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A PIECEWISE POLYNOMIAL APPROXIMATION TO THE SOLUTION OF AN INTEGRAL EQUATION WITH WEAKLY SINGULAR KERNEL

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Abstract

We construct collocation methods with an arbitrary degree of accuracy for integral equations with logarithmically or algebraically singular kernels. Superconvergence at collocation points is obtained. A grid is used, the degree of non-uniformity of which is in good conformity with the smoothness of the solution and the desired accuracy of the method.

1. The integral equation

Consider the integral equation

$$u(t) = \int_0^b \kappa(|t - s|)u(s)\,ds + f(t), \tag{1.1}$$

with an m times, $m \geqslant 2$, continuously differentiable absolute term on $[0, b]$ and with an $m - 1$ times continuously differentiable kernel on $(0, b]$, satisfying

$$|\kappa(t)| \leqslant c(|\ln t| + 1) \quad \text{and} \quad |\kappa^{(k)}(t)| \leqslant ct^{-k} \qquad \text{for } k = 1, \ldots, m - 1 \tag{1.2}$$

or

$$|\kappa^{(k)}(t)| \leqslant ct^{-k-\alpha}, \qquad 0 < \alpha < 1, \quad \text{for} \quad k = 0, 1, \ldots, m - 1. \tag{1.3}$$

We assume that the corresponding homogeneous integral equation has only the trivial solution. In this case equation (1.1) has a unique solution u, where

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$u \in C[0, b] \cap C^m(0, b)$ and (see [4])

$$|u^{(k)}(t)| \leqslant c_0\left(t^{-k-\alpha+1} + (b-t)^{-k-\alpha+1}\right),$$
$$k = 1, \ldots, m \quad \text{and} \quad c_0 = \text{constant}; \tag{1.4}$$

in the case (1.2) of logarithmic singularity, these estimates hold with $\alpha = 0$ for $k = 2, \ldots, m$, and

$$|u'(t)| \leqslant c_0(|\ln t| + |\ln(b-t)|).$$

On the basis of this information, collocation methods on non-uniform grids with piecewise polynomial approximation of the solution are constructed.

As break points of a piecewise polynomial approximation we choose

$$\left.\begin{aligned} t_i &= (b/2)(i/n)^r, \qquad i = 0, 1, \ldots, n, \\ \text{and} \quad t_{n+i} &= b - t_{n-i}, \qquad i = 1, \ldots, n, \end{aligned}\right\} \tag{1.5}$$

where $r \in R$, $r \geqslant 1$, characterizes the degree of non-uniformity of the grid. The break points are located symmetrically with regard to the centre of the interval $[0, b]$, with a greater density towards its ends, and

$$t_{i+1} - t_i \leqslant \frac{b}{2}\frac{r}{n}\left(\frac{i+1}{n}\right)^{r-1}, \qquad i = 0, 1, \ldots, n-1. \tag{1.6}$$

Analogous estimates are valid for the break points on the other half of the interval $[0, b]$.

2. The first method

We define some interpolation points in the standard interval $[-1, 1]$:

$$-1 < \tau_1 < \tau_2 < \cdots < \tau_m < 1. \tag{2.1}$$

By the linear transformation

$$\tau_{ik} := t_i + (\tau_k + 1)(t_{i+1} - t_i)/2, \qquad k = 1, \ldots, m, \quad i = 0, 1, \ldots, 2n-1, \tag{2.2}$$

we transfer these points into the interval $[t_i, t_{i+1}]$. It is clear that

$$t_i < \tau_{i1} < \tau_{i2} < \cdots < \tau_{im} < t_{i+1}, \qquad i = 0, 1, \ldots, 2n-1.$$

We construct the approximate solution u_n of equation (1.1) as a piecewise polynomial function of degree $m-1$ with break points (1.5); at points t_i, $i = 1, \ldots, 2n-1$, the function u_n may be discontinuous. It is required that u_n should satisfy equation (1.1) at the interpolation points:

$$u_n(\tau_{ik}) = \int_0^b \kappa(|\tau_{ik} - s|)u_n(s)\,ds + f(\tau_{ik}),$$
$$k = 1, \ldots, m, i = 0, 1, \ldots, 2n-1. \tag{2.3}$$

