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Small Flag Complexes with Torsion

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Abstract. We classify flag complexes on at most 12 vertices with torsion in the first homology group. The result is moderately computer-aided.

As a consequence we confirm a folklore conjecture that the smallest poset whose order complex is homotopy equivalent to the real projective plane (and also the smallest poset with torsion in the first homology group) has exactly 13 elements.

1 Introduction

There is a well-known 6-vertex simplicial triangulation of the real projective space $\mathbb{R}P^2$. It is smallest in terms of the number of vertices, and it is also the minimal simplicial complex with torsion in the first homology group. In this note we consider analogous minimization questions in the classes of flag complexes and order complexes, both of which are widely used combinatorial models of topological spaces.

If G is a simple, undirected graph, then the *clique complex* $\mathrm{Cl}(G)$ of G is the simplicial complex whose vertices are the vertices of G and whose faces correspond to the cliques in G. Clique complexes are also known as *flag complexes*. We will show the following fact.

Theorem 1.1 *We have the following classification:*

(i) *If G is a graph with at most 10 vertices, then $H_1(\mathrm{Cl}(G);\mathbb{Z})$ is torsion-free.*
(ii) *There are exactly four graphs, K_1, K_2, K_3, K_4, with 11 vertices for which $H_1(\mathrm{Cl}(G);\mathbb{Z})$ has torsion.*
(iii) *There exist 363 graphs, $L_1,\dots,L_{363}$, with 12 vertices and with the following property. If G is any 12-vertex graph for which $H_1(\mathrm{Cl}(G);\mathbb{Z})$ has torsion, then either G is one of the L_i or $G\setminus v$ is one of the K_i for some vertex v of G.*

Parts (i) and (ii) of the above theorem were also proved in [8]. Our main effort is in proving part (iii), but the method we use also verifies (i) and (ii). For a list of the graphs K_i, L_i, see Section 4.

Next, if P is a poset, then the *order complex* $\Delta(P)$ of P is the simplicial complex whose vertices are the elements of P and whose faces correspond to chains in P. This is the standard construction of the classifying space of P.

Note that $\Delta(P)$ is the clique complex of the *comparability graph* of P (this graph has vertex set P and an edge between every two comparable elements). Hence every

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order complex is a flag complex, and using the classification given by Theorem 1.1 we will obtain the following.

Theorem 1.2 *If P is a poset with at most 12 elements, then the group $H_1(\Delta(P);\mathbb{Z})$ is torsion-free.*

This confirms a conjecture stated in [7, Sect.4], [2, Ex.7.1.1], or [12, Conj.5.4] that any poset whose order complex is homotopy equivalent to $\mathbb{R}P^2$ (or indeed any poset with torsion in first homology) must have at least 13 elements. There is a known poset P with exactly 13 elements, for which $\Delta(P)$ is actually homeomorphic to $\mathbb{R}P^2$ (see the same references). Our result also determines that it is the minimal model of $\mathbb{R}P^2$ in finite T_0-spaces, in the sense of [3, 9].

The obvious way to prove Theorem 1.1 would be to compute $H_1(\mathrm{Cl}(G);\mathbb{Z})$ for all graphs G on at most 12 vertices. However, there are approximately $1.6 \cdot 10^{11}$ such graphs [11, A000088], which makes a direct check infeasible. An alternative approach to Theorem 1.2 is to go through all posets with at most 12 elements. There are approximately 10^9 of them [11, A000112], but this time the problem lies in the non-availability of good software for the generation of posets (at least to the author's knowledge). Our approach is to reduce the search space of graphs so that in the end homology only needs to be computed for fewer than 10^8 cases.

2 Enumeration of Graphs with Torsion

We will use some standard notation. If v is a vertex of a graph G, then $N_G(v)$ is the set of neighbors of v in G. We write $\mathrm{lk}_G v = G[N_G(v)]$ for the subgraph of G induced by the neighborhood of v. Observe that this notation coincides with the usual notion of link for simplicial complexes; *i.e.,* we have $\mathrm{lk}_{\mathrm{Cl}(G)} v = \mathrm{Cl}(\mathrm{lk}_G v)$, where $\mathrm{lk}_{\mathrm{Cl}(G)} v$ is understood in the simplicial sense. The degree of a vertex v is $\deg_G(v) = |N_G(v)|$. The complement $\overline{G}$ is the graph with vertex set $V(G)$ whose edges are the non-edges of G. The independence complex $\mathrm{Ind}(G)$ of G is defined as $\mathrm{Ind}(G) = \mathrm{Cl}(\overline{G})$. By K_m we denote the clique (complete graph) on m vertices. Then $\overline{K_m}$ is the graph with m vertices and no edges.

In order to prove Theorem 1.1 it suffices to characterize those graphs with torsion in first homology for which the removal of any vertex yields a graph without torsion. This motivates the next few definitions.

