THE EXISTENCE OF FANO SUBPLANES IN GENERALIZED HALL PLANES

Dedicated to the memory of Hanna Neumann

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1. Introduction

One of the best known classes of non-Desarguesian planes is the class of Hall planes (see Hall [2]). In [6] Hanna Neumann showed that the finite Hall planes of odd order possess subplanes of order two (i.e., Fano subplanes)[1]. Kirkpatrick [5] has considered a type of plane which is a generalization of the Hall planes and which he calls *generalized Hall planes*. In this paper we will give a sufficient condition that a finite generalized Hall plane possesses Fano subplanes. Some examples of odd order planes to which the condition applies shall be exhibited.

We shall assume that the reader is familiar with the elements of projective plane theory and particularly with finite translation planes and Veblen-Wedderburn systems. The reader is referred to Hall [2] and André [1] for this. We note that the method of choosing coordinates used in this paper is that due to Hall.

2. The kernel of a generalized Hall system

A projective plane π is said to be a generalized Hall plane with respect to the line l_∞ and the Baer subplane π_0 if π is a translation plane with respect to l_∞, l_∞ is a line of π_0 and there exists a collineation group G of π such that G is sharply transitive on the set of points $l_\infty \setminus (\pi_0 \cap l_\infty)$.

Kirkpatrick [5] shows that a generalized Hall plane of odd order[2] may be coordinatized by a Veblen-Wedderburn system F with the properties

(1) F is a right vector space of dimension two over a subfield F_0,

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[1] In [6] it is shown that there is a Hall plane of each possible finite odd order possessing Fano subplanes. Hughes (see [3]) has shown that there is a unique Hall plane of each possible finite order.

[2] The result is true for even order planes as well.

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(2) There exist two mappings P, Q of $F_0 \times F_0$ onto F_0 such that $(z\alpha + \beta)z = zP(\alpha, \beta) + Q(\alpha, \beta)$ for all $\alpha, \beta \in F_0$ and $z \in F \setminus F_0$, where

(i) P and Q are additive homomorphisms with $P(0, 1) = 1$ and $Q(0, 1) = 0$,

(ii) for any given γ and δ in F_0, the equation $(P(\alpha, \beta), Q(\alpha, \beta)) = (\gamma, \delta)$ has exactly one solution (α, β), and

(iii) for any given γ and δ in F_0, the equation $(P(\alpha, \beta), Q(\alpha, \beta)) = (\alpha\gamma, \beta\gamma + \delta)$ has exactly one solution (α, β); also, for this solution, $\alpha = 0$ if and only if $\delta = 0$.

It can be shown that a finite system satisfying (1) and (2) is a Veblen-Wedderburn system which coordinatizes a generalized Hall plane. Such a system is called a *generalized Hall system*.

The functions P and Q being additive may be written $P(\alpha, \beta) = f(\alpha) + h(\beta)$ and $Q(\alpha, \beta) = g(\alpha) + k(\beta)$ for all $\alpha, \beta \in F_0$ where f, g, h, k are additive endomorphisms of F_0 which we shall the *defining functions* of F. We note that the Hall systems are the special case where $f(\alpha) = r\alpha$, $g(\alpha) = s\alpha$, $h(\alpha) = \alpha$ and $k(\alpha) = 0$ for all $\alpha \in F_0$, where $x^2 - rx - s$ is an irreducible polynomial over F_0.

LEMMA 1. *The only generalized Hall system which is a field is $GF(4)$.*

PROOF. Suppose F is a generalized Hall system which is a field. Then $z^2 = z f(1) + g(1)$ for all $z \in F \setminus F_0$. So $F \setminus F_0$ contains less than three elements and it readily follows that $F_0 = GF(2)$ and $F = GF(4)$.

It is easy to verify that $GF(4)$ is, in fact, a generalized Hall system with defining functions $f(\alpha) = \alpha$, $h(\alpha) = \alpha$, $g(\alpha) = \alpha$ and $k(\alpha) = 0$ for all $\alpha \in GF(2)$.

THEOREM 1. *The kernel Ker(F) of a generalized Hall system $F \neq GF(4)$ is the set*

$$K = \{\lambda \in F_0 \mid f(\lambda\alpha) = \lambda f(\alpha),\ g(\lambda\alpha) = \lambda g(\alpha),$$
$$h(\lambda\alpha) = \lambda h(\alpha),\ k(\lambda\alpha) = \lambda k(\alpha) \text{ for all } \alpha \in F_0\}.$$

PROOF. If $z \in F \setminus F_0$ belongs to Ker(F) then $z(w_1 w_2) = (zw_1)w_2$ and $z(w_1 + w_2) = zw_1 + zw_2$ for all $w_1, w_2 \in F$. Suppose $w \in F \setminus F_0$. There is an automorphism θ of F such that $z^\theta = w$. So $w(w_1 w_2) = (ww_1)w_2$ and $w(w_1 + w_2) = ww_1 + ww_2$ for all $w_1, w_2 \in F$. Thus $w \in$ Ker(F) and so $F \setminus F_0 \subseteq$ Ker(F). It follows that F is a field and by Lemma 1, $F = GF(4)$. Hence, if $F \neq GF(4)$ then Ker(F) $\subseteq F_0$.

