projections of K on the axis and on the perpendicular through G to SP respectively, $GU = e\,.\,KI$.

From the similar triangles KGI, PZM, $\dfrac{KI}{KG} = \dfrac{PM}{PZ}$,

.... KGU, PSZ, $\dfrac{KG}{GU} = \dfrac{PZ}{SP}$.

$$\text{Therefore}\quad \frac{KI}{GU} = \frac{PM}{SP} = \frac{1}{e}, \quad \text{or } GU = e\,.\,KI.$$

If K is (r, θ) and if KV is perpendicular to SP,

$$KI = r \sin\theta, \quad KV = r\sin(\theta - a)$$

and from the equation of the conic $SP = \dfrac{l}{1 + e\cos a}$.

Again $\quad GJ = GS\,.\sin a = e\,.\,SP \sin a$.

Therefore $er\sin\theta + r\sin(\theta - a) = \dfrac{el\sin a}{1 + e\cos a}$

or
$$\frac{e\sin a}{1 + e\cos a}\cdot\frac{l}{r} = \sin(\theta - a) + e\sin\theta$$

is the equation of the normal.

A Problem in Confocals

By Robert J. T. Bell.

In the standard text books on Analytical Geometry the following problem occurs in the exercises on confocals:

The product of the four normals from a point P to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to $\dfrac{\lambda_1\lambda_2(\lambda_1 - \lambda_2)}{a^2 - b^2}$ where λ_1 and λ_2 are the parameters of the confocals through P.

In the volume of solutions to the examples in C. Smith's "Conic Sections" two methods of solving the problem are given, the second of which, it is pointed out, is due to A. R. Forsyth. The following method is on much the same lines as Forsyth's, but seems to be more direct.

If P is (α, β), λ_1, λ_2 are the roots of the equation

$$f(\lambda) \equiv (\lambda + a^2)(\lambda + b^2) - \alpha^2(\lambda + b^2) - \beta^2(\lambda + a^2) = 0,$$

and
$$\frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda + a^2)(\lambda + b^2)} \equiv 1 - \frac{\alpha^2}{\lambda + a^2} - \frac{\beta^2}{\lambda + b^2},$$

whence, by the rule for partial fractions,

$$\alpha^2 = \frac{(a^2 + \lambda_1)(a^2 + \lambda_2)}{a^2 - b^2} \text{ and } \beta^2 = -\frac{(b^2 + \lambda_1)(b^2 + \lambda_2)}{a^2 - b^2}.$$

If the normal at $Q, (x_1, y_1)$ passes through $P, (\alpha, \beta)$,

$$\frac{\alpha - x_1}{x_1/a^2} = \frac{\beta - y_1}{y_1/b^2} = k, \text{ say.} \tag{1}$$

Therefore $x_1 = \dfrac{a^2\alpha}{a^2 + k}$, $y_1 = \dfrac{b^2\beta}{b^2 + k}$, and hence

$$\frac{a^2\alpha^2}{(a^2 + k)^2} + \frac{b^2\beta^2}{(b^2 + k)^2} = 1,$$

or
$$\phi(k) \equiv (k + a^2)^2(k + b^2)^2 - a^2\alpha^2(b^2 + k)^2 - b^2\beta^2(a^2 + k)^2 = 0, \tag{2}$$
$$\equiv (k - k_1)(k - k_2)(k - k_3)(k - k_4),$$

so that k_1, k_2, k_3, k_4 give the first of the four normals from (α, β). Again if n is the distance from $Q(x_1, y_1)$ to $P(\alpha, \beta)$, we have from (1)

$$k = \frac{(\alpha - x_1)^2 + (\beta - y_1)^2}{(\alpha - x_1)x_1/a^2 + (\beta - y_1)y_1/b^2} = \frac{n^2}{\dfrac{\alpha x_1}{a^2} + \dfrac{\beta y^1}{b^2} - 1}.$$

Therefore
$$n^2 = k\left(\frac{\alpha^2}{a^2 + k} + \frac{\beta^2}{b^2 + k} - 1\right)$$
$$= -k\frac{f(k)}{(a^2 + k)(b^2 + k)} = -k\frac{(k - \lambda_1)(k - \lambda_2)}{(a^2 + k)(b^2 + k)},$$

whence
$$\Pi n^2 = \frac{k_1 k_2 k_3 k_4\, \Pi(\lambda_1 - k_1) \,.\, \Pi(\lambda_2 - k_1)}{\Pi(a^2 + k_1) \,.\, \Pi(b^2 + k_1)}$$
$$= \frac{k_1 k_2 k_3 k_4\ \phi(\lambda_1) \,.\, \phi(\lambda_2)}{\phi(-a^2) \,.\, \phi(-b^2)}.$$

But $k_1 k_2 k_3 k_4 = a^2 b^2 (a^2 b^2 - b^2\alpha^2 - a^2\beta^2) = a^2 b^2 \lambda_1 \lambda_2$; by (1) and (2);

$$\phi(-a^2) = -a^2\alpha^2(a^2 - b^2)^2, \quad \phi(-b^2) = -b^2\beta^2(a^2 - b^2)^2;$$

$$\phi(\lambda_1) = (\lambda_1 + a^2)^2(\lambda_1 + b^2)^2 - \frac{a^2(a^2 + \lambda_1)(a^2 + \lambda_2)(b^2 + \lambda_1)^2}{a^2 + b^2} + \frac{b^2(b^2 + \lambda_1)(b^2 + \lambda_2)(a^2 + \lambda_1)^2}{a^2 - b^2},$$
$$= (a^2 + \lambda_1)(b^2 + \lambda_1)\lambda_1(\lambda_1 - \lambda_2), \text{ on reduction.}$$

Similarly
$$\phi(\lambda_2) = (a^2 + \lambda_2)(b^2 + \lambda_2)\lambda_2(\lambda_2 - \lambda_1), \text{ and so}$$
$$\phi(\lambda_1) \,.\, \phi(\lambda_2) = (a^2 - b^2)^2\alpha^2\beta^2\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2.$$

Hence
$$\Pi n^2 = \frac{\lambda_1^2\lambda_2^2(\lambda_1 - \lambda_2)^2}{(a^2 - b^2)^2}.$$