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Abstract   Universal Taylor series are defined on simply connected domains, but they do not exist on an annulus. Instead we introduce universal Laurent or Laurent–Faber series on finitely connected domains in $\mathbb{C}$. These are generic universalities. Furthermore, we study some properties of universal Laurent series on an annulus.

1. Introduction

If $\sum_{n=0}^{\infty} \alpha_n z^n$ has radius of convergence $R = 1$, then for $|z_0| > 1$ the sequence $S_n(z_0) = \sum_{l=0}^{n} \alpha_l z_0^l$ is not bounded; thus, there exists a strictly increasing sequence $\mu_n \in \{0, 1, 2 \dots\}$, $n = 1, 2, \dots$, so that $S_{\mu_n}(z_0) \to \infty$, as $n \to +\infty$. Therefore, there are two possibilities: either $\lim_n S_n(z_0) = \infty$, or there exists another strictly increasing sequence $\lambda_n \in \{0, 1, 2, \dots\}$, $n = 1, 2, \dots$, such that the sequence $S_{\lambda_n}(z_0)$ has a finite limit (in $\mathbb{C}$). In this last case, we say that the series $\sum_{n=0}^{\infty} \alpha_n z^n$ overconverges at z_0. It is also possible to replace $\{z_0\}$ by a compact set $K \subset \{z \in \mathbb{C} : |z| \geqslant 1\}$ (K^c connected) and still the sequence S_{λ_n} converges uniformly on K towards to a function $g : K \to \mathbb{C}$. In this case we say that g is obtained by overconvergence by $\sum_{n=0}^{\infty} \alpha_n z^n$. Obviously, the function g is continuous on K and holomorphic in K^0, since g is a uniform limit of polynomials.

A power series $\sum_{n=0}^{\infty} \alpha_n z^n$ converging in $D = \{z : |z| < 1\}$ is called a universal Taylor series (in D), if we can obtain by overconvergence every function $g : K \to \mathbb{C}$ that is continuous on K and holomorphic in K^0 for all compact sets $K \subset \mathbb{C}$ satisfying $K \cap D = \emptyset$ and with K^c connected.

The set $U(D)$ of universal Taylor series in D has been proved to be non-void and especially G_δ (countable intersection of open sets) and dense in $H(D)$ endowed with the topology of uniform convergence on compacta [**11**].

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Furthermore, some properties of such universal Taylor series have been investigated in [**6**–**8**, **11**].

In the above consideration, the open unit disc may be replaced by any simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, and several kinds of universal Taylor series can be defined [**7**, **12**]. However, if Ω is an annulus, universal Taylor series do not exist [**2**, **7**]. Thus, it is natural to search for universal Laurent series. In fact such series exist on an annulus and this is a generic universality.

Properties of universal Taylor series on discs are easily transferred to universal Laurent series on an annulus (see § 3). However, we wish to handle the case of any finitely connected domain in $\mathbb{C}$; in fact it is possible to have several universalities, all of them being generic. Thus in § 2 we establish a unifying theorem.

Instead of partial sums we consider a sequence of maps T_n, $n = 0, 1, 2, \ldots$, defined on $H(\Omega)$ and take values on some function space. We say that a function $f \in H(\Omega)$ is universal if the sequence $T_n(f)$, $n = 0, 1, \ldots$, approximates a lot of functions outside Ω (see Definition 2.1 below). If the T_n are continuous, then the set U of universal functions is proved to be G_δ in $H(\Omega)$, endowed with the topology of uniform convergence on compacta. In order to combine Baire's theorem with Mergelyan's theorem and conclude that U is also dense, we use an assumption of the following type: if P is a rational function with prescribed poles, then $T_n(P)$ approaches P for some subsequence of n.

In the case of simply connected domains, when T_n are the partial sums of the Taylor development with respect to some centre, this condition obviously holds for all polynomials, since $T_n(P) = P$ for $n \geqslant \deg P$ and any polynomial P. Thus, the assumptions of the unifying theorem (§ 2) are rather natural.

In order to apply this unifying theorem to the case of finitely connected domains Ω, we observe that every $f \in H(\Omega)$ can be written as $f = f_0 + f_1 + \cdots + f_k$, where every f_j is holomorphic in a simply connected domain V_j of $\mathbb{C} \cup \{\infty\}$, with $V_0 \subset \mathbb{C}$, while $\infty \in V_1, \ldots, \infty \in V_k$. We consider the Taylor development of f_0 and partial sums. For f_j, $j = 1, \ldots, k$, we consider partial sums of Laurent expansions with centre $\alpha_j \notin V_j$. T_n is taken to be a finite sum of those partial sums. Then the dependence on f is continuous and $T_n(P) = P$ for all rational functions P with poles in $\{\infty, \alpha_1, \ldots, \alpha_k\}$ and $n \geqslant \deg P$. Thus there exist universal Laurent series and they form a dense and G_δ subset of $H(\Omega)$.

