,$$

where V^T is the transpose of $V = (y_{ij} - y_{0j})$, $i = 1, \ldots, r, j = 1, \ldots, n$. Application of the formula for the volume of an r-dimensional body in $\mathbf{R}^n$ given on page 453 of [1] yields the result.

If one does not like to use the quoted result for the linear case, then a direct, however, more complicated proof of (10) may perhaps be constructed

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by looking at all the projections of S into all the r-dimensional affine flats spanned by subsets of the major axes.

NOTE. The geometrical interpretation of Theorem 3 is that the square of the volume of S is the sum of squares of the volumes of its projections onto the coordinate planes. This is the n-dimensional generalisation of Pythagoras' Theorem.

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