# ON THE POTENTIAL THEORY OF COCLOSED HARMONIC FORMS

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**1. Introduction.** The potential theory of real harmonic tensors, which was first studied by Hodge **(5)**, offers a variety of problems by no means all of which have yet been examined. In the present paper there are formulated the solutions of some boundary value problems for the Poisson equations associated with coclosed harmonic forms. These problems include as special cases a number of previous results on coclosed harmonic forms and harmonic fields. In turn, however, they are themselves special cases of the mixed boundary value problem for harmonic forms which was studied in a preceding paper **(3)**. The method used with these boundary value theorems for coclosed harmonic forms is also applied to the differential equations of harmonic fields and of a new special class of harmonic forms which will be called biharmonic fields.

The notations and results of the theory of harmonic $p$-tensors as developed in **(4)** or **(8)** will be assumed known to the reader. The theory of the mixed boundary value problem for $\Delta\phi = 0$ **(3)** will also be used. We consider throughout a finite positive definite Rumannian manifold $M$ of class $C^\infty$, and dimension $N$ having a smooth boundary $B$ of dimension $N - 1$. The data of the various problems will also be supposed sufficiently differentiable.

**2. The mixed problem for Poisson's equation.** Since we shall need to apply the mixed problem to non-homogeneous equations, we formulate here the necessary and sufficient condition for the existence of a solution of the mixed problems in such cases. We recall that $\Delta = d\delta + \delta d$.

LEMMA 1. *There exists a solution $\phi$ of $\Delta\phi = \rho$, in $M$, with given values of $t\phi$ and $t\delta\phi$ on $B$, if and only if*

$$(\rho, \tau)_M - \int_B \delta\phi \wedge *\tau = 0 \tag{2.1}$$

*for every eigenform $\tau$ of the eigenspace $K = \{\tau \mid d\tau = 0, \delta\tau = 0, t\tau = 0\}$.*

*Proof.* The necessary and sufficient condition **(9)**

$$(\rho, \tau) = 0, \qquad d\tau = 0, \delta\tau = 0, t\tau = 0, n\tau = 0, \tag{2.2}$$

for the solvability of $\Delta\phi = \rho$ without any boundary conditions is satisfied in view of (2.1). Therefore a form $\phi_1$ with Laplacian $\Delta\phi_1 = \rho$ in $M$ exists. We now seek a harmonic form $\psi$ which satisfies the two boundary conditions

$$t\psi = t\phi - t\phi_1, \; t\delta\psi = t\delta\phi - t\delta\phi_1.$$

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By the theorem **(3)**, such a form $\psi$ exists if and only if for all $\tau \in K$ we have

$$\int_B t\delta(\phi - \phi_1) \wedge *\tau = 0. \tag{2.3}$$

But since $t\tau = 0$, we see from Green's formula that

$$\int_B t\delta\phi_1 \wedge *\tau = \int_B \delta\phi_1 \wedge *\tau - \tau \wedge *d\phi,$$
$$= - D(\phi_1, \tau) + (\tau, \Delta\phi_1) = (\tau, \rho),$$

and (2.3) is therefore equivalent to (2.1). This proves the lemma.

By adding if necessary elements of the eigenspace $K$ to the solution $\phi$ of the mixed boundary value problem we can arrange that $(\phi, \tau) = 0$, $\tau \in K$. The solution is then unique. For convenience this will be done in the following work. We also recall that the dimension of $K$ is the relative Betti number $R_p(M, B)$ of $M$ modulo its boundary $B$, and that this is equal to the absolute Betti number $R_q(M)$ of the complementary dimension $q = N - p$.

Let $g_K(x, y)$ be the Green's form of degree $p$ for the mixed problem, as defined in **(3)**. Then the solution of the above Poisson equation is given by

$$(g_K, \rho) - \int_B t\phi \wedge *dg_K - \int_B t\delta\phi \wedge *g_K. \tag{2.4}$$

In the boundary value problem dual to the above, we assign values of $n\phi$ and $nd\phi$. The condition of solvability for $\Delta\phi = \rho$ is in this case

$$(\rho, \tau)_M + \int_B \tau \wedge *d\phi = 0 \tag{2.5}$$

for all $\tau$ satisfying $d\tau = 0$, $\delta\tau = 0$, $n\tau = 0$. These eigenforms $\tau$ span the eigenspace $M$ of dimension $R_p(M)$.