The conditions (2.3) form a linear system of equations whose exact form is determined by the choice of a basis in the subspace of the piecewise polynomial functions. For example, taking u_n in each subinterval in the form

$$u_n(t) = \sum_{l=1}^{m} a_{jl}\varphi_{jl}(t), \qquad t_j \leqslant t \leqslant t_{j+1},$$

where φ_{jl} are the Lagrange fundamental polynomials ($\varphi_{jl}(\tau_{jk}) = \delta_{kl}$, *for* $k, l = 1, \ldots, m$) of degree $m - 1$, the conditions (2.3) lead to the system of equations

$$a_{ik} = \sum_{j=0}^{2n-1} \sum_{l=1}^{m} \int_{t_j}^{t_{j+1}} \kappa(|\tau_{ik} - s|)\varphi_{jl}(s)\, ds \cdot a_{jl} + f(\tau_{ik})$$

with respect to the unknown coefficients a_{ik}, $k = 1, \ldots, m$, $i = 0, 1, \ldots, 2n - 1$.

3. The second method

We choose the interpolation points τ_k, $k = 1, \ldots, m$, in the standard interval $[-1, 1]$ so that (compare with (2.1))

$$-1 = \tau_1 < \tau_2 < \cdots < \tau_m = 1, \tag{3.1}$$

and transfer them according to formula (2.2) into the interval $[t_i, t_{i+1}]$. It is clear that now

$$t_i = \tau_{i1} < \tau_{i2} < \cdots < \tau_{im} = t_{i+1}, \qquad i = 0, 1, \ldots, 2n - 1.$$

The approximate solution u_n of equation (1.1) is constructed in the form of a continuous piecewise polynomial function of degree $m - 1$, with break points (1.5). It is required that u_n should satisfy equation (1.1) in the interpolation points, that is, conditions (2.3) should be satisfied, with the reservation that these conditions are taken only once for the break points $t_i = \tau_{i-1,m} = \tau_{i1}$, $i = 1, \ldots, 2n - 1$.

4. Formulation of the main result

THEOREM. *Let the conditions for f, κ and equation* (1.1) *presented in Section* 1 *be satisfied. In the case of the validity of condition* (1.2) *put* $\alpha = 0$. *Then, for sufficiently large n, either of the two methods described in Sections* 2 *and* 3 *determines a unique approximate solution u_n. If*

$$r = \mu/(1 - \alpha) \geqslant 1, \qquad \mu \leqslant m, \tag{4.1}$$

then

$$\sup_{0 \leqslant t \leqslant b} |u_n(t) - u(t)| \leqslant (\text{constant}) n^{-\mu} \tag{4.2}$$

and

$$\max_{\substack{0 \leqslant i \leqslant 2n-1 \\ 1 \leqslant k \leqslant m}} |u_n(\tau_{ik}) - u(\tau_{ik})| \leqslant (\text{constant}) \varepsilon_n, \tag{4.3}$$

where

$$\varepsilon_n = \begin{cases} n^{-m}(\ln n)^\alpha & \textit{for } \mu > m/2, \\ n^{-m} \ln n & \textit{for } \mu = m/2, \\ n^{-2\mu}(\ln n)^\alpha & \textit{for } \mu < m/2, \end{cases} \tag{4.4}$$

in the case of (1.3) *and*

$$\varepsilon_n = \begin{cases} n^{-m} \ln n & \textit{for } \mu > m/2, \\ n^{-m}(\ln n)^2 & \textit{for } \mu = m/2, \\ n^{-2\mu} \ln n & \textit{for } \mu < m/2 \end{cases} \tag{4.5}$$

in the case of (1.2).

The proof is presented in Sections 5 and 6. We shall not specify the constants in (4.2) and (4.3), but note here that, by increasing r, they also increase, and thus the superconvergence at interpolation points is highly useful: to attain a method of mth degree of accuracy in the uniform norm we must choose $\mu = m$ and $r = m/(1 - \alpha)$ whereas, to attain nearly the same accuracy at the interpolation points, it is sufficient to put $\mu = m/2$ and $r = m/(2(1 - \alpha))$.

