NECESSARY AND SUFFICIENT CONDITIONS FOR UNITARY SIMILARITY

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1. Introduction

If A and B are two complex matrices and if U is a complex unitary matrix such that $UAU^{CT} = B$ (where U^{CT} denotes the conjugate transpose of U), then A and B are said to be unitarily similar. Necessary and sufficient conditions that two matrices be unitarily similar have been dealt with in [5] (from the point of view of group representation theory) and in [2] (from the point of view of developing a canonical form under unitary similarity).

Here the problem of the unitary similarity of two matrices with real quaternion elements is considered. Any matrix A with real quaternion elements can be written in the form $A = A_1 + jA_2$ (see [3] and [6], for example, for related properties) where A_1 and A_2 are matrices with complex elements. By definition, such a matrix $U = U_1 + jU_2$ is unitary if $UU^{CT} = I = U^{CT}U$ where $U^{CT} = U_1 + (jU_2)^{CT} = U_1^{CT} - jU_2^T$, (where U_2^T denotes the transpose of the complex matrix U_2), and where I denotes an identity matrix of proper order. Necessary and sufficient conditions on A and B are to be found so that there exists a unitary U such that $UAU^{CT} = B$. In the following the main theorem is obtained and some related results are then noted.

2. Unitary Similarity of Pairs of Matrices

There is a one-to-one correspondence between all $n \times n$ real quaternion matrices $A = A_1 + jA_2$ and all $2n \times 2n$ complex matrices of the form

$$A^* = \begin{bmatrix} A_1 & -A_2^C \\ A_2 & A_1^C \end{bmatrix},$$

where A_1^C denotes the complex conjugate of the matrix A_1. This correspondence is an isomorphism up to a point (see [3], for instance). For example, if A is hermitian or unitary then A^* is, respectively, hermitian or unitary. (The isomorphism does not hold up under the ordinary transpose operation; as an example see [7].) For purposes here any complex matrix of dimension $2n \times 2n$ which has the above form of A^* will be said to be "a (complex)

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matrix in *-form." Let $\mathscr{S}$ denote the set of all $2n$-rowed matrices in *-form, and let $\mathscr{M}$ denote the set of all $2n$-rowed complex matrices.

The following are true:

(a) Each X in $\mathscr{M}$ can be written uniquely in the form $X = P + iQ$ where P and Q are in $\mathscr{S}$. For if $X = (X_{ij})$, $P = (P_{ij})$, and $Q = (Q_{ij})$, where X_{ij}, P_{ij}, and Q_{ij} are $n \times n$ matrices for $i, j = 1, 2$ then $P_{11} = \frac{1}{2}(X_{11} + X_{22}{}^C) = P_{22}{}^C$, $P_{21} = \frac{1}{2}(X_{21} - X_{12}{}^C) = -P_{12}{}^C$, $Q_{11} = (2i)^{-1}(X_{11} - X_{22}{}^C) = Q_{22}{}^C$, and $Q_{21} = (2i)^{-1}(X_{21} + X_{12}{}^C) = -Q_{12}{}^C$ supplies such a representation for X which is seen to be unique.

(b) Every non-singular matrix Y in $\mathscr{S}$ can be written uniquely in the polar form $Y = UH$ where U and H are in *-form and U is unitary and H is positive definite hermitian (see Theorem 3, [6], where this is also seen to be true in the singular case).

(c) If W is a unitary matrix in $\mathscr{M}$, then W can be written in the form $W = \lambda U(I + iK)(I + K^2)^{-\frac{1}{2}}$ where λ is a complex number of modulus 1, U is a unitary matrix in *-form, and K is a hermitian matrix in *-form.