**3. A Dirichlet problem for coclosed harmonic forms.** The differential equations satisfied by coclosed harmonic forms are $\delta d\phi = 0$ and $\delta\phi = 0$. It was shown in **(3)** and **(4)** that there exist solutions of these equations with assigned values of $t\phi$. We now study the slightly more general problem with non-homogeneous differential equations.

THEOREM I. *Let $\rho = \rho_p$ and $\sigma = \sigma_{p-1}$ be given coderived forms defined in $M$ and let $\theta = \theta_p$ be a $p$-form defined on $B$. Then there exists a unique solution $\phi$ of the equations*

$$\delta d\phi = \rho, \quad \delta\phi = \sigma, \tag{3.1}$$

*satisfying the boundary condition*

$$t\phi = \theta \tag{3.2}$$

*and the orthogonality condition $(\phi, \tau) = 0$, $\tau \in K$.*

The uniqueness follows immediately since the difference $\phi = \phi_1 - \phi_2$ of of two solutions satisfies $\Delta\phi = 0$, $t\phi = 0$, $t\delta\phi = 0$ and $(\phi, \tau) = 0$, $\tau \in K$;

and so is zero from the theory of the mixed boundary value problem for harmonic forms.

*Proof.* We now formulate the appropriate mixed boundary value problem

$$\Delta\phi = \rho + d\sigma, \quad t\phi = \theta, \quad t\delta\phi = t\sigma, \tag{3.3}$$

with $(\phi, \tau) = 0$ for $\tau \in K$. We first show that a solution of this problem exists. The orthogonality condition of Lemma 1 is, for $\tau \in K$, the vanishing of

$$(\rho + d\sigma, \tau) - \int_B \sigma \wedge *\tau$$

$$= (\rho, \tau)_M + (\sigma, \delta\tau)_M + \int_B \sigma \wedge *\tau - \int_B \sigma \wedge *\tau = (\rho, \tau)_M.$$

Since $\rho$ is coderived, $\rho = \delta\pi$ say, we find

$$(\rho, \tau) = (\delta\pi, \tau) = (\pi, d\tau) - \int_B \tau \wedge *\pi = 0,$$

since $d\tau = 0$, $t\tau = 0$. Thus the condition holds and a unique solution $\phi$ exists.

Now let $\psi = \psi_{p-1}$ be defined as

$$\psi = \delta\phi - \sigma. \tag{3.4}$$

Then from (3.3),

$$t\psi = t\delta\phi - t\sigma = 0,$$

while, since $\sigma$ is coderived, $\sigma = \delta\xi$ say, we have $\psi = \delta(\phi - \xi)$ and hence $\delta\psi = 0$. Also,

$$d\psi = d\delta\phi - d\sigma = \rho - \delta d\phi, \tag{3.5}$$

from the differential equation in (3.3). Thus

$$\delta d\psi = \delta\rho = 0$$

so that

$$N(d\psi) = (\psi, \delta d\psi)_M + \int_B \psi \wedge *d\psi = 0,$$

since $t\psi$ is zero. Hence $d\psi \equiv 0$ in $M$. This shows that the first of (3.1) is satisfied by $\phi$. Finally,

$$N(\psi) = (\psi, \delta(\phi - \xi))_M = (d\psi, \phi - \xi) - \int_B \psi \wedge *(\phi - \xi) = 0$$

since $d\psi = 0$ and $t\psi = 0$. Therefore $\psi \equiv 0$ in $M$ and (3.4) shows that the second equation of (3.1) holds also. This concludes the proof of Theorem I.

We next consider the conditions which must be satisfied by the data of the problem if $\phi$ is to have further special properties. Obviously $\phi$ is coclosed if $\sigma$ is zero. We may however ask: when is $\phi$ coderived, closed, or derived? To answer the first of these questions we require the following:

LEMMA 2. *If a coclosed form $\alpha$ on $M$ is orthogonal to the eigenspace $K$, then $\alpha$ is coderived.*

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The proof is based on the bilinear formula for closed forms on a closed manifold (**6**, p. 85). Let $F$ be the double of our finite manifold $M$ (**4**). Then, if $\alpha = \alpha_p$ is coclosed and $\beta = \beta_p$ is closed on $F$, we may write the bilinear formula

$$(\alpha, \beta)_F = \sum_{i,j=1}^{R_p(F)} \epsilon^p_{ij}\, \omega^i \nu^j. \tag{3.6}$$