Numerical testing of the described methods will be undertaken in the future. In the case where $m = 2$, the method described in Section 3 reduces to the piecewise linear collocation mehod. This method for the uniform grid (in our notation $r = 1$) is investigated in [2]. Our result for $m = 2$ and $r = 1$ is consistent with the results of [2]. Numerical calculations confirm the superconvergence at the points of interpolation (see [2]). The theorem was announced in [5]. We refer also to Rice [3], who appears to have been the first to study graded grids for approximation of functions with singularities.

5. Transition to the operator equation

Let us denote by T the integral operator of equation (1.1). Then (1.1) can be considered as the equation

$$u = Tu + f \tag{5.1}$$

in the Banach space $E = L_\infty$ with the norm $\|u\| = \sup_{0<t<b}|u(t)|$. Both methods for the solution of (1.1) described above are equivalent to the solution of equation

$$u_n = P_n T u_n + P_n f, \tag{5.2}$$

where $P_n = P_{n,m}$ is the interpolation projector assigning to any continuous function u its piecewise polynomial interpolant:

$$(P_n u)(t) = \sum_{k=1}^{m} u(\tau_{ik})\varphi_{ik}(t) \quad \text{for } t_i < t < t_{i+1},\ i = 0, 1, \dots, 2n-1;$$

the interpolant is determined in each interval $[t_i, t_{i+1}]$ independently; $P_n u$ is discontinuous or continuous in break points t_i, depending on the choice of (2.1) or (3.1), respectively.

The norms $\|P_n\|$ are uniformly bounded, $\|P_n\| = \|P\|$, $n = 1, 2, \dots,$ where P is the Lagrange interpolation projector of degree $m - 1$ on $[-1, 1]$ defined by interpolation points (2.1) or (3.1). It is easy to see that $\|P_n u - u\|_{L_\infty} \to 0$ as $n \to \infty$ for $u \in E' = C[0, b]$.

Since T is a compact operator from L_∞ into C, we conclude by means of standard arguments (see [1], Lemma 15.5) that $\|P_n T - T\|_{L_\infty \to L_\infty} \to 0$ as $n \to \infty$. Now, from the unique solvability of (5.1), it follows that (5.2) is uniquely solvable for sufficiently large n, $n \geqslant n_0$, whereby

$$\|u_n - u\|_{L_\infty} \leqslant c_1 \|u - P_n u\|_{L_\infty}, \qquad c_1 = \sup_{n \geqslant n_0} \|(I - P_n T)^{-1}\| < \infty. \tag{5.3}$$

In addition to this traditional estimate we need an estimate

$$\|u_n - P_n u\|_{L_\infty} \leqslant c_2 \|T(u - P_n u)\|_{L_\infty}, \qquad c_2 = c_1 \|P\|, \tag{5.4}$$

which follows from equalities $u_n - P_n u = P_n T(u_n - u)$, $u_n - u = (I - P_n T)^{-1}(P_n u - u)$ and $u_n - P_n u = (I - P_n T)^{-1} P_n T(P_n u - u)$.

6. Error estimates for the piecewise polynomial interpolant

Let u be any function satisfying (1.4).

PROPOSITION 1. *If* $r = \mu/(1 - \alpha) \geqslant 1$ *and* $\mu \leqslant m$ *then*

$$\|u - P_n u\|_{L_\infty} \leqslant c_3 n^{-\mu} \quad \textit{where } c_3 = \text{constant}. \tag{6.1}$$

PROOF. The well-known inequality $\|u - P_n u\| \leqslant (1 + \|P_n\|)\operatorname{dist}(u, P_n E)$ can be reduced to the form

$$\|u - P_n u\|_{L_\infty} \leqslant (1 + \|P\|) \max_{0 \leqslant i \leqslant 2n-1} \eta_i,$$

$$\eta_i = \inf_{v \in \pi_{m-1}} \max_{t_i \leqslant t \leqslant t_{i+1}} |u(t) - v(t)|,$$

