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A GENERALIZATION OF RADON'S THEOREM II

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A new proof is given of the following result:

Let m and d be positive integers, and let a set of $md + m - d$ points be given in d-dimensional space. Then the set can be partitioned into m sets such that the m convex polytopes spanned by the sets have a non-empty intersection.

Let n-set mean a set of n points in R^d . We shall say that an n-set is m-divisible if it can be divided into m sets in such a way that the convex hulls of the m sets have a non-empty intersection. In 1964, [6], I proved the following:

THEOREM. *Any* $(m(d+1)-d)$*-set is* m*-divisible.*

The proof has been regarded as difficult and it is therefore a pleasure to be able to present a much simpler proof below. It is also a pleasure to acknowledge my debt to Imre Bárány, whose proof of Theorem 2.2 in [1] inspired the present one.

The proof is by induction on m . The case $m = 1$ is trivial and so, assuming that the theorem holds for $m = k > 0$, we are to prove that it holds for $m = k + 1$. Put $K = (k+1)(d+1) - d$ and let a K-set $\Omega_0 = \left\{p_1^0, \ldots, p_K^0\right\}$ be given. If the theorem is false for Ω_0 , then there is an $\varepsilon > 0$ such that it is also false for any set $\Omega = \{p_1, \ldots, p_K\}$,

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where $|p_i - p_i^0| < \varepsilon$, $i = 1, \ldots, K$. We choose Ω strongly independent, as defined by Reay [4], and prove the theorem for Ω , which suffices. Strong independence of Ω means that for each t and for any t affine subspaces $B_1, \ldots, B_t$ of R^d , spanned by pairwise disjoint subsets of Ω , $\dim(B_1 \cap \ldots \cap B_t) = \max(-1, \dim B_1 + \ldots + \dim B_t - (t-1)d)$.

Consider a partition in Ω , that is a partition of a subset of Ω , consisting of disjoint non-empty sets $\Omega_0, \ldots, \Omega_k$, having the property that $\operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k \neq \emptyset$. Such partitions exist by the case $m = k$ of the theorem. Choose such a partition for which the distance from $\operatorname{conv} \Omega_0$ to $\operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k$ is minimal. If this distance is zero, we are through, and so we assume it to be positive. A contradiction will now be obtained by showing that some other partition in Ω will make the distance considered smaller.

Let $q \in \operatorname{conv} \Omega_0$ and $r \in \operatorname{conv} \Omega_1 \cap \ldots \cap \operatorname{conv} \Omega_k$ be a pair of points realizing the distance. By Carathéodory's theorem there is a simplex, with vertex set $\Omega_0' \subseteq \Omega_0$, such that q is in the relative interior of $\operatorname{conv} \Omega_0'$. Replacing Ω_0 by Ω_0' , we may thus assume that q is in the relative interior of $\operatorname{conv} \Omega_0$. Similarly, we may assume that each $\operatorname{conv} \Omega_i$, $i = 1, \ldots, k$, is a simplex with r in its relative interior. Putting $A_i = \operatorname{aff} \Omega_i$, the affine hull of Ω_i , we then get $\dim A_i = |\Omega_i| - 1$, $i = 0, \ldots, k$. Furthermore, by the condition of strong independence,

$$\dim(A_1 \cap \ldots \cap A_k) = \dim A_1 + \ldots + \dim A_k - (k-1)d$$
$$= |\Omega_1 \cup \ldots \cup \Omega_d| - k + d - kd .$$

We now want to prove that $\Omega_0 \cup \ldots \cup \Omega_k$ is a proper subset of Ω , which will leave us some point p_j to add to a suitable Ω_i so as to lower the distance in question. Consider the parallel hyperplanes H_q through q and H_r through r , both orthogonal to $q - r$. The open slab between them clearly separates conv Ω_0 from conv $\Omega_1 \cap \ldots \cap$ conv Ω_k . Furthermore $\Omega_0 \subset H_q$, while conv $\Omega_1 \cap \ldots \cap$ conv $\Omega_k \subset H_r$. The first of these inclusions holds true because H_q is a supporting hyperplane of conv Ω_0 in q and q is in the relative interior of the simplex conv Ω_0 . The second one holds because r is in the relative interior of each simplex conv $\Omega_1, \ldots,$ conv Ω_k , so that some neighbourhood of r in $A_1 \cap \ldots \cap A_k$ is contained in conv $\Omega_1 \cap \ldots \cap$ conv Ω_k , and is thus supported by H_r in R . (If it was not, it would meet the open slab mentioned above.) These two inclusions show that $A_0 \subset H_q$ and $A_1 \cap \ldots \cap A_k \subset H_r$, which, by the strong independence in Ω can happen only if

$$\dim A_0 + \dim\left(A_1 \cap \ldots \cap A_k\right) < d .$$

Thus

$$|\Omega_0| - 1 + |\Omega_1 \cup \ldots \cup \Omega_k| - k + d - kd < d ,$$

so that

$$|\Omega_0 \cup \ldots \cup \Omega_k| < k(d+1) + 1 = |\Omega| .$$

It is no restriction to assume that $p_1 \notin \Omega_0 \cup \ldots \cup \Omega_k$.