Here the matrix $\epsilon^p$ is the transposed inverse of the intersection matrix $\alpha^p$ of the $p$-cycles of a fundamental base of $F$ with the $q$-cycles of a complementary fundamental base; the $\omega^i$ are the periods of $*\alpha$ on the $q$-cycles; the $\nu^i$ are the periods of $\beta$ on the $p$-cycles; and

$$q = N - p.$$

We may suppose that the cycles of the two fundamental bases have been chosen as follows. The $q$-cycles consist of $R_q(M)$ independent absolute $q$-cycles of $M$, forming a fundamental base for $M$, together with certain additional cycles of $F - M$. The $p$-cycles, dual to the above $q$-cycles of $F$, which lie all or partly in $M$ constitute a base for the relative $p$-cycles of $M$, mod $B$. To see this, we note that any relative $p$-cycle of $M$ mod $B$, together with its image in $F - M$, constitutes an absolute $p$-cycle of $F$ which is expressible as a sum of the fundamental $p$-cycles of $F$. The intersection submatrix of these $q$-cycles and relative $p$-cycles of $M$ is again non-singular.

Since $\tau_p \in K$ is closed, and $t\tau_p = 0$, $\tau$ has zero period over any cycle lying in a sufficiently small neighbourhood $B \times I$ of $B$ in $M$. Thus $\tau$ is derived in $B \times I$ (**1**); let $\tau = d\gamma_{p-1}$ there.

Let $\rho_\epsilon$ be a $C^\infty$ scalar function of a real variable $t$ which satisfies

$$\rho_\epsilon = 0 \text{ for } t < 0, \quad 0 \leqslant \rho_\epsilon \leqslant 1, \quad \rho_\epsilon = 1, \text{ for } t > \epsilon.$$

Such a function is easily constructed. Now let $x^N$ be a variable which parametrizes the interval $I$, uniformly over the neighbourhood $B \times I$, and define

$$\beta_\epsilon = \begin{cases} 0 & \text{in } F - M, \\ d(\rho_\epsilon(x^N)\, \gamma_{p-1}) & \text{in } B \times I, \\ \tau & \text{in } M - B \times I. \end{cases}$$

Then $\beta_\epsilon$ is closed, of class $C^\infty$ in $F$, and vanishes outside of $M$. Moreover $\lim_{\epsilon \to 0} \beta_\epsilon = \tau$ in $M$, while for any cycle $A_p{}^i$ of $F$,

$$\gamma^i = \lim_{\epsilon \to 0} \nu^i(\epsilon) = \lim_{\epsilon \to 0} \int_{A_p{}^i} \beta_\epsilon = \int_{R_p{}^i} \tau,$$

where $R_p{}^i$ is the relative $p$-cycle of $M$ mod $B$ which is the part of $A_p{}^i$ lying in $M$.

From (3.6) we find

$$\begin{aligned} (\alpha, \tau)_M &= \lim_{\epsilon \to 0} (\alpha, \beta_\epsilon)_F = \lim_{\epsilon \to 0} \sum_{i,j}^{R_p(F)} \epsilon^p_{ij} \omega^i \nu^j(\epsilon) \\ &= \sum_{i,j}^{R_p(M,B)} \epsilon^p_{ij} \omega^i \nu^j(0). \end{aligned}$$

Thus

$$(\alpha, \tau)_M = \sum_{i,j}^{R_p(M,B)} \epsilon^p_{ij} \omega^i \nu^j,$$

where the $\nu^j$ are the relative periods of $\tau$ on a fundamental base for $M$ mod $B$. But we can find a form $\tau \in K$ having assigned periods on these cycles. Since $(\alpha, \tau) = 0$, $\tau \in K$, we see that

$$\sum_{i,j}^{R_p(M,B)} \epsilon^p_{ij} \omega^i \nu^j = 0,$$

for arbitrary $\nu^j$. Because $\epsilon^p_{ij}$ $(i, j = 1, \ldots, R_p(M, B))$ is non-singular, we conclude that $\omega^i = 0$. Thus $*\alpha$ has zero periods on the absolute $q$-cycles of $M$ and so is a derived form **(1)**. We therefore conclude that $\alpha$ is coderived, as stated in the lemma.