The disadvantage is that $T_n(f)$ does not converge to f in Ω, unless it is an annulus. We can obtain this convergence if we consider Faber–Laurent expansions. This is possible when Ω is bounded by a finite number of analytic Jordan curves. It is also possible to have matrix transforms of the previous T_n, and still have generic universalities. Extensions of results of the present paper have been obtained by Vlachou in [**14**, **15**], where approximation outside Ω and simultaneously inside Ω is obtained.

2. A unification theorem

In this section we consider domains $\Omega \subset \mathbb{C}$, such that $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ has a finite number of components $A_0, A_1, \ldots, A_k$, where $k \geqslant 0$. We assume that $\infty \in A_0$ and we fix points $\alpha_j \in A_j$, $j = 1, \ldots, k$. Let also Y be a hemicompact space; that is, there exists an increasing sequence of compact subsets $Y_p \subset Y$, $p = 1, 2, \ldots$, such that every compact set $J \subset Y$

is contained in some Y_p. Let $H(\Omega)$ be the space of holomorphic functions in Ω, with the topology of uniform convergence on compacta. In addition, we consider the space $C(Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_k\}]))$ of complex continuous functions on $Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_k\}])$ with the topology of uniform convergence on compacta. Both spaces are complete and metrizable; in fact, they are Fréchet spaces. We finally fix a sequence of maps

$$T_n : H(\Omega) \to C(Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_k\}])), \quad n = 0, 1, 2 \ldots.$$

Definition 2.1. Under the above notation and assumptions a holomorphic function $f \in H(\Omega)$ belongs to the class $U \equiv U(\Omega, \{T_n\})$ if the following holds: for every compact set $K \subset \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_k\}$ with $K \cap \Omega = \emptyset$ and K^c connected and every function $g : K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \ldots\}$, $n = 1, 2, \ldots$, such that for every compact set $J \subset Y$ we have

$$\sup_{x \in J} \sup_{z \in K} |[T_{\lambda_n}(f)](x, z) - g(z)| \to 0 \quad \text{as } n \to +\infty.$$

In the previous definition it is equivalent to require $\lambda_n < \lambda_{n+1}$.

In order to obtain a denumerable description of the class U we consider q_j, $j = 1, 2, \ldots$, an enumeration of all polynomials with coefficients in $\mathbf{Q} + \mathrm{i}\mathbf{Q}$, and on the other hand we use Lemma 2.2 below.

Lemma 2.2. *Let Ω be a domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$. We assume that $(\mathbf{C} \cup \{\infty\}) \setminus \Omega$ has components $A_0, \ldots, A_k$, $k \geqslant 0$, and we fix $\infty \in A_0$, $\alpha_1 \in A_1$, $\ldots$, $\alpha_k \in A_k$. Then there exists a sequence of compact sets $K_m \subset \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_k\}$, $m = 1, 2, \ldots$, with $K_m \cap \Omega = \emptyset$ and K_m^c connected such that the following holds.*

Every compact set $K \subset \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_k\}$ with $K \cap \Omega = \emptyset$ and K^c connected is contained in K_m, for some $m = 1, 2, \ldots$.

Proof. We fix a point ζ_0 in Ω. Let $K \subset \mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_k\}$ be a compact set satisfying $K \cap \Omega = \emptyset$ and with K^c connected. Since $\zeta_0, \infty, \alpha_1, \ldots, \alpha_k$ belong to $(\mathbb{C} \cup \{\infty\}) \setminus K$, which is connected, we can find polygonal lines $\gamma_0, \gamma_1, \ldots, \gamma_k$ in $(C \cup \{\infty\}) \setminus K$ starting from ζ_0 and ending at $\infty, \alpha_1, \ldots, \alpha_k$, respectively (the last edge of γ_0 is a closed half-line of the form $\{x \in \mathbb{R} : x \geqslant n\} \cup \{\infty\}$, for some natural number n). We assume that all other vertices of $\gamma_0, \gamma_1, \ldots, \gamma_k$ are in $\mathbf{Q} + \mathrm{i}\mathbf{Q}$. The set $\Gamma = \gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_k$ is compact and disjoint from K. If 'dist' denotes the distance in $\mathbb{C} \cup \{\infty\}$ induced by Euclidean distance in $\mathbb{R}^3$ via stereographical projection, then $\mathrm{dist}(\Gamma, K) \geqslant (1/s) > 0$ for some $s \in \{1, 2, \ldots\}$. Thus $K \subset L(\Gamma, s)$, where

$$L(\Gamma, s) = \{z \in \mathbb{C} \cup \{\infty\} : z \in \Omega^c \text{ and } \mathrm{dist}(z, \Gamma) \geqslant (1/s)\}.$$

The set $L(\Gamma, s)$ is compact in $\mathbb{C} \setminus \{\alpha_1, \ldots, \alpha_k\}$ and is disjoint from Ω, its complement is

$$\Omega \cup \{z \in C \cup \{\infty\} : \mathrm{dist}(z, \Gamma) < (1/s)\},$$

which is connected as the union of two connected sets containing the same point ζ_0.