Suppose $\lambda \in$ Ker(F). Then $\lambda(z\alpha + \beta) = \lambda(z\alpha) + \lambda\beta = (\lambda z)\alpha + \lambda\beta$ for all α, $\beta \in F_0$. So $(z\alpha + \beta)\ h(\lambda) + k(\lambda) = (zh(\lambda) + k(\lambda))\alpha + \lambda\beta$ for all $\alpha, \beta \in F_0$, whence $(h(\lambda) - \lambda)\beta = k(\lambda)\ (\alpha - 1)$ for all $\alpha, \beta \in F_0$ which implies $h(\lambda) = \lambda$ and $k(\lambda) = 0$. Thus $\lambda x = x\lambda$ for all $x \in F$.

Now $\lambda((z\alpha+\beta)z) = (\lambda(z\alpha+\beta))z$ for all α, $\beta \in F_0$ and therefore $[z(f(\alpha) + h(\beta)) + g(\alpha) + k(\beta)]\lambda = (z\alpha\lambda + \beta\lambda)z = z(f(\alpha\lambda) + h(\beta\lambda)) + g(\alpha\lambda) + k(\beta\lambda)$ for all α, $\beta \in F_0$. So $(f(\alpha) + h(\beta))\lambda = f(\alpha\lambda) + h(\beta\lambda)$ and $(g(\alpha) + k(\beta))\lambda = g(\alpha\lambda) + k(\beta\lambda)$ for all α, $\beta \in F_0$. It readily follows that $f(\alpha)\lambda = f(\alpha\lambda)$, $g(\alpha)\lambda = g(\alpha\lambda)$, $h(\alpha)\lambda = h(\alpha\lambda)$ and $k(\alpha)\lambda = k(\alpha\lambda)$ for all $\alpha \in F_0$.

Suppose $\lambda \in K$. Then $\lambda = h(1)\lambda = h(\lambda)$ and $0 = k(1)\lambda = k(\lambda)$. So $\lambda x = x\lambda$ for all $x \in F$. It is easy to verify that $\lambda(x+y) = \lambda x + \lambda y$ for all x, $y \in F$. Now $(\lambda(z\alpha + \beta))z = z(f(\alpha\lambda) + h(\beta\lambda)) + g(\alpha\lambda) + k(\beta\lambda) = [z(f(\alpha) + h(\beta)) + g(\alpha) + k(\beta)]\lambda = ((z\alpha+\beta)z)\lambda = \lambda((z\alpha+\beta)z)$ for all α, $\beta \in F_0$ and $z \in F \setminus F_0$. Clearly $\lambda(x\mu) = (\lambda x)\mu$ for all $x \in F$ and $\mu \in F_0$ and so $\lambda \in \mathrm{Ker}(F)$.

COROLLARY. *If $\lambda \in Ker(F)$, where F is a generalized Hall system, $\lambda x = x\lambda$ for all $x \in F$.*

3. Fano subplanes

LEMMA 2. *Let F be a generalized Hall system with defining functions f, g, h, k on the field F_0. Denote $f(1)$ and $g(1)$ by r and s respectively. If r and $s \in Ker(F)$, $r + 2s - 1 = 0$ and $l = r + s - 1$ then*

(i) $(l^{-1}(u+s))(u+1) - u = 0$ *and*

(ii) $(l^{-1}(u+s))\,u - u - l^{-1}(u+s) = 0$

for all $u \in F \setminus F_0$.

PROOF. Firstly, $l \neq 0$ since $r + s - 1 = r + 2s - 1$ implies $s = 0$ and so $z^2 = zr$ for all $z \in F \setminus F_0$ which implies $z = 0$ or r for all $z \in F \setminus F_0$, a contradiction.

We note also that $l \in \mathrm{Ker}(F)$ since r and $s \in \mathrm{Ker}(F)$.