The latter may be seen as follows. Let W be expressed in the form of (a) so that $W = P + iQ$. If P is singular, the matrix λW, where λ is a suitable scalar of absolute value 1, will have such a representation in which the matrix P is non-singular; this can be seen by considering the determinant of the matrix P (for λW) which is a function (of λ and $\bar{\lambda}$) of such a nature that this is evident. Let $P = UH$ be the polar form of P as described in (b). Then $U^{CT}W = H + iU^{CT}Q = H + iU^{CT}QH^{-1}H = (I + iK)H$ is unitary, and since $(U^{CT}W)^{CT}(U^{CT}W) = I$, it follows that

$$H(I + K^{CT}K)H + iH(K - K^{CT})H = I,$$

where H and K are in *-form. Since the sums and products of matrices in $\mathscr{S}$ are in $\mathscr{S}$, the above is the representation of I in the form developed in (a). Therefore $K = K^{CT}$ and $I + K^2 = (H^{-1})^2 = H^{-2}$, so that W (or λW) $= U(I + iK)H = U(I + iK)(I + K^2)^{-\frac{1}{2}}$, (where $X^{-\frac{1}{2}}$ denotes the inverse of the positive definite square root of a positive definite hermitian matrix X) where all matrices have the required form.

If A, B, and U are quaternion matrices such that $UAU^{CT} = B$ where U is unitary, then it is easily seen that $U^*A^*(U^*)^{CT} = B^*$, i.e., that the corresponding complex *-matrices are (complex) unitarily similar. On the other hand, suppose that A^* and B^* are two $2n \times 2n$ complex matrices in *-form which are (complex) unitarily similar under a $2n \times 2n$ complex unitary matrix which is not necessarily in *-form. The question arises as to whether or not they are similar under a complex unitary matrix which is in *-form. This is seen to be true as follows:

Theorem 1. If A^* and B^* are two $2n \times 2n$ complex matrices which are unitarily similar, then there exists a complex unitary matrix U which is in *-form such that $UA^*U^{CT} = B^*$.

Suppose that $W^{CT}A^*W = B^*$ where A^* and B^* are in *-form and W is unitary but not in *-form. (That this can occur can be seen by a simple example.) Expressing W in the form developed in (c), this relation becomes:

$$(I + K^2)^{-\frac{1}{2}}(I - iK)U^{CT}A^*U(I + iK)(I + K^2)^{-\frac{1}{2}} = B^*.$$

Let $U^{CT}A^*U = C$. From part (a) it follows that

$$(I + K^2)^{-\frac{1}{2}}[C + KCK](I + K^2)^{-\frac{1}{2}} = B^*$$
$$(I + K^2)^{-\frac{1}{2}}[CK - KC](I + K^2)^{-\frac{1}{2}} = 0$$

since all matrices are in *-form. From the second equation $CK = KC$ and from the first $B^* = C$. Hence $U^{CT}A^*U = C = B^*$ where U is in *-form.

From the above there follows:

THEOREM 2. Two real quaternion matrices A and B are unitarily similar if and only if the corresponding complex matrices A^* and B^* are unitarily similar.

There is a theorem due to Specht [5] which states the following: Two complex matrices M and N are unitarily similar if and only if for all functions $G(x, y) = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_n}y^{\beta_n}$ (where α_i and β_i are non-negative integer exponents) the trace of $G(M, M^{CT})$ = the trace of $G(N, N^{CT})$. If $M = A^*$ and if $N = B^*$ where $A = A_1 + jA_2$ and $B = B_1 + jB_2$ are quaternion matrices, $G(A^*, A^{*CT})$ is a matrix in *-form with two diagonal block matrices such that one is the conjugate of the other so that the trace of such a matrix is twice the sum of the real parts of the complex numbers which appear along the diagonal of one of these block matrices. This may be translated in terms of matrices with real quaternion elements as follows. Any quaternion matrix A can be written in the form $A = A_1 + jA_2 = (A_{11} + iA_{12}) + j(A_{21} + iA_{22})$ where the A_{ij} are real matrices. Let A_{11} be referred to as the "real part of A". Then the following theorem relates the concept of trace with that of semi-groups of matrices with non-commutative elements (which is of interest in that the concept of trace generally fails in the non-commutative case).

THEOREM 3. Two real quaternion matrices A and B are unitarily similar if and only if for all functions $G(x, y) = x^{\alpha_1}y^{\beta_1} \dots x^{\alpha_n}y^{\beta_n}$ (where α_i and β_i are non-negative integer exponents) the trace of the real part of $G(A, A^{CT})$ = the trace of the real part of $G(B, B^{CT})$.