Since the collection of all $L(\Gamma, s)$ is denumerable, an enumeration of them gives the desired sequence K_m, $m = 1, 2, \ldots$. $\qquad\square$

We fix a sequence K_m, $m = 1, 2, \ldots$, as in Lemma 2.2.

Also let L_p, $p = 1, 2, \ldots$, be an exhausting family of compact sets in Ω. By our assumption in Ω, we can assume that $(\mathbb{C} \cup \{\infty\}) \setminus L_p$ has $k+1$ components, one containing ∞ and the others $\alpha_1, \ldots, \alpha_k$, respectively [**13**, Theorem 13.3].

For $J \subset Y$ compact, $K \subset \Omega^c$ compact with K^c connected, g a complex function on K, $n \in \{0, 1, 2, \ldots\}$ and $s \in \{1, 2, \ldots\}$ we consider the set

$$E(\Omega, J, K, g, n, s) = \left\{ f \in H(\Omega) : \sup_{x \in J} \sup_{z \in K} |[T_n(f)](x, z) - g(z)| < (1/s) \right\}.$$

Proposition 2.3. *Under the above assumptions and using the same notation we have*

$$U = \bigcap_p \bigcap_m \bigcap_j \bigcap_s \bigcup_{n=0}^{\infty} E(\Omega, Y_p, K_m, q_j, n, s).$$

The proof of Proposition 2.3 is based on Mergelyan's theorem [**13**] and is similar to the proof of Lemma 3.5 of [**7**].

In order to prove that U is G_δ and dense in the complete space $H(\Omega)$, Baire's theorem can be applied. It suffices to prove that under some conditions, the set $E(\Omega, Y_p, K_m, q_j, n, s)$ is open and that $\bigcup_{n=0}^{\infty} E(\Omega, Y_\rho, K_m, q_j, n, s)$ is dense in $H(\Omega)$.

Proposition 2.4. *Under the above assumptions and using the same notation, we also assume that the maps $T_n : H(\Omega) \to C(Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots \alpha_k\}]))$ are continuous. Then $E(\Omega, Y_p, K_m, q_j, n, s)$ is open in $H(\Omega)$ and the class U is G_δ in $H(\Omega)$.*

For the proof it suffices to observe that the set

$$A = \left\{ \omega \in C(Y \times (\mathbb{C} \setminus [\Omega \cup \{a_1, \ldots, a_k\}])) : \sup_{x \in Y_p} \sup_{z \in K_m} |\omega(x, z) - q_j(z)| < (1/s) \right\}$$

is open and $T_n^{-1}(A) = E(\Omega, Y_p, K_m, q_j, n, s)$. In order to establish that

$$\bigcup_{n=0}^{\infty} E(\Omega, Y_p, K_m, q_j, n, s)$$

is dense in $H(\Omega)$, we make the assumption that the applications T_n 'respect in the limit' each rational function whose poles are included in $\{\infty, \alpha_1, \ldots, \alpha_k\}$.

Theorem 2.5. *Let $\Omega \subset \mathbb{C}$ be a domain of finite connectivity with $\infty, \alpha_1, \ldots, \alpha_k$ fixed points in each component of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$. Let Y be a hemicompact topological space. We endow the spaces $H(\Omega)$ and $C(Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_k\}]))$ with the topology of uniform convergence on compacta. We consider a sequence of maps $T_n : H(\Omega) \to C(Y \times (\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots \alpha_k\}]))$ and the corresponding class U defined in Definition 2.1.*

We assume that the T_n are continuous and that there exists a function $h : \mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_k\}] \to \mathbb{C}$ continuous on its domain of definition and holomorphic on the interior, satisfying $h(z) \neq 0$ for all z in $\mathbb{C} \setminus [\Omega \cup \{\alpha_1, \ldots, \alpha_2\}]$, such that the following holds.

For every rational function P, whose poles are included in $\{\infty, \alpha_1, \dots, \alpha_k\}$, and every compact set $K \subset \mathbb{C}$ with $K \cap [\Omega \cup \{\alpha_1, \dots, \alpha_k\}] = \emptyset$ and K^c connected there exists a sequence $n_\tau \in \{0, 1, 2, \dots\}$, $\tau = 1, 2, \dots$, such that, for every compact set $J \subset Y$ we have

$$\sup_{x \in J} \sup_{z \in K} |[T_{n_\tau}(P)](x, z) - h(z)P(z)| \to 0 \quad \text{as } \tau \to \infty.$$

Then the class U is G_δ and dense in $H(\Omega)$.