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where π_{m-1} denotes the set of the polynomials of degree $\leqslant m-1$. We prove the inequalities

$$\eta_i \leqslant c_4 n^{-\mu}(i+1)^{\mu-m}, \qquad i=0,1,\dots,n-1, \tag{6.2}$$

and similar inequalities for the other half of the interval $[0, b]$, that is, for $i = n, \dots, 2n-1$. By (1.4) to (1.6), the known estimate $\eta_i \leqslant \gamma_m \max_{t_i \leqslant t \leqslant t_{i+1}} |u^{(m)}(t)|(t_{i+1}-t_i)^m$, where $\gamma_m = 2^{1-2m}/(m!)$ for $1 \leqslant i \leqslant n-1$, can be rewritten as

$$\begin{aligned}\eta_i &\leqslant \gamma_m 2c_0(b/2)^{-m-\alpha+1}(n/i)^{r(m+\alpha-1)}(b/2)^m(r/n)^m((i+1)/n)^{(r-1)m}\\ &= 2c_0\gamma_m(b/2)^{1-\alpha}r^m n^{-\mu}(i+1)^{\mu-m}((i+1)/i)^{rm-\mu} \leqslant c_4 n^{-\mu}(i+1)^{\mu-m}.\end{aligned}$$

To estimate η_0, it is sufficient to take $v(t)$ as a constant or a linear function. In the case of $\alpha > 0$, by (1.4) and (1.5),

$$\begin{aligned}\eta_0 &\leqslant \max_{0 \leqslant t \leqslant t_1} |u(t)-u(0)| \leqslant \int_0^{t_1} |u'(s)|\, ds\\ &\leqslant \frac{c_0}{1-\alpha} t_1^{1-\alpha} = \frac{c_0}{1-\alpha}(b/2)^{1-\alpha} n^{-\mu};\end{aligned}$$

in the case of $\alpha = 0$, we put $v(t) = u(0) + (u(t_1) - u(0))(t/t_1)$; therefore

$$\begin{aligned}|u(t)-v(t)| &= \left|\int_0^t u'(s)\, ds - \frac{t}{t_1}\int_0^{t_1} u'(s)\, ds\right|\\ &= \left|\int_0^t \left[u'(s) - u'\left(\frac{t_1}{t}s\right)\right] ds\right| = \left|\int_0^t ds \int_s^{t_1 s/t} u''(\tau)\, d\tau\right|\\ &\leqslant c_0\int_0^t ds \int_s^{t_1 s/t} \frac{d\tau}{\tau} = c_0\int_0^t \left(\ln\left(\frac{t_1}{t}s\right) - \ln(s)\right) ds = c_0 t \ln\frac{t_1}{t}\end{aligned}$$

and

$$\begin{aligned}\eta_0 &\leqslant \max_{0 \leqslant t \leqslant t_1} |u(t)-v(t)| \leqslant c_0 \max_{0 \leqslant t \leqslant t_1} t \ln\frac{t_1}{t}\\ &= c_0 e^{-1} t_1 = c_0 e^{-1}(b/2) n^{-\mu}.\end{aligned}$$

That completes the proof of the estimate (6.2). Estimate (6.1) follows from (6.2) and similar estimates for $i = n, \dots, 2n-1$. Thus Proposition 1 is proved.

PROPOSITION 2. *If $r = \mu/(1-\alpha) \geqslant 1$, $\mu \leqslant m$ and $p = 1/(1-\alpha)$, then*

$$\|u - P_n u\|_{L_p(0,b)} \leqslant c_{5,\mu}\delta_n,$$

where

$$\delta_n = \begin{cases} n^{-m} & \text{for } \mu > m/2,\\ n^{-m}(\ln n)^{1-\alpha} & \text{for } \mu = m/2,\\ n^{-2\mu} & \text{for } \mu < m/2.\end{cases} \tag{6.3}$$

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PROOF. It is clear that

$$\|u - P_n u\|_{L_p(0,b)} \leqslant \left\{ \sum_{i=0}^{2n-1} (t_{i+1} - t_i) \max_{t_i \leqslant t \leqslant t_{i+1}} |u(t) - (P_n u)(t)|^p \right\}^{1/p}$$
$$\leqslant (1 + \|P\|)\left\{ \sum_{i=0}^{2n-1} (t_{i+1} - t_i)\eta_i^p \right\}^{1/p}.$$

