Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 276–285

FREE DECOMPOSITIONS OF A LATTICE

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1. Introduction. Two basic questions have been raised for free products of lattices:

1. Do any two free products have a common refinement?
2. Can every lattice be decomposed into a free product of freely indecomposable lattices?

Both questions have been around for some time and attempts at solving them were made especially after the Structure Theorem for Free Products was discovered (see G. Grätzer, H. Lasker, and C. R. Platt [**3**]). Partial answer to question one was supplied in A. Kostinsky [**7**].

In this paper we answer both questions. Our basic observation is that the proper framework for these results is the theory of free **K**-products, that is, free products in an arbitrary equational class **K** of lattices. This approach has the advantage that the answers are supplied for all equational classes of lattices.

It is especially simple to answer Question 1 for equational classes having a special property (J) (see §2) describing certain sublattices of free products. We also show that many equational classes fail to have (J).

2. Results. An equational class **K** of lattices is called *trivial* if it is the class of all one element lattices; otherwise it is *nontrivial.*

THEOREM 1. *Let* **K** *be a nontrivial equational class of lattices. For any* L *in* **K**, *any two representations of* L *as a free* **K**-*product have a common refinement.*

It is easy to state what the common refinement is. To simplify our notation, we agree that we use the "internal" definition of free **K**-product, that is, the free **K**-factors are considered as sublattices of the free **K**-products. To further simplify our notation, let us agree that if L is a lattice and $(A_i | i \in I)$ is a family of subsets of L where each A_i is either a sublattice or the empty set, then we say that L is a free **K**-product of $(A_i | i \in I)$ if and only if L is a free **K**-product of $(A_i | i \in I$ and $A_i \neq \emptyset)$.

THEOREM 1′. *Let* L *be a free* **K**-*product of* $(A_i | i \in I)$ *and of* $(B_j | j \in J)$. *Then* L *is a free* **K**-*product of* $(A_i \cap B_j | i \in I, j \in J)$ *and, for* $i \in I$, A_i *is a free* **K**-*product of* $(A_i \cap B_j | j \in J)$, *and, for* $j \in J$, B_j *is a free* **K**-*product of* $(A_i \cap B_j | i \in I)$.

Theorem 1′ has many important consequences.

Received August 20, 1973. This research was supported by the National Research Council of Canada.

COROLLARY 1. *If L is a free* $\mathbf{K}$*-product of A and B, and also of A and C, then $B = C$.*

Corollary 1 is the *Cancellation Property for Free Products*. Observe that in Corollary 1 we assumed that A, B, and C are sublattices of L and we concluded the unexpectedly strong $B = C$. It is natural to ask whether the isomorphism of the free $\mathbf{K}$-products would imply B isomorphic to C? To state it more precisely, let L_i, $i = 1, 2$, be the free $\mathbf{K}$-products of A_i and B_i; does $L_1 \cong L_2$ and $A_1 \cong A_2$ imply that $B_1 \cong B_2$? It was remarked by S. Comer that no result of this sort can hold for an equational class $\mathbf{K}$; we can always choose $L_1 \cong L_2 \cong A_1 \cong A_2 \cong F_{\mathbf{K}}(\aleph_0)$, $B_1 = F_{\mathbf{K}}(1)$, and $B_2 = F_{\mathbf{K}}(2)$. ($F_{\mathbf{K}}(\mathfrak{m})$ is the free lattice over $\mathbf{K}$ on $\mathfrak{m}$ generators). S. Comer raised the question what happens if A_1 is finitely generated.

COROLLARY 2. *Let A be a finitely generated lattice in* $\mathbf{K}$. *If L_i, $i = 1, 2$, is a free* $\mathbf{K}$*-product of A and B_i, and $L_1 \cong L_2$, then $B_1 \cong B_2$.*

Call a lattice L in $\mathbf{K}$ *freely* $\mathbf{K}$*-indecomposable* if and only if L cannot be represented as a free $\mathbf{K}$-product of two lattices. A result, related to Corollary 2, is the following:

COROLLARY 3. *Let A be a freely* $\mathbf{K}$*-indecomposable lattice in* $\mathbf{K}$. *If L_i, $i = 1, 2$, is a free* $\mathbf{K}$*-product of A and B_i, and $L_1 \cong L_2$, then $B_1 \cong B_2$.*