3. Unitary Similarity of Sets of Matrices.

In a recent result [1] A. A. Albert has obtained necessary and sufficient conditions for the orthogonal equivalence of sets of real $n \times n$ symmetric matrices, $\{A_1, A_2, \dots, A_m\}$ and $\{\bar{A}_1, \bar{A}_2, \dots, \bar{A}_m\}$. The vector space, φ, of all matrices $A = \sum_{i=1}^{m} a_i A_i$, where the a_i are real, is considered, and $\bar{\varphi}$ is similarly defined. The necessary and sufficient condition involves, among other properties, a determinant condition, a Jordan algebra isomorphism, and a trace condition, and it is noted there that the result generalizes to complex hermitian matrices. By using a combination of preceding results (above) necessary and sufficient conditions can be obtained here which take a somewhat different direction but which apply to sets of general complex matrices and, because of § 2, to real quaternion matrices also.

Let $\{A_1, A_2, \ldots, A_m\}$ and $\{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_m]$ be two sets of m $n \times n$ matrices with complex elements, and let $S = \{A_1, A_2, \ldots, A_m, A_1{}^{CT}, A_2{}^{CT}, \ldots, A_m{}^{CT}\}$ and $T = \{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_m, \bar{A}_1{}^{CT}, \bar{A}_2{}^{CT}, \ldots, \bar{A}_m{}^{CT}\}$. Let $[S]$ denote the semigroup generated by S under matrix multiplication and let $[T]$ be similarly defined. Then:

THEOREM 4. Let $\{A_1, A_2, \ldots, A_m\}$ and $\{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_m\}$ be two sets of m $n \times n$ matrices with complex elements. There exists a unitary matrix U such that $UA_iU^{CT} = \bar{A}_i$, $i = 1, 2, \ldots, m$, if and only if there exists a (semi-group) isomorphism between $[S]$ and $[T]$ in which $A_i \to \bar{A}_i$, $A_i{}^{CT} \to \bar{A}_i{}^{CT}$ for $i = 1, 2, \ldots, m$, and such that the traces of corresponding matrices are equal.

The semi-group $[S]$ is such that if a matrix M is in $[S]$, so is M^{CT}; therefore $[S]$ is completely reducible by Theorem 3 in (4). The same is true of $[T]$. If the traces of corresponding elements are equal, there is a non-singular matrix P such that, in particular, $P^{-1}A_iP = \bar{A}_i$ and $P^{-1}A_i{}^{CT}P = \bar{A}_i{}^{CT}$ for $i = 1, 2, \ldots, m$. Let H be the uniquely determined positive definite matrix H such that $H^2 = PP^{CT}$. Set $U = H^{-1}P$; this can be verified to be a unitary matrix such that $U^{CT}A_iU = \bar{A}_i$ for $i = 1, 2, \ldots, m$. The converse is immediate.

This result extends directly to matrices with real quaternion elements if it is observed that the matrix W in part (c) determines the U in *-form which is finally used; if the same W relates all pairs A_i and $\bar{A}_i$, as above, then the same U so determined will do the same. It is also evident that the above need not be restricted to finite sets of A_i. (In the statement of Theorem 4 for quaternion matrices "traces of corresponding matrices" is replaced by "traces of the real parts of corresponding matrices".)

References

[1] Albert, A., *On the Orthogonal Equivalence of Sets of Real Symmetric Matrices*, J. Math. and Mech., Vol. 7, (1958), pp. 219—235.

[2] Brenner, J., *The Problem of Unitary Equivalence*, Acta Math., Vol. 86 (1951), pp. 297—308.

[3] Lee, H. C., *Eigenvalues and Canonical Forms of Matrices with Quaternion Coefficients*, Proc. Roy. Irish Acad. Sect. A, Vol. 52 (1949), pp. 253—260.

[4] Specht, W., *Zur Theorie der Gruppen linearer Substitutionen*, II, Jber. Deutschen Math. Verein., Vol. 49 (1939), pp. 207—215.

[5] Specht, W., *Zur Theorie der Matrizen*, II, Jber. Deutschen Math. Verein., Vol. 50 (1940), pp. 19—23.

[6] Wiegmann, N., *Some Theorems on Matrices with Real Quaternion Elements*, Canad. J. Math., Vol. 7 (1955), pp. 191—201.

[7] Wiegmann, N., *The Structure of Unitary and Orthogonal Quaternion Matrices*, Ill. J. Math., 2 (1958) pp. 402—407.

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