Proof. Combining Baire's theorem with Propositions 2.3 and 2.4 it suffices to prove that $\bigcup_{n=0}^{\infty} E(\Omega, J, K, g, n, s)$ is dense in $H(\Omega)$ for every compact set $J \subset Y$, $K \subset \mathbb{C}$ compact with K^c connected and $K \cap [\Omega \cup \{\alpha_1, \dots, \alpha_k\}] = \emptyset$, $g : K \to \mathbb{C}$ continuous on K and holomorphic in K^0 and $s \in \{1, 2, \dots\}$.

Let $f \in H(\Omega)$, $L \subset \Omega$ compact and $\varepsilon > 0$. We consider an open neighbourhood $V(f)$ of f in $H(\Omega)$ of the form $V(f) = \{P \in H(\Omega) : \sup_{w \in L} |P(w) - f(w)| < \varepsilon\}$. Without loss of generality we may assume that $(\mathbb{C} \cup \{\infty\}) \setminus L$ has $k + 1$ components for some $k \geqslant 0$, each one containing $\infty, \alpha_1, \dots, \alpha_k$, respectively. We have to prove that $V(f)$ meets $\bigcup_{n=0}^{\infty} E(\Omega, J, K, g, n, s)$: that is, find $P \in H(\Omega)$ and $n \geqslant 0$ such that $P \in V(f)$ and $P \in E(\Omega, J, K, g, n, s)$.

We set $M = \sup_{z \in K} |h(z)| \in (0, +\infty)$ and we observe that the function g/h is continuous on K and holomorphic in K^0. Since L and K are disjoint compact sets, our assumption on the number of components of their complements in $\mathbb{C} \cup \{\infty\}$ implies that $L \cup K$ is a compact set in $\mathbb{C}$ and $(\mathbb{C} \cup \{\infty\}) \setminus (L \cup K)$ has $k + 1$ components, each one containing $\infty, \alpha_1, \dots, \alpha_k$, respectively. We consider the function $F : L \cup K \to \mathbb{C}$ defined by $F(w) = f(w)$ for $w \in L$ and $F(z) = (g(z)/h(z))$ for $z \in K$. By an extension of Mergelyan's theorem [**13**, Exercise 1, p. 394], there exists a rational function P whose poles are included in $\{\infty, \alpha_1, \dots, \alpha_k\}$ such that

$$\sup_{w \in L} |P(w) - f(w)| < \varepsilon \tag{$*$}$$

and

$$\sup_{z \in K} \left| P(z) - \frac{g(z)}{h(z)} \right| < \frac{1}{2sM},$$

which implies

$$\sup_{z \in K} |h(z)P(z) - g(z)| < \frac{1}{2s}. \tag{$**$}$$

By assumption there exists $n_\tau \in \{0, 1, 2, \dots\}$ such that

$$\sup_{x \in J} \sup_{z \in K} |[T_{n_\tau}(P)](x, z) - h(z)P(z)| < \frac{1}{2s}. \tag{$***$}$$

We set $n = n_\tau$.

First, P belongs to $H(\Omega)$, since it is rational with poles outside of Ω, and $P \in V(f)$ by (∗). Further we have

$$\sup_{x \in J} \sup_{z \in K} |[T_n(P)](x, z) - g(z)|$$

$$\leqslant \sup_{x \in J} \sup_{z \in K} |[T_{n_\tau}(P)](x, z) - h(z)P(z)| + \sup_{z \in K} |h(z)P(z) - g(z)|$$

$$< \frac{1}{2s} + \frac{1}{2s} = \frac{1}{s},$$

where we have used relations (∗∗∗) and (∗∗). This completes the proof.   □

3. The case of an annulus

We consider the special case where Ω is an annulus $\Delta = \Delta(0, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ with $0 < R_1 < R_2 < +\infty$. We set $\alpha_1 = 0$ and Y to be a singleton. If $f(z) = \sum_{n=-\infty}^{+\infty} \alpha_n(f)z^n$ belongs to $H(\Delta)$, we set $T_n(f)(z) = \sum_{k=-n}^{n} \alpha_k(f)z^k$. We denote by $U(\Delta)$ the class $U(\Delta, \{T_n\})$ defined in Definition 2.1. It is also easy to see that the assumptions of Theorem 2.5 are satisfied in this case. Therefore, the class $U(\Delta)$ is dense and G_δ in $H(\Delta)$.

We also need the notion of universal Taylor series in a disc introduced in [11].

Definition 3.1. Let $D_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R < +\infty$, be a disc. A holomorphic function $f(z) = \sum_{n=0}^{+\infty} \delta_n z^n$ in $H(D_R)$ belongs to the class $U(D_R)$ if the following holds.

For every compact set $K \subset \mathbb{C}$, $K \cap D_R = \emptyset$ with K^c connected and every function $g : K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, \dots\}$, $n = 1, 2, \dots$, such that

$$\sup_{z \in K} \left| \sum_{k=0}^{\lambda_n} \delta_k z^k - g(z) \right| \to 0 \quad \text{as } n \to +\infty.$$

In the above definition it is equivalent to require $\lambda_n < \lambda_{n+1}$.