The standard consequences of Theorem 1 are as follows:

COROLLARY 4. *If L is a free* $\mathbf{K}$*-product of A and B, and of A_1 and B_1, then $A \subseteq A_1$ implies that $B \supseteq B_1$.*

COROLLARY 5. *Let L be the free* $\mathbf{K}$*-product of $(A_i | i \in I)$ and of $(B_j | j \in J)$. Assume that all A_i, $i \in I$, and B_j, $j \in J$, are freely* $\mathbf{K}$*-indecomposable. Then there is a bijection φ between I and J such that $A_i = B_{i\varphi}$ for all $i \in I$.*

In other words, decomposition into freely $\mathbf{K}$-indecomposable components is absolutely unique.

COROLLARY 6. *Let L be the free* $\mathbf{K}$*-product of $(A_i | i \in I)$ and of $(B_j | j \in J)$. Assume that each A_i, $i \in I$, is freely* $\mathbf{K}$*-indecomposable. Then there is a partition $(I_j | j \in J)$ of I into nonvoid blocks such that, for each $j \in J$, B_j is the free* $\mathbf{K}$*-product of $(A_i | i \in I_j)$.*

The next result shows that there are many lattices to which Corollaries 5 and 6 do not apply.

THEOREM 2. *Let* $\mathbf{K}$ *be a nontrivial equational class of lattices. Then for each infinite cardinal $\mathfrak{m}$, there are $2^{\mathfrak{m}}$ pairwise nonisomorphic lattices of cardinality $\mathfrak{m}$ in* $\mathbf{K}$ *that cannot be represented as a free* $\mathbf{K}$*-product of freely* $\mathbf{K}$*-indecomposable lattices.*

https://doi.org/10.4153/CJM-1975-034-5 Published online by Cambridge University Press

Another way of stating Theorem 2 is that the Common Refinement Property does not apply to infinitely many decompositions.

The proof of Theorem 1′ is especially simple if the equational class **K** satisfies the following property:

(J) If L is a free **K**-product of the $(L_i | i \in I)$, A_i is a sublattice of L_i for $i \in I$, and A is the sublattice of L generated by $\bigcup (A_i | i \in I)$, then A is a free **K**-product of $(A_i | i \in I)$.

Observe that Theorem 1′ is a special case of (J); it requires (J) to hold provided that A is a free **K**-factor.

(J) was proved by B. Jónsson [**5**] for any equational class **K** having the Amalgamation Property. One can ask whether (J) holds for all equational classes. The following result shows that it is not the case; in fact, it provides $2^{\aleph_0}$ equational classes failing (J). Recall that an equational class is *arguesian* (see [**4**]) if it satisfies a special identity in six variables which reflect the Desargues Theorem for the lattice of all sublattices of a projective space.

THEOREM 3. *Let* **K** *be an equational class of modular lattices. If* **K** *satisfies property* (J), *then* **K** *is arguesian.*

Theorem 3 yields $2^{\aleph_0}$ equational classes of modular lattices failing (J). A related result can be found in B. Jónsson [**6**] in which it is proved that if **K** is an equational class of modular lattices with the property that every subdirectly irreducible member of **K** has dimension at most n for some fixed integer n, the **K** fails (J).

3. Proof of Theorem 1. Let L be a free **K**-product of $(A_i | i \in I)$. We assume that the A_i are sublattices of L. We denote by L^0 the lattice obtained from L by adjoining a new zero, denoted by 0. Observe that if $L \in \mathbf{K}$, then so is L^0. So for every $i \in I$ we can consider the homomorphisms φ_i determined by

$$a\varphi_i = a \quad \text{for } a \in A_i;$$
$$a\varphi_i = 0 \quad \text{for } a \in A_j,\ j \neq i,\ j \in I.$$

For $a \in L$ we will use the notation:

$$a\varphi_i = a_{A_i}$$

and call a_{A_i} the *lower cover* of a in A_i. It follows from the definition that

$$a_{A_i} \leqq a$$

and if $a_{A_i} \in A_i$, then a_{A_i} is the largest element of A_i below a (see B. Jónsson [**6**]).