(i)
$$\begin{aligned}(l^{-1}(u+s))(u+1) - u &= l^{-1}((u+s)(u+1)) - u\\ &= l^{-1}((u+1+s-1)(u+1)) - u\\ &= l^{-1}((u+1)^2 + (s-1)(u+1)) - u\\ &= l^{-1}(lu + r + 2s - 1) - u \text{ (using the corollary to Theorem 1)}\\ &= 0,\end{aligned}$$

(ii)
$$\begin{aligned}&(l^{-1}(u+s))u - u - l^{-1}(u+s)\\ &= l^{-1}((u+s)u) - u - l^{-1}(u+s)\\ &= l^{-1}((u+s)u - (u+s)) - u\\ &= l^{-1}(lu) - u \text{ (using the corollary to Theorem 1 again)}\\ &= 0.\end{aligned}$$

THEOREM 2. *Let π be a generalized Hall plane coordinatized by the generalized Hall system F with defining functions f, g, h, k on the field F_0. Suppose $f(1) = r$ and $g(1) = s$ belong to $Ker(F)$ and $r + 2s - 1 = 0$. Then the quadrangle $Y = (\infty)$, $0 = (0,0)$, $I = (1,1)$, $R = (l^{-1}(u+s), 0)$, where $l = r + s - 1$, generates a Fano subplane of π for all u in $F \setminus F_0$.*

PROOF. Clearly $YI \cap OR = (1,0)$ and $OI \cap YR = (l^{-1}(u+s), l^{-1}(u+s))$. The line IR has equation $y = x(u+1) - u$ since the coordinates of R satisfy it (by Lemma 2 (i)) and those of I clearly do. Thus $IR \cap OY = (0, -u)$.

The line $(0, -u)(1,0)$ has equation $y = xu - u$. Now $(l^{-1}(u+s), l^{-1}(u+s))$ lies on this line (by Lemma 2(ii)). So the quadrangle Y, 0, I, R generates a Fano subplane of π.

4. Examples

In this section we shall give some examples of odd order generalized Hall planes to which Theorem 2 applies.

1. Hall planes. If F_0 is a finite field of odd order r and s may be chosen such that $r + 2s - 1 = 0$ and $x^2 - rx - s$ is irreducible over F_0 (see [6], p. 39). The kernel of the Hall system with defining functions $f(\alpha) = r\alpha$, $g(\alpha) = s\alpha$, $h(\alpha) = \alpha$ and $k(\alpha) = 0$ is F_0. So we see by Theorem 2 that the planes over such systems possess Fano subplanes. These planes, in fact, constitute the entire class of odd order Hall planes. The Fano subplanes just given are those discovered by Hanna Neumann [6].

2. Let $F_0 = GF(p^{2n})$ where p is an odd prime.
(a) $f(\alpha) = r^{\theta-1}\alpha$, $g(\alpha) = s\alpha^{\theta}$ $h(\alpha) = \alpha^{\theta-1}$ and $k(\alpha) = 0$ and
(b) $f(\alpha) = r^{\theta-1}\alpha^{\theta-1}$, $g(\alpha) = s\alpha^{\theta-1}$, $h(\alpha) = \alpha^{\theta-1}$ and $k(\alpha) = 0$,
where θ is an automorphism of F_0, are sets of defining functions giving rise to two classes of generalized Hall systems provided that $x^{\theta}x - rx - s$ is irreducible over F_0. These systems appear in Johnson [4] in another form. The planes they coordinatize are the planes derived from some well known semifield planes using Ostrom's derivation process.[3]

Suppose in (a) and (b) we choose θ as the involutory automorphism ϕ of F_0, $s \in GF(p^n) = F'$ such that $x^2 - (1-2s)x - s$ is irreducible over F' ([6] p. 39) and $r = 1 - 2s$. Then $p(x) = x^{\phi}x - rx - s$ is irreducible over F_0. If $r \neq 0$, $p(x)$ is irreducible over F_0 since $x^{\phi}x \in F'$ for all $x \in F_0 \setminus F'$. If $r = 0, p(x)$ is reducible over F_0. However the restriction $r \neq 0$ is not a serious one.

The generalized Hall systems defined by this choice of r, s and θ have $f(1) = (1-2s)^{\phi} = 1 - 2s$ and $g(1) = s$ and so $f(1) + 2g(1) - 1 = 0$. Also it is easy to show by applying Theorem 1 that $\text{Ker}(F) = F'$. From Theorem 2 we see that

[3] Ostrom [7] is a good reference for the process of derivation of planes.

the planes coordinatized by these generalized Hall systems possess Fano subplanes.

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References

[1] J. André, 'Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe', *Math. Z.* 60 (1954), 156–186.

[2] M. Hall, Jr. *The Theory of Groups* (Macmillan, New York, 1959).

[3] D. R. Hughes, 'Collineation groups of non-desarguesian planes. I. The Hall Veblen-Wedderburn systems', *Amer. J. Math.* 81 (1959), 921–938.

[4] N. L. Johnson, 'Translation planes constructed from semifield planes', *Pacific J. Math.* (1971), 701–712.

[5] P. B. Kirkpatrick, 'Generalization of Hall planes of odd order', *Bull. Austral. Math. Soc.* 4 (1971), 205–209.

[6] Hanna Neumann, 'On some finite non-desarguesian planes', *Arch. Math.* 6 (1955), 36–40.

[7] T. G. Ostrom, 'Semi-translation planes', *Trans. Amer. Math. Soc.* 111 (1964), 1–18.

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