The set $U(D_R)$ has been proved to be G_δ and dense in $H(D_R)$ [11]. Properties of the elements of $U(D_R)$ have been established in [1, 6–8]. In the rest of this section we investigate properties of the elements of the class $U(\Delta)$.

Remark 3.2. Let

$$\sum_{k=-\infty}^{+\infty} c_k z^k \in U(\Delta), \quad \Delta = \Delta(0, R_1, R_2).$$

Then

$$\sum_{k=0}^{\infty} c_k z^k \in U(D_{R_2}) \quad \text{and} \quad \sum_{k=1}^{\infty} c_{-k} w^k \in U(D_{1/R_1}).$$

This is obvious if we recall that $\sum_{k=1}^{\infty} c_k z^k$ converges uniformly on compacta of $|z| < R_2$ and $\sum_{k=-\infty}^{-1} c_k z^k$ converges uniformly on compacta of $|z| > R_1$.

Remark 3.3. For every

$$\sum_{l=-\infty}^{\infty} c_l z^l \in U(\Delta), \quad \Delta = \Delta(0, R_1, R_2),$$

and every z_0 with $|z_0| = R_1$ or $|z_0| = R_2$, the series $\sum_{l=-\infty}^{+\infty} c_l z_0^l$ is not (C, k) summable for any $k = 1, 2, \dots$.

If $|z_0| = R_2$ we observe that $\sum_{l=1}^{\infty} c_{-l} z_0^{-l}$ converges and that $\sum_{l=0}^{\infty} c_k z^l \in U(D_{R_2})$, and then, using the corresponding property of a universal Taylor series on a disc [**7**, **8**], the result follows.

If $u \in U(D_R)$ and $|z_0| = R$, it is also true that

$$\lim_{\substack{\zeta \to z_0 \\ z \in D_R}} u(z)$$

does not exist in $\mathbb{C}$ [**7**].

Thus we obtain the following.

Remark 3.4. If $f \in U(\Delta)$, $\Delta = \Delta(0, R_1, R_2)$, and $z_0 \in \partial\Delta$, then

$$\lim_{\substack{\zeta \to z_0 \\ z \in \mathcal{D}}} f(z)$$

does not exist in $\mathbb{C}$. In particular, f cannot be continuously extended in $\bar{\Delta}$ and f cannot be holomorphically extended in any domain strictly containing Δ.

Remark 3.5. Let

$$\sum_{k=-\infty}^{+\infty} c_k z^k \in U(\Delta), \quad \Delta = \Delta(0, R_1, R_2),$$

and let g_1 and g_2 be two complex-valued measurable 2π-periodic functions. Then there exists a sequence $0 \leqslant \lambda_n < \lambda_{n+1}$ such that

(i) $\displaystyle \lim_n \sum_{k=-\lambda_n}^{\lambda_n} c_k R_2^k e^{ik\vartheta} = g_2(\vartheta)$ and

(ii) $\displaystyle \lim_n \sum_{k=-\lambda_n}^{\lambda_n} c_k R_1^k e^{ik\vartheta} = g_1(\vartheta)$

for almost all ϑ.

It suffices to use Lusin's theorem to approximate (in the almost everywhere sense) g_1 and g_2 by continuous functions φ_n^1, φ_n^2 and then apply the definition of $U(\Delta)$ to the compact sets

$$K_n = \left\{ z \in \mathbb{C}; \ |z| = R_2, \ \frac{1}{n} \leqslant \arg z \leqslant 2\pi \right\} \cup \left\{ z \in \mathbb{C} : |z| = R_1, \ \frac{1}{n} \leqslant \arg z \leqslant 2\pi \right\}$$

(see also [**11**]).

The above is an extension of Menchoff's result on universal trigonometric series [**3**,**7**,**9**].

Remark 3.6. Let

$$\sum_{k=-\infty}^{+\infty} c_k z^k \in U(\Delta), \quad \Delta = \Delta(0, R_1, R_2).$$

Then we have the following.

(i) $\limsup_{k \to +\infty} \sqrt[k]{|c_k|} = 1/R_2$.

(ii) $\limsup_{k \to +\infty} \sqrt[k]{|c_{-k}|} = R_1$.

(iii) For every decreasing sequence $b_n \downarrow 0$ such that $\sum_{n=1}^{\infty} (b_n/n) < +\infty$ we have

$$\limsup_{k \to +\infty} \frac{|c_n| R_2^n}{\mathrm{e}^{nb_n}} = +\infty.$$

In particular, $c_n R_2^n$ does not have polynomial growth, as $n \to +\infty$.

(iv) For every decreasing sequence $b_n \downarrow 0$, such that $\sum_{n=1}^{\infty} (b_n/n) < +\infty$ we have

$$\lim_{n \to +\infty} \frac{|c_{-n}|}{R_1^n \mathrm{e}^{nb_n}} = +\infty.$$

In particular, c_{-n}/R_1^n does not have polynomial growth, as $n \to +\infty$.