Now let L be also the free **K**-product of $(B_j | j \in J)$. Take $a \in A_i$. We claim that

$$a_{B_j} \in (A_i \cap B_j) \cup \{0\}.$$

Since L is generated by $\bigcup (B_j | j \in J)$ we can represent a in the form

$$a = p(b_{j_1,1}, \ldots, b_{j_1,n_1}, \ldots, b_{j_k,1}, \ldots, b_{j_k,n_k}),$$

where p is an $(n_1 + \ldots + n_k)$-ary lattice polynomial, $j_1, \ldots, j_k \in J$, and $b_{j_i,m} \in B_{j_i}$ for $i = 1, \ldots, k$, $1 \leqq m \leqq n_i$. Computing the lower A_i-covers and observing that $a_{A_i} = a$ we obtain

$$a = p((b_{j_1,1})_{A_i}, \ldots, (b_{j_k,n_k})_{A_i}).$$

We can assume, without any loss of generality, that $j = j_1$. Forming lower B_j-covers we get

$$a_{B_j} = p(b_{j_1,1}, \ldots, b_{j_1,n_1}, 0, \ldots, 0, \ldots, 0),$$

and

$$a_{B_j} = (a_{A_i})_{B_j} = p(((b_{j_1,1})_{A_i})_{B_j}, \ldots, ((b_{j_k,n_k})_{A_i})_{B_j}).$$

Observe, however, that for any $r \neq 1$ and $1 \leqq t \leqq n_r$,

$$((b_{j_r,t})_{A_i})_{B_j} = 0,$$

so

$$a_{B_j} = p(b_{j_1,1}, \ldots, b_{j_1,n_1}, 0, \ldots, 0) = p(((b_{j_1,1})_{A_i})_{B_j}, \ldots, ((b_{j_1,n_1})_{A_i})_{B_j}, 0, \ldots, 0).$$

Since p is isotone and for all $1 \leqq t \leqq n_1$

$$b_{j_1,t} \geqq (b_{j_1,t})_{A_i} \geqq ((b_{j_1,t})_{A_i})_{B_j},$$

we obtain

$$\begin{aligned} a_{B_j} &= p(b_{j_1,1}, \ldots, b_{j_1,n_1}, 0, \ldots, 0) \geqq p((b_{j_1,1})_{A_i}, \ldots, (b_{j_1,n_1})_{A_i}, 0, \ldots, 0) \\ &\geqq p(((b_{j_1,n_1})_{A_i})_{B_j}, \ldots, ((b_{j_1,n_1})_{A_i})_{B_j}, 0, \ldots, 0) = a_{B_j}, \end{aligned}$$

and so

$$a_{B_j} = p((b_{j_1,1})_{A_i}, \ldots, (b_{j_1,n_1})_{A_i}, 0, \ldots, 0) \in A_i \cup \{0\}.$$

By definition, $a_{B_j} \in B_j \cup \{0\}$, hence

$$a_{B_j} \in (A_i \cap B_j) \cup \{0\},$$

as claimed.

This implies immediately that

$$a = p((b_{j_1,1})_{A_i}, \ldots, (b_{j_1,n_1})_{A_i}, \ldots, (b_{j_k,n_k})_{A_i}),$$

and so A_i is generated by $\bigcup (A_i \cap B_j | j \in J) \cup \{0\}$.

Now a simple induction on the rank of a polynomial proves that for $a \in A_i$ and for the polynomial of smallest rank p representing a in the form

$$a = p(a_1, \ldots, a_n), a_1, \ldots, a_n \in \bigcup (A_i \cap B_j | j \in J) \cup \{0\};$$

no a_i is 0, and therefore A_i is generated by $\bigcup (A_i \cap B_j | j \in J)$.

If **K** satisfies property (J) then we are done: A_i is a free **K**-product of $(A_i \cap B_j | j \in J)$.

In the general case, we form the lattice $\bar{L}$, a free **K**-product of $(A_i \cap B_j | i \in I, j \in J)$ and take the natural homomorphism

$$\alpha : \bar{L} \rightarrow L$$

that is the identity map on each $A_i \cap B_j$ ($i \in I$ and $j \in J$). We wish to show that α is an isomorphism. Since α is obviously onto, it is sufficient to show that α is one-to-one. It is easily seen that it is sufficient to verify this on the sublattice of $\bar{L}$ generated by $\bigcup (A_i \cap B_j | j \in J)$ (because this implies that A_i is a free **K**-product of $(A_i \cap B_j | j \in J)$).