Taking $\sigma = 0$ in Theorem I and applying the lemma, we have

COROLLARY Ia. *When $\sigma = 0$ the solution $\phi$ is coderived.*

In order that $\phi$ be closed it is clearly necessary to have $\rho = 0$. Also $\theta = t\phi$ must be admissible in the sense of Tucker, that is,

$$d_B \theta = 0 \text{ and } \int_{bR^i_{p+1}} \theta = 0,$$

for every relative $(p + 1)$-cycle $R^i_{p+1}$. Conversely, these conditions are sufficient, because, if they are satisfied, we see that $t\phi = \theta = t\xi$, say, where $\xi$ is a form closed in $M$ **(1)**. Then, since $\delta d\phi = 0$, we have

$$\begin{aligned} N(d\phi) &= (\phi, \delta d\phi) + \int_B \phi \wedge *d\phi \\ &= \int_B \xi \wedge *d\phi = (d\xi, d\phi) - (\xi, \delta d\phi) = 0, \end{aligned}$$

and so $d\phi \equiv 0$ in $M$.

COROLLARY Ib. *The form $\phi$ is closed if and only if $\rho$ vanishes and $\theta$ is admissible.*

For $\phi$ to be a derived form it must in addition have vanishing periods on all $p$-cycles of $M$. In particular $\theta$ must be a derived $p$-form in $B$, $\theta = d\zeta$ say. The remaining conditions may then be expressed

$$\int_{R_p{}^i} \phi = \int_{bR_p{}^i} \zeta,$$

but in this formula $\phi$ itself still appears. Replacing $\phi$ by its value as given by (2.4) and (3.3) we could find the explicit conditions on the data of the problem, but these are too cumbrous to be useful.

**4. A mixed problem for coclosed harmonic forms.** The two mixed boundary value problems of **(3)** are equivalent under dualization. However, since the equations $\delta d\phi = 0$, $\delta\phi = 0$ of coclosed harmonic forms are not self-dual, the

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two lead to different problems for coclosed harmonic forms. The second of these is

THEOREM II. *Let $\rho = \rho_p$ and $\sigma = \sigma_{p-1}$ be two coderived forms in $M$. Let $\xi = \xi_q$ and $\eta = \eta_{q-1}$ be forms defined on $B$ such that*

$$d_B\xi = (-1)^N t{*}\sigma, \quad \int_{bR_{q+1}} \xi = (-1)^N \int_{R_{q+1}} {*}\sigma, \tag{4.1}$$

*and*

$$d_B\eta = (-1)^N t{*}\rho, \quad \int_{bR_q} \eta = (-1)^N \int_{R_q} {*}\rho. \tag{4.2}$$

*Then there exists a unique $p$-form $\phi$ satisfying the differential equations*

$$\delta d\phi = \rho, \qquad \delta\phi = \sigma, \tag{4.3}$$

*the boundary conditions*

$$t{*}\phi = \xi, \quad t{*}d\phi = \eta, \tag{4.4}$$

*and the orthogonality condition $(\phi, \tau) = 0$, $\tau \in M$.*

The conditions (4.1) and (4.2) are necessary consequences of (4.3) and (4.4); we wish to show their sufficiency.

*Proof.* Consider the $M$-problem **(3)**

$$\Delta\phi = \rho + d\sigma, \quad t{*}\phi = \xi, \quad t{*}d\phi = \eta; \quad (\phi, \tau) = 0, \ \tau \in M. \tag{4.5}$$

According to (2.5), a solution exists provided that for $\tau \in M$, the quantity

$$\begin{aligned}(\rho + d\sigma, \tau) &+ \int_B \tau \wedge {*}d\phi \\ &= (\rho, \tau) + (\sigma, \delta\tau) + \int_B \sigma \wedge {*}\tau + \int_B \tau \wedge \eta = (\rho, \tau)_M + \int_B \tau \wedge \eta\end{aligned}$$

vanishes. Since $\rho$ is coderived, $\rho = \delta\pi$ say, the condition becomes

$$0 = (\delta\pi, \tau) + \int_B \tau \wedge \eta = (\pi, d\tau) + \int_B \tau \wedge (\eta - {*}\pi).$$

The volume integral on the right vanishes since $d\tau = 0$. We shall now show that $\eta - t{*}\pi$ is an admissible tangential boundary value on $B$. Indeed,

$$d_B(\eta - t{*}\pi) = d_B\eta - td{*}\pi = (-1)^N t{*}\rho - (-1)^N t{*}\delta\pi = 0,$$

since $\rho = \delta\pi$. Likewise,

$$\begin{aligned}\int_{bR_q{}^i} (\eta - {*}\pi) &= \int_{bR_q{}^i} \eta - \int_{R_q{}^i} d{*}\pi = \int_{bR_q{}^i} \eta - (-1)^N \int_{R_q{}^i} {*}\delta\pi \\ &= \int_{bR_q{}^i} \eta - (-1)^N \int_{R_q{}^i} {*}\rho = 0,\end{aligned}$$

from (4.2). Thus $\eta - t{*}\pi = t\alpha$ where $d\alpha = 0$ in $M$. In consequence we find

$$\int_B \tau \wedge (\eta - {*}\pi) = \int_B \tau \wedge \alpha = \int_M d(\tau \wedge \alpha) = 0,$$

since $d\alpha = 0$ and $d\tau = 0$. Therefore the necessary condition is satisfied and a solution $\phi$ of (4.5) exists.