The proof follows from [**8**] and from the fact that

$$\sum_{k=0}^{\infty} c_k z^k \in U(\mathcal{D}_{R_2}) \quad \text{and} \quad \sum_{k=1}^{\infty} c_{-k} w^k \in U(D_{1/R_1}).$$

Finally, we have a great Picard type property.
The proof follows the lines of [**1**].

Proposition 3.7. *Let $f \in H(\Delta)$, where $\Delta = \Delta(0, R, 1)$, $0 < R < 1$, and let n be a non-negative integer. If the holomorphic functions f and $f^{(n)} - 1$ have at most finitely many zeros in Δ, then $f \notin U(\Delta)$.*

Proof. Let f satisfy the hypotheses of this proposition. Then, there is $0 < p_0 < 1$ with $p_0 > R$ such that f and $f^{(n)} - 1$ have no zeros in the annulus $\Delta(0, p_0, 1)$. Let $p = \frac{1}{2}(1 + p_0)$, $\delta = \frac{1}{2}(1 - p_0) > 0$ and fix $z_0 = p\mathrm{e}^{\mathrm{i}\vartheta}$. Define

$$F(z) = \delta^{-n} f(z_0 + \delta z) \quad \text{for } z \in D = \{z \in \mathbb{C} : |z| < 1\}.$$

Then F and $F^{(n)} - 1$ have no zeros in D. Hence, by Miranda's theorem [**10**] we conclude that

$$\log |F(z)| \leqslant \frac{1}{1 - |z|} \left[64 \log^+ |F(0)| + \frac{M_n}{(1 - |z|)^4} \right] \quad \text{for every } z \in D,$$

where M_n depends only on n. Let $p_0 < r < 1$ be such that $1 - r < \delta$ and take $z = r_1 e^{i\vartheta}$ in the above inequality where $r_1 \in (0, 1)$ is such that $\delta(1 - r_1)1 - r$. We have $F(r_1 e^{i\vartheta}) = \delta^{-n} f(r e^{i\theta})$ and there is a constant c_1 depending only on δ and n such that

$$\log |f(r e^{i\theta})| \leqslant \frac{c_1}{1 - r} \left[\log^+ |f(p e^{i\theta})| + \frac{1}{(1 - r)^4} \right],$$

whenever $0 < 1 - r < \delta$ and $p_0 < r < 1$.

Thus $\log |f(r e^{i\theta})| \leqslant c/(1 - r)^5$ for all $r \in (0, 1)$ such that $0 < 1 - r < \delta$ and $p_0 < r < 1$. Hence the proof will be completed after the following lemma is established, which follows by a combination of Remark 3.6 of the present paper and Lemma 2.2 of [**1**].

Lemma 3.8. *Let c, k be positive real numbers and $0 < R < 1$. Then for every $f \in H(\Delta(0, R, 1))$ such that*

$$|f(z)| = O(e^{c/(1 - |z|)^k}) \quad \text{as } |z| \to 1$$

we have $f \notin U(\Delta(0, R, 1))$.

And the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.9. The previous proof implies easily that if $u \in U(\Delta(0, R_1, R_2))$, then for every $\alpha \in \mathbb{C}$ with at most one exception the following holds: the roots of the equation $u(z) = \alpha$, $R_1 < |z| < R_2$, have at least two cluster points z_1, z_2 with $|z_1| = R_1$ and $|z_2| = R_2$.

4. General Laurent expansions

Let Ω be a domain in $\mathbb{C}$, such that $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ has a finite number of components $A_0, A_1, \ldots, A_k$ $(k \geqslant 0)$ with $\infty \in A_0$, $\alpha_1 \in A_1, \ldots, \alpha_k \in A_k$ being fixed.

We consider the space $H(A_0^c)$ with the topology of uniform convergence on compacta. For $j \in \{1, \ldots, k\}$ let $H_0(A_j^c)$ be the subspace of $H(A_j^c)$ consisting of all $f \in H(A_j^c)$ satisfying $f(\infty) = 0$. We endow $H_0(A_j^c)$ with the topology of uniform convergence on compacta. All spaces $H(A_0^c), H_0(A_1^c), \ldots, H_0(A_k^c)$ are Fréchet spaces.

Since $\Omega = A_0^c \cap A_1^c \cap \cdots \cap A_k^c$ we can consider the map

$$T : H(A_0^c) \times H_0(A_1^c) \times \cdots \times H_0(A_k^c) \to H(\Omega)$$

defined by $T(f_0, f_1, \ldots, f_k) = f_0 + f_1 + \cdots + f_k$.

Obviously, T is linear and continuous (where $H(\Omega)$ is endowed with the Fréchet topology of uniform convergence on compacta). Liouville's theorem implies easily that T is one to one. We will show that T is also onto (hence an isomorphism).