First, some notation. Let $\overline{A_i}$ be the sublattice of $\bar{L}$ generated by $\bigcup (A_i \cap B_j | j \in J)$; let α_i be the restriction of α to $\overline{A_i}$;

$$\alpha_i : \overline{A_i} \rightarrow A_i$$

is again an onto homomorphism, and let Θ and Θ_i be the kernels of α and α_i respectively. Note that $\Theta_i = \Theta \cap (\overline{A_i})^2$ for $i \in I$.

Claim 1. Θ is the smallest congruence relation of $\bar{L}$ containing all the Θ_i, $i \in I$.

Proof. Θ is a congruence relation containing all Θ_i, $i \in I$, hence if Φ is the smallest one with this property, then $\Phi \leqq \Theta$. Factor α through $\bar{L}/\Phi$; $\alpha = \beta\gamma$, where β is the natural homomorphism from $\bar{L}$ onto $\bar{L}/\Phi$. Then $(\overline{A_i})\beta \cong A_i$ and so γ is an isomorphism from $(\overline{A_i})\beta$ onto A_i. It is routine to check that $\bar{L}/\Phi$ satisfies all the properties of a free **K**-product of the $(A_i | i \in I)$ and so γ is an isomorphism and $\Theta = \Phi$, as claimed.

Claim 2. For $a \in \overline{A_i}$,

$$a_{A_i \cap B_j} = (a\alpha)_{B_j}.$$

Proof. Since $\overline{A_i}$ is generated by $\bigcup (A_i \cap B_j | j \in J)$ we can write a in the form

$$a = p(a_1, \ldots, a_n, b_1, \ldots, b_m),$$

where $a_1, \ldots, a_n \in A_i \cap B_j$ and $b_1, \ldots, b_m \in \bigcup (A_i \cap B_k | k \in J,\ k \neq j)$. Then

$$a_{A_i \cap B_j} = p(a_1, \ldots, a_n, 0, \ldots, 0).$$

Applying α to the representation of a, we obtain

$$a\alpha = p(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

since α is the identity map on $\bigcup (A_i \cap B_n | i \in I, n \in J)$.
Therefore,

$$(a\alpha)_{B_j} = p(a_1, \ldots, a_n, 0, \ldots, 0),$$

since $(b_n)_{B_j} = 0$ in view of $b_n \in A_i \cap B_k$ for some $k \neq j$. This proves Claim 2.

Claim 3. Let A be a free $\mathbf{K}$-product of B, C, and D, and let a be in the sublattice generated by B and C. Then

$$a_C = a_{[C \cup D]},$$

where a_C is formed in $[B \cup C]$.

Proof. This is clear from the definition of lower cover.

Now we return to the proof of Theorem 1′. Let us fix $j \in J$ and let $\bar{B}_j$ be the sublattice of $\bar{L}$ generated by $\bigcup (A_i \cap B_j | i \in I)$.

For $a \in A_i$,

$$a_{A_i \cap B_j} = (a\alpha)_{B_j}$$

by Claim 2, and so by Claim 3,

$$a_{\bar{B}_j} = (a\alpha)_{B_j}.$$

So if $a, b \in A_i$, $a \equiv b(\Theta_i)$, then $a\alpha = b\alpha$ and thus $(a\alpha)_{B_j} = (b\alpha)_{B_j}$. Therefore,

$$a \equiv b(\Theta_i) \text{ implies that } a_{\bar{B}_j} = b_{\bar{B}_j}.$$

Using Claim 1, and the description of the minimal congruence relation containing a set of pairs of elements (see, e.g. [**1**]) we obtain:

$$a \equiv b(\Theta) \text{ implies that } a_{\bar{B}_j} = b_{\bar{B}_j}.$$

So if $a, b \in \bar{B}_j$ and $a \equiv b(\Theta)$, then

$$a = a_{\bar{B}_j} = b_{\bar{B}_j} = b,$$

that is, α is one-to-one on $\bar{B}_j$. This shows that $\bar{B}_j$ and B_j are isomorphic, completing the proof of Theorem 1′.