Next we define

$$\psi = \delta\phi - \sigma = \delta(\phi - \theta),$$

where $\theta$ is such that $\sigma = \delta\theta$. Then

$$d\psi = d\delta\phi - d\sigma = \rho - \delta d\rho = \delta(\pi - d\phi)$$

by the differential equation (4.5). Hence $\delta d\psi = 0$.

From (4.5), we see that

$$\begin{aligned} t{*}d\psi &= t{*}(\rho - \delta d\phi) = (-1)^N d_B\eta - t{*}\delta d\phi \\ &= (-1)^N d_B\eta - (-1)^N t d{*}d\phi \\ &= (-1)^N\{d_B\eta - d_B t{*}d\phi\} = 0. \end{aligned}$$

Hence

$$N(d\psi) = (\psi, \delta d\psi) + \int_B \psi \wedge {*}d\psi = 0,$$

and so $d\psi \equiv 0$ in $M$. Next we see that

$$N(\psi) = (\psi, \delta(\phi - \theta)) = (d\psi, \phi - \theta) - \int_B \psi \wedge {*}(\phi - \theta)$$

and the volume integral contains the vanishing factor $d\psi$. We show next that $t{*}(\phi - \theta)$ is admissible on $B$. In fact,

$$\begin{aligned} d_B t{*}(\phi - \theta) &= d_B t{*}\phi - t d{*}\theta = d_B\xi - (-1)^N t{*}\delta\theta \\ &= d_B\xi - (-1)^N t{*}\sigma = 0, \end{aligned}$$

by (4.1). Also,

$$\begin{aligned} \int_{bR_{q+1}} {*}(\phi - \theta) &= \int_{bR_{q+1}} \xi - \int_{R_{q+1}} d{*}\theta = \int_{bR_{q+1}} \xi - (-1)^N \int_{R_{q+1}} {*}\delta\theta \\ &= \int_{bR_{q+1}} \xi - (-1)^N \int_{R_{q+1}} {*}\sigma = 0, \end{aligned}$$

by the second of (4.1). Thus $t{*}(\phi - \theta)$ is admissible and so equal to $t\beta$ for some closed form $\beta$ in $M$. That is,

$$N(\psi) = - \int_B \psi \wedge {*}(\phi - \theta) = - \int_B \psi \wedge \beta = - \int_M d(\psi \wedge \beta) = 0$$

since $d\beta = 0$ and $d\psi = 0$. Hence $\psi \equiv 0$ and the two differential equations (4.3) are satisfied. This completes the proof of the theorem.

In order that $\phi$ should be coderived, it is necessary that $\sigma = 0$, that $\xi$ should be a derived form on $B$, and that the absolute periods of ${*}\phi$ should vanish. This last condition cannot be expressed without the Green's form $g_M$ which is dual to the $g_K$ of (2.4).

When $\phi$ is closed, $\rho$ and $\eta$ vanish. Conversely, if $\rho$ and $\eta$ are zero, we see from (4.3) and (4.4) that $\delta d\phi = 0$ and $t{*}d\phi = 0$. Thus

$$N(d\phi) = (\phi, \delta d\phi) + \int_B \phi \wedge {*}d\phi = 0$$

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and $\phi$ is closed. Now the dual of Lemma 2 states that a closed form orthogonal to $M$ is derived.

COROLLARY IIa. *The solution $\phi$ is derived if and only if $\rho$ and $\eta$ vanish.*

We also state as a corollary the special case when the differential equations are homogeneous. If $\rho$ and $\sigma$ are zero the conditions (4.1) and (4.2) show that $\xi$ and $\eta$ are admissible on $B$. Since $\phi$ is coclosed the periods of $*\phi$ are defined, and by adding a suitable eigenform $\tau \in M$ the relative period of $*\phi$ can be given assigned values on given relative $q$-cycles $R_q^i$. These periods will depend only on the $bB_q^i$.