By (6.2) and (1.6)

$$\|u - P_n u\|_{L_p(0,b/2)} \leqslant (1 + \|P\|)c_4\left(\frac{br}{2}\right)^{1/p} n^{-\mu} \left\{ \sum_{i=0}^{n-1} n^{-r}(i + 1)^{r-1-(m-\mu)p} \right\}^{1/p},$$

and a similar estimate holds for the other half of the interval. Now estimates (6.3) follow, because

$$\mu > m/2 \Rightarrow r - 1 - (m - \mu)p > -1, \quad \sum_{i=0}^{n-1} (i + 1)^{r-1-(m-\mu)p} \leqslant c_{6,\mu} n^{r-(m-\mu)p};$$

$$\mu = m/2 \Rightarrow r - 1 - (m - \mu)p = -1, \quad \sum_{i=0}^{n-1} (i + 1)^{r-1-(m-\mu)p} \leqslant c_6 \ln n;$$

$$\mu < m/2 \Rightarrow r - 1 - (m - \mu)p < -1, \quad \sum_{i=0}^{n-1} (i + 1)^{r-1-(m-\mu)p} \leqslant c_{6,\mu}.$$

Thus Proposition 2 is proved.

PROPOSITION 3. *If* $r = \mu/(1 - \alpha) \geqslant 1$, $\mu \leqslant m$, *then*

$$\|T(u - P_n u)\|_{L_\infty} \leqslant c_{7,\mu}\varepsilon_n, \tag{6.4}$$

where ε_n *is determined by* (4.4) *or* (4.5).

PROOF. Let $\alpha > 0$. For $p = 1/(1 - \alpha)$ and $q = 1/\alpha$ it holds that

$$\|T(u - P_n u)\|_{L_\infty} = \sup_{0 < t < b} \left| \int_0^b \kappa(|t - s|)[u(s) - (P_n u)(s)]\, ds \right|$$
$$\leqslant \|u - P_n u\|_{L_\infty} \sup_{0 < t < b} \int_{\substack{s \in [0,\, b] \\ |s-t| < h}} |\kappa(|t - s|)|\, ds$$
$$+ \|u - P_n u\|_{L_p} \left\{ \int_{\substack{s \in [0,\, b] \\ |s-t| > h}} |\kappa(|t - s|)|^q\, ds \right\}^{1/q}.$$

By means of (1.3) and Propositions 1 and 2 this can be reduced to

$$\|T(u - P_n u)\|_{L_\infty} \leqslant c_{8,\mu}\left(n^{-\mu}h^{1-\alpha} + \delta_n|\ln h|^\alpha\right). \tag{6.5}$$

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Choosing h in the case $\mu \geqslant m/2$ such that $h^{1-\alpha} = n^{\mu-m}$ and in the case $\mu < m/2$ such that $h^{1-\alpha} = n^{-\mu}$, we obtain the estimates (6.4) and (4.4).

In the case of $\alpha = 0$, instead of (6.5) we have $\|T(u - P_n u)\|_{L_\infty} \leqslant c_{8,\mu}(n^{-\mu}h|\ln h| + \delta_n|\ln h|)$ and the proof of the estimates (6.4) and (4.5) is analogous to the one above. Thus Proposition 3 is proved.

To complete the proof of the theorem, note that we obtain (4.2) immediately from (5.3) and (6.1). From (5.4) and (6.4) we get (4.3), since

$$|u_n(\tau_{ik}) - u(\tau_{ik})| \leqslant \|u_n - P_n u\|_{L_\infty}.$$

The proof of the theorem is now complete.

References

[1] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii and V. Ya. Stetsenko, *Approximate solution of operator equations* (Wolters–Noordhoff, 1972).

[2] A. Pedas, "Piecewise linear approximation of solution of integral equation with logarithmical singular kernel", *Tartu Riikl. Ul. Toimetised, Vih.* 500 (1979), 33–42.

[3] J. R. Rice, "On the degree of convergence of nonlinear spline approximation", in *Approximations with special emphasis on spline functions* (Academic Press, New York, 1969).

[4] G. Vainikko and A. Pedas, "The properties of solutions of weakly singular integral equations", *J. Austral. Math. Soc. B* 22 (1981), 419–430.

[5] G. Vainikko and P. Uba, "A piecewise polynomial approximation to the solution of an integral equation with weak singularity", in theses of conference *Theoretical and applied problems of mathematics*, Tartu (1980), 196–198.

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