4. Proof of the Corollaries. We need a simple lemma:

LEMMA 1. *Let L be the free $\mathbf{K}$-product of A and B. If for a sublattice A_1 of A, L is generated by $A_1 \cup B$, then $A = A_1$.*

Proof. Let $a \in A$. Since $A_1 \cup B$ generates L,

$$a = p(a_1, \ldots, a_n, b_1, \ldots, b_m),$$

for some polynomial p and $a_1, \ldots, a_n \in A_1$, $b_1, \ldots, b_m \in B$. Therefore,

$$a = a_A = p(a_1, \ldots, a_n, 0, \ldots, 0)$$

and so $a \in (A_1 \cup \{0\}) \cap A$. Thus $A \subseteq A_1$, proving $A = A_1$.

We start proving the Corollaries with Corollary 4. Let A, B, A_1, B_1, and L be given as in Corollary 4, and let $A \subseteq A_1$. By Theorem 1′,

$$A \cap A_1, \quad A \cap B_1, \quad A_1 \cap B, \quad B \cap B_1$$

is the common refinement, and $A = A \cap A_1$, $A \cap B_1 = \emptyset$, so

$$A, \quad A_1 \cap B, \quad B \cap B_1$$

gives a free **K**-product decomposition of L. Thus, applying Lemma 1 to A_1, B_1 and to the sublattice $B \cap B_1$ of B_1, we conclude that $B \cap B_1 = B_1$, that is $B \supseteq B_1$, proving Corollary 4.

Now assume that A, B, and C are given as in Corollary 1. Then by Corollary 4, $B \subseteq C$ and $C \subseteq B$, so $B = C$.

To verify Corollary 3, let L be a free **K**-product of A, B_1 and of A', B_2, where $A \cong A'$ is freely **K**-indecomposable. By Theorem 1′ we get that A is a free **K**-product of $A \cap A'$ and $A \cap B_2$, and so

$$A \cap A' = \emptyset \quad \text{or} \quad A \cap B_2 = \emptyset.$$

Similarly,

$$A' \cap A = \emptyset \quad \text{or} \quad A' \cap B_1 = \emptyset.$$

If $A \cap A' = \emptyset$, then by Theorem 1′ we get the decomposition

$$A \cap B_2, \quad B_1 \cap A', \quad B_2 \cap B_1.$$

Applying Lemma 1 to A, B_1, and the sublattice $A \cap B_2$ of A, we conclude that $A \subseteq B_2$. By Corollary 4, $A' \subseteq B_1$, and so L is a free **K**-product of A, A', and $B_1 \cap B_2$. Thus B_1 is a free **K**-product of A' and $B_1 \cap B_2$, B_2 is a free **K**-product of A and $B_1 \cap B_2$ and therefore $B_1 \cong B_2$.

If $A \cap A' \neq \emptyset$, then $A \cap B_2 = \emptyset$ and $A' \cap B_1 = \emptyset$, and so L is a free **K**-product of $A \cap A'$ and $B_1 \cap B_2$. Applying Lemma 1 twice we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$, and so $B_1 = B_2$. The proof of Corollary 3 is thus complete.

Before we proceed to Corollary 2 we need a lemma:

LEMMA 2. *A finitely generated lattice L in* **K** *is a free* **K**-*product of finitely many freely* **K**-*indecomposable lattices.*†

Proof. Let L be n-generated. We prove Lemma 2 by induction on n. If $n = 1$, then $|L| = 1$ and so L is **K**-indecomposable. Let us assume the statement proved for lattices with less than n generators. If L is **K**-indecomposable, we have nothing to prove. So let L be a free **K**-product of A and B. Let $a_1, \ldots, a_n$ be a generating set of L. Take $a \in A$ and $b \in B$; then

$$a = p(a_1, \ldots, a_n), \quad b = q(a_1, \ldots, a_n),$$

for suitable polynomials p and q. Hence

$$a = p((a_1)_A, \ldots, (a_n)_A), \; b_A = 0 = q((a_1)_A, \ldots, (a_n)_A).$$

†A sharper result follows from the main result of G. Grätzer and J. Sichler, Proc. Amer. Math. Soc. *46* (1974), 9–14, namely that an n-generated lattice is a free K-product of at most n lattices.