Let G be a graph with n vertices. We say that G is an H_1*-torsion* graph if the group $H_1(\mathrm{Cl}(G);\mathbb{Z})$ has torsion. We will say that G *has cyclic links* if for every vertex v of G the group $H_1(\mathrm{Cl}(\mathrm{lk}_G v);\mathbb{Z})$ is nontrivial. If G is H_1-torsion, then we will say G is *irreducible* if for every vertex v of G the graph $G \setminus v$ is not H_1-torsion. Finally, we say G is *tame* if every vertex v satisfies $4 \leq \deg_G(v) \leq n-4$.

The next lemma is a straightforward compilation of the statements of [5, Lemma 2.5] and [1, Theorem 3.4].

Lemma 2.1 *If a graph H has a vertex v such that $\mathrm{lk}_H v$ is either K_m or $\overline{K_m}$ for $m \geq 1$, then $\mathrm{Ind}(H)$ is homotopy equivalent to the suspension of some space.*

It has the following direct consequence.

Lemma 2.2 *If G is a graph with n vertices and v is a vertex with $\deg_G(v) \geq n-3$, then $\mathrm{Cl}(G)$ is homotopy equivalent to the suspension of some space.*

Proof Let $H = \overline{G}$. We have $\deg_H(v) \leq 2$ and $\mathrm{Cl}(G) = \mathrm{Ind}(H)$.

If $\deg_H(v) = 0$, then $\mathrm{Ind}(H)$ is a cone with apex v, hence a contractible space. If $\deg_H(v) = 1$ or $\deg_H(v) = 2$, then $\mathrm{lk}_H v$ is one of the graphs K_1, K_2, or $\overline{K_2}$, and Lemma 2.1 applies. ■

The key to (fairly) efficient enumeration of irreducible H_1-torsion graphs is the following observation.

Lemma 2.3 *If G is an irreducible H_1-torsion graph, then G is connected, tame, and has cyclic links.*

Proof It is clear that G is connected. We start by proving that G has cyclic links. Let v be any vertex of G. We have a cofibre sequence

$$\mathrm{Cl}(\mathrm{lk}_G v) \to \mathrm{Cl}(G \setminus v) \to \mathrm{Cl}(G)$$

and hence a long exact sequence of homology groups (with $\mathbb{Z}$ coefficients):

$$\cdots \to H_1(\mathrm{Cl}(\mathrm{lk}_G v)) \to H_1(\mathrm{Cl}(G \setminus v)) \to H_1(\mathrm{Cl}(G)) \to H_0(\mathrm{Cl}(\mathrm{lk}_G v)) \to \cdots .$$

The conclusion $H_1(\mathrm{Cl}(\mathrm{lk}_G v)) \neq 0$ follows by a standard exact sequence argument from the fact that $H_1(\mathrm{Cl}(G))$ has torsion, while $H_1(\mathrm{Cl}(G \setminus v))$ and $H_0(\mathrm{Cl}(\mathrm{lk}_G v))$ are torsion-free.

Next we prove that G is tame. Since $H_1(\mathrm{Cl}(H); \mathbb{Z}) = 0$ for all graphs H with at most 3 vertices, the condition $H_1(\mathrm{Cl}(\mathrm{lk}_G v)) \neq 0$ means that for every vertex v of G we have $|N_G(v)| \geq 4$, *i.e.*, $\deg_G(v) \geq 4$. The inequality $\deg_G(v) \leq n-4$, where n is the number of vertices of G, follows from Lemma 2.2, since the first homology group of a suspension is torsion-free. ■

The next lemma records the computer-assisted part of the argument.

Lemma 2.4 *If G is an irreducible H_1-torsion graph with at most 12 vertices, then G is one of the graphs $K_1, \dots, K_4, L_1, \dots, L_{363}$ appearing in Theorem 1.1.*

Proof By Lemma 2.3 all irreducible H_1-torsion graphs can be found among connected tame graphs with cyclic links. Let $n \leq 12$ be the number of vertices we are considering. If $n \leq 7$, then there are no tame graphs. For each $8 \leq n \leq 12$ we generate all n-vertex connected graphs, pick the tame ones and among those pick the ones with cyclic links. In the resulting set of graphs we then check for torsion in the first homology of the clique complex, and finally, in the case $n = 12$, we eliminate the graphs that are reducible.

The numbers of graphs that arise at the consecutive steps of this reduction are shown in Table 1. More specific implementation details are given in Section 3. ■

Lemma 2.4 clearly implies Theorem 1.1. We now proceed with the proof of Theorem 1.2.

Lemma 2.5 *Each of the graphs K_i, L_i of Theorem 1.1 contains an induced 5-cycle.*

n	connected graphs [11, A001349]	connected, tame	connected, tame with cyclic links	connected, tame with cyclic links and H_1-torsion	irreducible H_1-torsion
8	11 117	6	0	0	0
9	261 080	634	2	0	0
10	11 716 571	194 917	492	0	0
11	1 006 700 565	64 434 518	207 839	4	4
12	164 059 830 476	26 169 627 695	93 453 159	394	363

Table 1: Various graph classes appearing in the consecutive steps of the computation.