The easier case is when p_1 is in that open halfspace, bounded by H_q , in which H_r lies. The segment qp_1 will then be in $\operatorname{conv}\left(\Omega_0 \cup \{p_1\}\right)$ and for any point q' on it sufficiently near q , but not equal to q , we will have $|q'-r| < |q-r|$. Thus the distance from $\operatorname{conv}\left(\Omega_0 \cup \{p_1\}\right)$ to conv $\Omega_1 \cap \ldots \cap$ conv Ω_k will be smaller than that from conv Ω_0 .

In the second, more difficult, case, p_1 is separated (weakly) from H_r by H_q . We shall see that for some $i \in \{1, \ldots, k\}$ there is a ray from r , contained in $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k$, which passes through the simplex $\operatorname{conv}\left(\Omega_i \cup \{p_1\}\right)$ and also lies in the halfspace bounded by H_r

and containing H_q . Any point r' $(\neq r)$ on this ray, sufficiently near r , will then satisfy $|q-r'| < |q-r|$ and also be in $\text{conv}\,\Omega_1 \cap \ldots \cap \text{conv}(\Omega_i \cup \{p_1\}) \cap \ldots \cap \text{conv}\,\Omega_k$, so that the desired contradiction will be obtained once again.

For the proof of the existence of the ray just mentioned, we introduce an affine coordinate system in which $r = (0, \ldots, 0)$, $p_1 = (1, \ldots, 1)$, A_1 is given by $x_j = 0$, $j \in C_1, \ldots, A_k$ by $x_j = 0$, $j \in C_k$. Here $C_1, \ldots, C_k$, form a partition in $\{1, \ldots, d\}$ (with possibly some empty parts). The disjointness of the C_i is a consequence of the strong independence in Ω . Furthermore H_r is given by $a_1x_1 + \ldots + a_dx_d = 0$, where $a_j = 0$ for $j \notin (C_1 \cup \ldots \cup C_k)$, as we have seen that $A_1 \cap \ldots \cap A_k \subset H_r$. The a_j's are normalized so that p_1 (which is not in H_r) lies in the hyperplane $a_1x_1 + \ldots + a_dx_d = 1$. Thus H_q will have the equation $a_1x_1 + \ldots + a_dx_d = a$, with $0 < a \leq 1$.

Now consider the flat $\text{aff}(\Omega_1 \cup \{p_1\}) \cap A_2 \cap \ldots \cap A_s$. It is no restriction to assume that $C_1 = \{1, \ldots, |C_1|\}$, $C_2 = \{|C_1|+1, \ldots, |C_1|+|C_2|\}$ and so on. Then the point $p_1' = (1, \ldots, 1, 0, \ldots, 0)$ ($|C_1|$ 1's) is in the flat, as it equals $p_1 + (0, \ldots, 0, -1, \ldots, -1)$. Furthermore p_1' and p_1 are in the same open halfspace of $\text{aff}(\Omega_1 \cup \{p_1\})$, bounded by A_1 , as $(0, \ldots, -1, \ldots, -1)$ is in A_1 . Hence the ray from r through p_1' passes through the simplex $\text{conv}(\Omega_1 \cup \{p_1\})$. It will also lie in the halfspace bounded by H_r and containing H_q , provided $(a_1, \ldots, a_d) \cdot (1, \ldots, 1, 0, \ldots, 0) = a_1 + \ldots + a_{|C_1|} > 0$. Thus, if $a_1 + \ldots + a_{|C_1|} > 0$, we have what we want. Similarly, we shall be satisfied with $a_{|C_1|+1} + \ldots + a_{|C_1|+|C_2|} > 0$, and so on. But one of these equalities must hold, as $a_1 + \ldots + a_{|C_1|+\ldots+|C_k|} = 1$.

Doignon and Valette [2] have proved that our theorem remains valid in any affine space over an ordered division ring. The proof given above can be modified so as to show this. Finally I would like to call the reader's attention to the recent survey papers by Eckhoff [3] and Reay [5], which give a lot of information on Radon's theorem and related matters.

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