From the first equation we conclude that A is generated by $(a_1)_A, \ldots, (a_n)_A$, and from the second we see that so is $\{0\}$, and therefore $(a_i)_A = 0$ for some $1 \leqq i \leqq n$. Therefore, A is $(n-1)$-generated. Similarly, B is $(n-1)$-generated, and by induction hypothesis, A and B are free **K**-products of freely **K**-indecomposable lattices. And, therefore, so is L.

Now Corollary 2 is trivial. By Lemma 2 we write A as a free **K**-product of freely **K**-indecomposable lattices and the free **K**-factors can be cancelled one-by-one by Corollary 3.

Corollaries 5 and 6 require no proof.

5. Proof of Theorem 2. Let A be a lattice in **K**. We define a partial lattice $P(A)$ as follows:

$P(A) = A \cup \{a_0, a_1, a_2, \ldots\} \cup \{b_0, b_1, b_2, \ldots\}$;
if $x \wedge y = z$ in A, then $x \wedge y = z$ in $P(A)$;
if $x \vee y = z$ in A, then $x \vee y = z$ in $P(A)$;
$a_0 \wedge x = a_0$ and $a_0 \vee x = x$, for all $x \in A$;
$b_n = a_n \wedge b_{n+1}$ for $n = 0, 1, 2, \ldots$.

All the meets and joins are the ones listed above and in addition the meets and joins of comparable elements, namely, $b_0 \wedge x = x$, $b_0 \vee x = x$ for all $x \in P(A)$, $b_n \wedge b_m = b_n$, $b_n \vee b_m = b_m$, $b_n \wedge a_m = b_n$ and $b_n \vee a_m = a_m$ for $n \leqq m$.
Let $G(A)$ denote the lattice freely generated by $P(A)$ in **K**.

Claim 1. $G(A)$ contains $P(A)$.

Proof. For an integer n, let

$$P_n(A) = A \cup \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$$

be the partial sublattice of $P(A)$. It is sufficient to show that for any integer n, $P_n(A)$ can be isomorphically represented in a lattice in **K**. Let L_n be the free **K**-product of A and $F_{\mathbf{K}}(n+1)$, the latter freely generated by $x_0, \ldots, x_n$. We view A and $F_{\mathbf{K}}(n+1)$ as sublattices of L_n. We map $P_n(A)$ into L_n:

$$\begin{aligned}
&a \to x_0 \vee a \text{ for } a \in A\\
&a_0 \to x_0\\
&\quad\vdots\\
&a_{n-1} \to x_{n-1}\\
&b_{n-1} \to x_{n-1} \wedge x_n\\
&b_{n-2} \to x_{n-2} \wedge x_{n-1} \wedge x_n\\
&\quad\vdots\\
&b_0 \quad \to x_0 \wedge x_1 \wedge \ldots \wedge x_n.
\end{aligned}$$

This map is obviously one-to-one and preserves all meets and joins of $P_n(A)$.

Claim 2. For any integer n, $G(A)$ is a free **K**-product of the lattices

$$A \cup \{a_0\}, \{a_1\}, \ldots, \{a_{n-1}\}, \text{ and } G_n,$$

where G_n is the sublattice of $G(A)$ generated by $\{a_n, b_n, a_{n+1}, b_{n+1}, \ldots\}$.

Proof. This follows trivially from the definition of free **K**-product as the lattice in **K** that is freely generated by the partial lattice formed as a disjoint union of the factors.

Claim 3. $G(A)$ is not the free **K**-product of freely **K**-indecomposable lattices.

Proof. Claim 2 yields that $G(A)$ can be written as a free **K**-product of n lattices for every integer n. So by Corollary 6, any decomposition of $G(A)$ into freely **K**-indecomposable factors has to have infinitely many factors. However, a free **K**-product of infinitely many lattices does not have a least element. Since b_0 is the zero of $G(A)$, this verifies the claim.

Claim 4. $G(A) \cong G(B)$ implies that $A \cong B$.