Proof This is an immediate brute-force computer check. ∎

Proof of Theorem 1.2 Suppose, on the contrary, that P is a poset with at most 12 vertices and with torsion in $H_1(\Delta(P);\mathbb{Z})$. Let G be the comparability graph of P so that $\Delta(P) = \mathrm{Cl}(G)$. By Theorem 1.1 the graph G contains, as an induced subgraph, one of K_i or L_i, and therefore, by Lemma 2.5, it also contains an induced 5-cycle. However, the comparability graph of a poset cannot have an induced cycle of odd length greater than three, and this contradiction ends the proof. ∎

3 Implementation Outline

There are some implementation tricks that speed up the computation (which, otherwise, is a straightforward translation of the proof of Lemma 2.4). For the interested readers we give an outline for the most time-consuming case of 12-vertex graphs.

Let $\mathcal{C}_8$ be the set of all graphs H with exactly 8 vertices (not necessarily connected) and with $H_1(\mathrm{Cl}(H);\mathbb{Z}) \neq 0$. This set can be computed by a brute-force algorithm that checks all 12346 of the 8-vertex graphs. We have $|\mathcal{C}_8| = 7702$.

If G is a graph and k is an integer, let $G + k$ denote G with additional k isolated vertices. If $\mathcal{C}$ is a class of graphs, then $\mathcal{C} + k = \{G + k \,:\, G \in \mathcal{C}\}$. When G is a graph, and v is its vertex, let G_v denote the graph with the same vertices as G and with the edge set $\{xy \,:\, x, y \in N_G(v),\ xy \in E(G)\}$. Using our previous notation,

$$G_v = \mathrm{lk}_G v + \big(|V(G)| - \deg_G(v)\big)\,.$$

In the first phase we use the program `geng` from the `nauty` package [10] to generate all connected, tame 12-vertex graphs (`geng -c -d4 -D8`). From this set we need to choose graphs with cyclic links. This condition is easily verified as follows. If G is a graph and v is its vertex, then the graph G_v is easily computable in the internal representation of `nauty` using quick bit operations. We have that $H_1(\mathrm{Cl}(G_v);\mathbb{Z}) \neq 0$ if and only if $H_1(\mathrm{Cl}(\mathrm{lk}_G v);\mathbb{Z}) \neq 0$. Since G is tame, we have $G_v = H + 4$ for some 8-vertex graph H. From this we conclude that to verify $H_1(\mathrm{Cl}(\mathrm{lk}_G v);\mathbb{Z}) \neq 0$ it suffices to check that $G_v \in \mathcal{C}_8 + 4$. We now use the capability of `nauty` to compute canonical representations of graphs with the property that two graphs are isomorphic if and only if their canonical representations are equal. We precompute the canonical representations of graphs in $\mathcal{C}_8 + 4$, sort them, and for every vertex v of a graph G under consideration we perform a binary search for the canonical representation of G_v in that list.

The first phase required in total approximately 184 hours of processor time and yielded approximately $9 \cdot 10^7$ graphs (Table 1). This quantity is surprisingly close to what is predicted by the appealing (but not correct) heuristic that assumes that all vertex links in G are independent random graphs, with each isomorphism class equally likely. Then the probability of G having cyclic links would be $(\frac{7702}{12346})^{12} \approx 0.003474$, giving the expected number of such graphs (among tame graphs) as approximately $26 \cdot 10^9 \cdot 0.003474 \approx 9 \cdot 10^7$.

In the second phase we used the `chomp` program [4] to compute $H_1(\mathrm{Cl}(G); \mathbb{Z})$ for all the graphs obtained in the first phase. In fact, it is faster to check H_1 of the two-dimensional simplicial complex whose maximal faces are the triangles in G. This does not influence the existence of torsion in H_1. This phase required approximately 30 hours and produced 394 graphs. At the end we eliminated the graphs that still contained one of the K_i as an induced subgraph and this left the final 363 graphs L_i.

Note that at the same speed the brute-force check by `chomp` of the homology of all tame, connected, 12-vertex graphs would take approximately 350 days. This illustrates the power of the test for cyclic links.

4 Conclusion

It seems natural to expect that the minimal simplicial complex (flag complex / order complex) with torsion in homology would be somehow related to the two-dimensional real projective space $\mathbb{R}P^2$. Indeed, the complexes $\mathrm{Cl}(K_i)$, $\mathrm{Cl}(L_i)$ are obtained from a small number of triangulations of $\mathbb{R}P^2$ by various extensions. One checks directly with tools such as `polymake` [6] and `chomp` [4] that

- two of the complexes $\mathrm{Cl}(K_i)$ are homeomorphic to $\mathbb{R}P^2$, and the remaining two collapse to $\mathbb{R}P^2$;
- among the complexes $\mathrm{Cl}(L_i)$ there are
  - 14 spaces homeomorphic to $\mathbb{R}P^2$,
  - 344 complexes that simplicially collapse to $\mathbb{R}P^2$,
  - 5 spaces homotopy equivalent to $\mathbb{R}P^2 \vee S^1$.

A full list of the graphs K_i and L_i in `nauty` [10] format is available from the author's website, www.mimuw.edu.pl/~aszek/ or from the source file of this paper at the `arXiv` repository, arxiv:1208.3892.

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