COROLLARY IIb. *There exists a unique coclosed harmonic $p$-form $\phi$ such that $t*\phi$ and $t*d\phi$ have given admissible boundary values, and $*\phi$ has given periods on $R_p(M)$ independent relative $q$-cycles whose boundaries are fixed.*

**5. The Poisson equations associated with harmonic fields.** The differential equations satisfied by harmonic fields are the self-dual pair $d\phi = 0$, $\delta\phi = 0$. It has been shown that there exists a unique harmonic field having a given admissible tangential boundary value and given relative periods **(4)**. We therefore formulate the following

THEOREM III. *Let $\rho = \rho_{p+1}$ and $\sigma = \sigma_{p-1}$ be given forms, derived and coderived on $M$, respectively. Let $\xi$ be a form given on $B$ such that*

$$d_B\xi = t\rho, \quad \int_{bR^i_{p+1}} \xi = \int_{R_{p+1}} \rho \tag{5.1}$$

*for all relative $(p + 1)$-cycles. Then there exists a unique form $\phi$ satisfying the differential equations*

$$d\phi = \rho, \quad \delta\phi = \sigma, \tag{5.2}$$

*the boundary condition*

$$t\phi = \xi \tag{5.3}$$

*and the orthogonality condition $(\phi, \tau) = 0$, $\tau \in K$.*

For the proof we consider the problem

$$\Delta\phi = \delta\rho + d\sigma, \quad t\rho = \xi, \quad t\delta\phi = t\sigma; \quad (\phi, \tau) = 0, \ \tau \in K. \tag{5.4}$$

A solution exists if and only if for each $\tau \in K$,

$$(\delta\rho + d\sigma, \tau) - \int_B \sigma \wedge *\tau = 0.$$

However a simple calculation using Green's formula and the fact that $t\tau = 0$ shows that this condition is in fact satisfied. Thus a solution $\phi$ of (5.4) exists.

Again we define

$$\psi = \delta\phi - \sigma = \delta(\phi - \alpha), \tag{5.5}$$

since $\sigma$ is derived, $\sigma = \delta\alpha$ say. We find as before

$$d\psi = d\delta\phi - d\sigma = \delta\rho - \delta d\phi$$

so that $\delta d\psi = 0$. In addition, $t\psi = t\delta\phi - t\sigma = 0$, by (5.4). Thus

$$N(d\psi) = (\psi, \delta d\psi) + \int_B \psi \wedge *d\psi = 0,$$

and we find that $d\psi \equiv 0$. Again,

$$N(\psi) = (\psi, \delta(\phi - \alpha)) = (d\psi, \phi - \alpha) - \int_B \psi \wedge *(\phi - \alpha) = 0,$$

so that $\psi$ vanishes identically. This shows that the second of (5.2) holds.

Let $\chi = d\phi - \rho$; then $\delta\chi = \delta d\phi - \delta\rho = d\sigma - d\delta\phi = 0$, since $\delta\phi = \sigma$. Since $\rho$ is derived, $\rho = d\zeta$ say, we find $\chi = d(\phi - \zeta)$. Now

$$N(\chi) = (\chi, d(\phi - \zeta)) = (\delta\chi, \phi - \zeta) + \int_B (\phi - \zeta) \wedge *\chi.$$

The volume integral vanishes since $\delta\chi = 0$. Now $t(\phi - \zeta)$ is an admissible boundary value since, first,

$$d_B(t\phi - t\zeta) = d_B t\phi - td\zeta = d_B\xi - t\rho = 0,$$

from (5.1), and second,

$$\int_{bR_{p+1}} t(\phi - \zeta) = \int_{bR_{p+1}} t\phi - \int_{R_{p+1}} d\zeta = \int_{bR_{p+1}} \xi - \int_{R_{p+1}} \rho = 0,$$

by (5.4) and (5.1). Hence $t(\phi - \zeta) = t\gamma$ say, where $d\gamma = 0$ in $M$, and

$$\int_B (\phi - \zeta) \wedge *\chi = \int_B \gamma \wedge *\chi = \int_M d(\gamma \wedge *\chi) = 0,$$

since $d\gamma = 0$, $\delta\chi = 0$. Thus $\chi \equiv 0$ in $M$ and so the equations (5.2) are both valid. This completes the proof of the theorem.

From Lemma 2 it is evident that the solution normalized orthogonal to $K$ is coderived if $\sigma = 0$. This problem has been studied in Euclidean space by Miranda **(7)**.