Proof. By Claim 2, $A \cup \{a_0\}$ is a free factor of $G(A)$. $A \cup \{a_0\}$ has a zero, a_0, which is meet-irreducible, therefore $A \cup \{a_0\}$ is freely **K**-indecomposable. Conversely, if A_1 is a freely **K**-indecomposable factor of $G(A)$ with more than one element, then by Theorem 1′ and Claim 2, A_1 has to be contained in one of $A \cup \{a_0\}$, $\{a_1\}$, $\ldots$, and G_n. Since $|A_1| > 1$, $A_1 \subseteq A \cup \{a_0\}$ or $A_1 \subseteq G_n$. If $A_1 \subseteq A \cup \{a_0\}$, then $A_1 = A \cup \{a_0\}$, otherwise $A \cup \{a_0\}$ would not be freely **K**-indecomposable. Now if $A_1 \not\subseteq A \cup \{a_0\}$, then $A_1 \subseteq G_n$ for all n, and so $A_1 \subseteq \bigcap (G_n | n = 1, 2, \ldots) = \emptyset$, a contradiction

Therefore, we recognize $A \cup \{a_0\}$ in $G(A)$ as the only free factor with more than one element, verifying the claim.

Now Theorem 2 is trivial. If A ranges over a set of pairwise nonisomorphic lattices of cardinality $\mathfrak{m}$ (all belonging to **K**), then $G(A)$ will give the same number of pairwise nonisomorphic lattices of cardinality $\mathfrak{m}$ each of which satisfies Claim 3. It is well known that there are $2^{\mathfrak{m}}$ pairwise nonisomorphic lattices of cardinality $\mathfrak{m}$ in **K**.

6. Proof of Theorem 3. The proof of Theorem 3 is based on a lemma of [**2**].

LEMMA 3. *Let L be a modular lattice and let $M_5 = \{o, a, b, c, i\}$ be a sublattice of L which is the five element nondistributive lattice with o as its least and i its greatest element. Then the interval $[o, a]$ as a lattice satisfies the arguesian identity.*

Now let **K** be an equational class of modular lattices. Let us assume that **K** does not satisfy the arguesian identity but that **K** satisfies (J).

Since $\mathbf{K}$ is not arguesian and the arguesian identity has six variables, $F_{\mathbf{K}}(6)$ is not arguesian; let $x_1, \ldots, x_6$ be the free generators of $F_{\mathbf{K}}(6)$.

Now let L be the free K-product of M_5 and $F_{\mathbf{K}}(6)$. Let $A = \{o, a, b, i\}$ be a four-element sublattice of M_5. Since (J) is assumed to hold in $\mathbf{K}$, the sublattice L_1 of L generated by A and $F_{\mathbf{K}}(6)$ is the free $\mathbf{K}$-product of A and $F_{\mathbf{K}}(6)$. But A is $F_{\mathbf{K}}(2)$, and so L_1 is the free lattice over $\mathbf{K}$ on the generators $a, b, x_1, \ldots, x_6$.

Now let L_2 be the sublattice of L_1 generated by the elements $y_i = (a \wedge x_i) \vee o, i = 1, 2, \ldots, 6$. We claim that $L_2 \cong F_{\mathbf{K}}(6)$ with $y_1, \ldots, y_6$ as the free generators. Indeed, consider the homomorphism of L_1 onto $F_{\mathbf{K}}(6)$ defined by

$$a \rightarrow x_1 \vee x_2 \vee \ldots \vee x_6, b \rightarrow x_1 \wedge x_2 \wedge \ldots \wedge x_6, x_i \rightarrow x_i, i = 1, 2, \ldots, 6.$$

This homomorphism maps $y_i \rightarrow x_i$, $i = 1, 2, \ldots, 6$, and so L_2 onto $F_{\mathbf{K}}(6)$; therefore $L_2 \cong F_{\mathbf{K}}(6)$.

We conclude that in L_1, the interval $[o, a]$ is not arguesian (since it contains a copy of $F_{\mathbf{K}}(6)$ which is not arguesian). But the interval $[o, a]$ of L_1 is a sublattice of the interval $[o, a]$ of L, and by Lemma 3 the latter interval is arguesian, a contradiction.

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