Let $f \in H(\Omega)$ and $z \in A_0^c$. We consider any cycle $\gamma_{0,z}$ in Ω, such that $\text{ind}(\gamma_{0,z}, z) = 1$, $\text{ind}(\gamma_{0,z}, \alpha_j) = 1$ for all $j = 1, \ldots, k$ and obviously $\text{ind}(\gamma_{0,z}, \infty) = 0$. Then the integral

$$f_0(z) = \frac{1}{2\pi i} \int_{\gamma_{0,z}} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

is independent of the choice of the cycle $\gamma_{0,z}$ and defines a holomorphic function $f_0 \in H(A_0^c)$; thus f_0 is given by a Cauchy transform on varying cycles.

Let $j_0 \in \{1, \ldots, k\}$, $f \in H(\Omega)$ and $z \in A_{j_0}^c$; we consider any cycle $\gamma_{j_0,z}$ in Ω, such that

$$\operatorname{ind}(\gamma_{j_0,z}, z) = 0, \qquad \operatorname{ind}(\gamma_{j_0,z}, a_{j_0}) = -1, \qquad \operatorname{ind}(\gamma_{j_0,z}, \alpha_j) = 0$$

for all $j \in \{1, \ldots, k\} \setminus \{j_0\}$ and obviously $\operatorname{ind}(\gamma_{j_0,z}, \infty) = 0$. Then the integral

$$f_{j_0}(z) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_{j_0,z}} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

is independent of the choice of the cycle $\gamma_{j_0,z}$ and defines a holomorphic function $f_{j_0} \in H_0(A_{j_0}^c)$; thus f_{j_0} is defined by a Cauchy transform on varying cycles.

Cauchy's theorem implies that for $z \in \Omega$ we have $f(z) = f_0(z) + f_1(z) + \cdots + f_k(z)$. Thus we have proved the following proposition.

Proposition 4.1. *The operator* $T : H(A_0^c) \times H_0(A_1^c) \times \cdots \times H_0(A_k^c) \to H(\Omega)$ *is an isomorphism. If* $f \in H(\Omega)$, $f_0 \in H(A_0^c)$, $f_1 \in H_0(A_1^c)$, $\ldots$, $f_k \in H_0(A_k^c)$ *are related by* $f = f_0 + f_1 + \cdots + f_k$, *then* $f_0, f_1, \ldots, f_k$ *are called the components of* f *and depend continuously on* f.

In view of Theorem 2.5 the map T_n can be constructed as a sum of quantities, each one varying continuously on $f_0, f_1, \ldots, f_k$, respectively.

We restrict our attention to the following example.

Let $\phi_0 : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ be such that $\lim_{n \to +\infty} \phi_0(n) = +\infty$ and let $\zeta \in A_0^c$. Then for $Y = \{\zeta\}$ or $Y = A_0^c$ we set

$$T_n^0(f_0) = \sum_{l=0}^{\phi_0(n)} \frac{f_0^{(l)}(\zeta)}{l!} (z - \zeta)^l.$$

Let $j_0 \in \{1, \ldots, k\}$ and $\phi_{j_0} : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ be such that $\lim_{n \to +\infty} \phi_{j_0}(n) = +\infty$. The function f_{j_0} has a Laurent expansion for $|z - \alpha_{j_0}|$ large of the form

$$f_{j_0}(z) = \sum_{-\infty}^{-1} c_l(f_{j_0})(z - \alpha_{j_0})^l,$$

where $c_l(f_{j_0})$ depends continuously on f_{j_0} by the well-known formula for Laurent coefficients. We set

$$T_n^{j_0}(f_{j_0})(z) = \sum_{l=-1}^{-\phi_{j_0}(n)} c_l(f_{j_0})(z - \alpha_{j_0})^l.$$

Finally, setting $T_n(f) = T_n^0(f_0) + T_n^1(f_1) + \cdots + T_n^k(f_k)$ one can easily check that the assumptions of Theorem 2.5 are satisfied with $h \equiv 1$. Thus we have the following theorem.

Theorem 4.2. *Under the above assumptions and notation, the set of Laurent universal series* $U(\Omega, \{T_n\})$ *is a* G_δ *and dense subset of* $H(\Omega)$.

Remark 4.3. The operators $T_n^0(f_0)$, $T_n^{j_0}(f_{j_0})$ may be replaced by Cesàro means depending continuously on a parameter $\alpha \in (-1, +\infty)$ (see [**7**]); then the parameter space Y may be $(-1, +\infty) \times \{\zeta\}$ or $(-1, +\infty) \times (A_0^c)$.

In all previous examples the function h in the assumptions of Theorem 2.5 is identically equal to 1. More general functions h arise if we consider holomorphically varying matrix transforms in accordance with [**7**]; the general case of h has been used in the proof of Theorem 4.3 in [**7**].