**6. A mixed problem for biharmonic fields.** We consider here a new class of harmonic forms, which satisfy the self-dual equations $\delta d\phi = 0$, $d\delta\phi = 0$. Since these have the same relation to biharmonic forms ($\Delta^2\phi = 0$) as do harmonic fields to harmonic forms, they shall be called biharmonic fields. The mixed problem for harmonic forms leads to the following result on biharmonic fields.

THEOREM IV. *Let $\rho$ and $\sigma$ be forms of degree $p$ on $M$, coderived and derived respectively. Let $\xi = \xi_p$ and $\eta = \eta_{p-1}$ be defined on $B$, with*

(6.1) $$d_B\eta = t\sigma, \qquad \int_{bR_p{}^i} \eta = \int_{R_p{}^i} \sigma.$$

*Then there exists a unique form $\phi = \phi_p$ satisfying the differential equations*

(6.2) $$\delta d\phi = \rho, \quad d\delta\phi = \sigma,$$

*the boundary conditions*

(6.3) $$t\phi = \xi, \qquad t\delta\phi = \eta,$$

*and the orthogonality condition* $(\phi, \tau) = 0$, $\tau \in K$.

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*Proof.* The uniqueness can be verified at once, from Lemma 1. To establish the existence of a solution, we set up the mixed problem

(6.4) $$\Delta\phi = \rho + \sigma, \quad t\phi = \xi, \quad t\delta\phi = \eta; \quad (\phi, \tau) = 0, \ \tau \in K.$$

A solution exists if and only if for all $\tau \in K$,

$$(\rho + \sigma, \tau) - \int_B \eta \wedge *\tau$$

vanishes. Since $\rho$ is coderived, $\rho = \delta\theta$ say, and also $\sigma = d\zeta$; this quantity is equal to

$$\begin{aligned}&(\delta\theta + d\zeta, \tau) - \int_B \eta \wedge *\tau \\ &= (\theta, d\tau) - \int_B \tau \wedge *\theta + (\zeta, \delta\tau) + \int_B \zeta \wedge *\tau - \int_B \eta \wedge *\tau \\ &= \int_B (\zeta - \eta) \wedge *\tau.\end{aligned}$$

We show that $t\zeta - \eta$ is admissible. Indeed,

$$d_B(t\zeta - \eta) = td\zeta - d_B\eta = t\sigma - d_B\eta = 0,$$

by the first of (6.1). Then also

$$\int_{bR_p{}^i} (\zeta - \eta) = \int_{R_p{}^i} d\zeta - \int_{bR_p{}^i} \eta = \int_{R_p{}^i} \sigma - \int_{bR_p{}^i} \eta = 0,$$

by the second of (6.1). Hence $t\zeta - \eta = t\alpha$, where $d\alpha = 0$ in $M$, and

$$\int_B (\zeta - \eta) \wedge *\tau = \int_B \alpha \wedge *\tau = \int_M d(\alpha \wedge *\tau) = 0,$$

as in previous calculations. Thus the condition relative to (6.4) is satisfied, and a solution $\phi$ exists. It remains to be shown that (6.2) are satisfied.

Now let

(6.5) $$\psi = \delta d\phi - \rho = \sigma - d\delta\phi.$$

Since $\rho = \delta\theta$, we find that $\psi = \delta(d\phi - \theta)$ is coderived. Also

$$t\psi = t\sigma - td\delta\phi = t\sigma - d_B t\delta\phi = t\sigma - d_B\eta = 0,$$

from (6.1). Thus we find

$$N(\psi) = (\psi, \delta(d\phi - \theta)) = (d\psi, d\phi - \theta) - \int_B \psi \wedge *(d\phi - \theta).$$

The surface integral vanishes since $t\psi = 0$. Now $d\psi = d\sigma - dd\delta\phi = 0$ since $\sigma$ is derived. Thus $N(\psi) = 0$, so $\psi$ vanishes identically. That is, the equations (6.2) are both valid. This completes the proof.

We note two special cases of Theorem IV. First, let $\rho$ vanish so that $\delta d\phi = 0$, and let $\xi$ be an admissible tangential boundary value. It then follows easily from Green's formula that $\phi$ is closed. Second, let $\sigma$ and $\eta$ both vanish. Green's formula then shows that $\phi$ is coclosed, and from Lemma 2 it follows that $\phi$ is coderived.

**7. A Neumann problem for biharmonic fields.** The Neumann boundary value problem for harmonic forms **(2)** yields an independent result in connection with biharmonic fields. We state first the Neumann theorem in the non-homogeneous case.