5. The case of analytic boundaries

If $\Omega = \Delta = \{z : R_1 < |z| < R_2\}$ with $0 < R_1 < R_2 < +\infty$ and $\alpha_1 = 0$, then the constructed sequence T_n, $n = 0, 1, 2$, has the property that for every $f \in H(\Omega)$ the sequence $T_n(f)$ is also defined in Ω and converges to f on Ω. However, in the general setting of the previous section, $T_n(f)$ is also defined on Ω, but it does not converge anywhere on Ω. Instead, under some assumptions on Ω, we will modify T_n by means of Faber-type expansions to obtain $T_n(f) \to f$ on Ω.

In the rest of this section we assume that Ω is a domain in $\mathbb{C}$ bounded by a finite number of analytic Jordan curves.

Let $A_0, A_1, \ldots, A_k$, where $k \geqslant 0$, be the components of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ and $\infty \in A_0$, $\alpha_1 \in A_1, \ldots, \alpha_k \in A_k$ be fixed.

If $f \in H(\Omega)$, then $f = f_0 + f_1 + \cdots + f_k$ with $f_0 \in H(A_0^c)$, $f_1 \in H_0(A_1^c)$, $\ldots$, $f_k \in H_0(A_k^c)$ according to Proposition 4.1.

The simply connected domain A_0^c is bounded by an analytic Jordan curve; thus, if the F_l^0, $l = 0, 1, 2, \ldots$, denote its Faber polynomials (see [**4**, **5**]), then f_0 has a unique Faber expansion $f_0(z) = \sum_{l=0}^{\infty} d_l(f_0) F_l^0(z)$ valid on the whole A_0^c.

Let $\phi_0 : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ be such that $\lim_n \phi_0(n) = +\infty$. We set

$$T_n^0(f_0) = \sum_{l=0}^{\phi_0(n)} d_l(f_0) F_l^0(z)$$

and Y is taken to be a singleton.

Let $j_0 \in \{1, \ldots, k\}$. Let D_{j_0} be the image of $A_{j_0}^c \cup \{\infty\}$ under the map $w = 1/(z - \alpha_{j_0})$ and let the $F_l^{j_0}$, $l = 0, 1, 2, \ldots$, denote the Faber polynomials of D_{j_0}; then one can easily deduce the unique expansion

$$f_{j_0}(z) = \sum_{l=0}^{\infty} d_l^{j_0}(f_{j_0}) F_l^{j_0} \left(\frac{1}{z - \alpha_{j_0}} \right)$$

valid in $A_{j_0}^c \cup \{\infty\}$.

Let $\phi_{j_0} : \{0, 1, 2, \ldots\} \to \{0, 1, 2, \ldots\}$ be such that $\lim_n \phi_{j_0}(n) = +\infty$. Then we set

$$T_n^{j_0}(f_{j_0})(z) = \sum_{l=0}^{\phi_{j_0}(n)} d_l^{j_0}(f_{j_0}) F_l^{j_0} \left(\frac{1}{z - \alpha_{j_0}} \right).$$

Finally, we set $T_n(f) = T_n^0(f_0) + T_n^1(f_1) + \cdots + T_n^k(f_k)$ and using the results of [**4**] we can easily check that the assumptions of Theorem 2.5 are satisfied with $h \equiv 1$. It is also true that $T_n(f)(z) \to f(z)$ uniformly on compact subsets of Ω, for all $f \in H(\Omega)$.

The universal functions $U(\Omega, \{T_n\})$ are called universal Faber–Laurent series; their set is a G_δ and dense subset of $H(\Omega)$, provided that Ω is a domain bounded by a finite number of analytic Jordan curves.

It is also possible to use matrix transforms or to use Faber, Taylor and Laurent expansions simultaneously, and still have generic universalities.

Using properties of universal Faber series in simply connected domains bounded by an analytic curve [**4**], one can easily deduce the following proposition.

Proposition 5.1. *Let $z_0 \in \partial\Omega$ be any boundary point of Ω. Under the above assumptions and considerations every universal Faber–Laurent series $f \in U(\Omega, \{T_n\})$ does not extend continuously on the point $z_0 \in \partial\Omega$. In particular, f cannot be holomorphically extended to any domain strictly containing Ω.*

For the proof it suffices to observe that if $f \in U(\Omega, \{T_n\})$, then f_0 is a universal Faber series in A_0^c in the sense of [**4**] and each f_j, $j \in \{1, \ldots, k\}$, is continuous on $\partial(A_0^c)$. Thus, for $z_0 \in \partial(A_0^c)$ the existence of

$$\lim_{\substack{\zeta \to z_0 \\ z \in A_0^c}} f(z)$$

is equivalent to that of

$$\lim_{\substack{\zeta \to z_0 \\ z \in A_0^c}} f_0(z).$$

But the last limit does not exist in $\mathbb{C}$ [**4**,**7**].

The proof for the other boundaries of Ω is similar.

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