LEMMA 3. *There exists a unique solution $\phi$ of*

$$\Delta\phi = \rho \tag{7.1}$$

*which satisfies the boundary conditions*

$$t{*}d\phi = \xi, \quad t\delta\phi = \eta \tag{7.2}$$

*and the orthogonality condition*

$$(\rho, \tau) = 0, \qquad\qquad d\tau = 0, \quad \delta\tau = 0, \tag{7.3}$$

*if and only if for every harmonic field $\tau$ we have*

$$(\rho, \tau) - \int_B (\tau \wedge \xi - \eta \wedge {*}\tau) = 0. \tag{7.4}$$

The proof is similar to that of Lemma 1 and will therefore be omitted.

We state the application to the biharmonic field equations as follows.

THEOREM V. *Let $\rho$ and $\sigma$ be $p$-forms defined on $M$ which are coderived and derived, respectively. Let $\xi = \xi_{q+1}$ and $\eta = \eta_{p-1}$ be forms defined on $B$ such that*

$$d_B\xi = (-1)^N t{*}\rho, \quad \int_{bR_q{}^i} \xi = (-1)^N \int_{R_q{}^i} {*}\rho, \tag{7.5}$$

*and*

$$d_B\eta = t\sigma, \qquad \int_{bR_p{}^i} \eta = \int_{R_p{}^i} \sigma. \tag{7.6}$$

*Then there exists a unique $p$-form $\phi$ satisfying the differential equations*

$$\delta d\phi = \rho, \quad d\delta\phi = \sigma, \tag{7.7}$$

*the boundary conditions*

$$t{*}d\phi = \xi, \quad t\delta\phi = \eta, \tag{7.8}$$

*and the orthogonality condition*

$$(\phi, \tau)_M = 0 \tag{7.9}$$

*for every harmonic field $\tau$ defined on $M$.*

*Proof.* We formulate the problem

$$\Delta\phi = \rho + \sigma, \quad t{*}d\phi = \xi, \quad t\delta\phi = \eta; \quad (\phi, \tau) = 0 \text{ if } d\tau = 0, \delta\tau = 0. \tag{7.10}$$

According to Lemma 3, a solution exists if and only if for every harmonic field $\tau$, the quantity

$$(\rho + \sigma, \tau) + \int_B (\tau \wedge \xi - \eta \wedge {*}\tau)$$

vanishes. Writing $\rho = \delta\theta$ and $\sigma = d\zeta$ in view of our hypotheses, and making use of Green's formula we find this expression can be put in the form

$$(7.11) \qquad \int_B [\tau \wedge (\xi - *\theta) - (\eta - \zeta) \wedge *\tau].$$

As in our previous work, we may show that the conditions (7.5) imply that $\xi - *\theta$ is admissible; and that (7.6) imply that $\eta - \zeta$ is admissible. A further application of Stokes' formula then shows that the integral (7.11) vanishes for all harmonic fields $\tau$. Thus the condition of Lemma 3 is satisfied and a solution of (7.10) exists.

As in §6 we set $\psi = \delta d\phi - \rho = \sigma - d\delta\phi$. It again follows that $\psi = \delta(d\phi - \theta)$, and, from the first of (7.6), that $t\psi = 0$. Since $d\psi = 0$, $\delta\psi = 0$, we can again show that $\psi$ vanishes identically. This shows that (7.7) holds and establishes Theorem V.

In contrast to Theorem IV, this last result is self-dual. When $\rho$ and $\xi$ vanish, we see by Green's formula that $\phi$ is closed, and, by the dual of Lemma 1 (since $\phi$ is orthogonal to the eigenspace $d\tau = 0$, $\delta\tau = 0$, $n\tau = 0$) that $\phi$ is derived. Dually, $\phi$ is coderived if $\sigma$ and $\eta$ are zero.

**8. Concluding remarks.** Four of the five theorems have been deduced from the mixed boundary value theorem for harmonic forms, and one from the Neumann theorem. Corresponding special cases of the Dirichlet theorem for harmonic forms are not known.

The conditions on the data of the theorems for harmonic forms all involve the eigenforms which are harmonic fields having some special properties. However, in the special cases considered in this paper, all of the conditions are expressible directly in terms of the data without the appearance of any class of eigenforms. The normalizations such as $(\phi, \tau) = 0$, $\tau \in K$ could be discarded, and the solutions would then lose the property of uniqueness. Or they could be replaced by suitable conditions on the periods of the solutions.

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