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A NOTE ON BOUNDS OF CARDINAL FUNCTIONS IN THE CLOSED PREIMAGE

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Let $\mathbf{X}$ and $\mathbf{Y}$ be T_1 spaces and $f: \mathbf{X} \to \mathbf{Y}$ be a closed and onto mapping. If a fiber of the mapping f is defined to be the inverse image of a singleton in the range, then a bound for the tightness of the domain is the product of the tightness of the range and the supremum of the tightness of the fibers of f. Similar bounds can also be shown for the Lindelöf degree and the extent of $\mathbf{X}$. Examples are provided to demonstrate that such results are not possible for open maps. Cellularity and spread are discussed briefly.

The following result given by Sakai in [**3**] suggested this line of investigation. The underlying idea is to weaken the conditions placed on fibers as much as possible and still be assured that the domain of the closed map is neat.

DEFINITION: [3] A space X is called *neat* if for every free closed ultrafilter $\mathcal{H}$ with cip on X there is a system $\langle X_\gamma, \mathcal{V}_\gamma, F_\gamma \rangle_\gamma \in \Gamma$ such that

(1) $\Gamma < \lambda(\mathcal{H})$ and $\bigcup_{\gamma \in \Gamma} X_\gamma = X$,

(2) for each $\gamma \in \Gamma$, $\mathcal{V}_\gamma$ is an open collection in X and $X_\gamma \subseteq \cup \mathcal{V}_\gamma$,

(3) each $f_\gamma: X_\gamma \to \mathcal{V}_\gamma$ is such that if $A \in [X]^{\leqslant \omega}$ and $f_\gamma|_A$ is injective, then $\overline{A}^{\cup \mathcal{V}_\gamma} \subseteq \bigcup_{x \in A} f_\gamma(x)$,

(4) for each $\gamma \in \Gamma$ and $x \in X_\gamma$, $\exists H \in \mathcal{H}$ such that $f_\gamma(x) \cap X_\gamma \cap H = \emptyset$.

THEOREM 0. [**3**, Theorem 3.7] *Let f be a closed map from $\mathbb{X}$ onto a closed complete space $\mathbb{Y}$. If each fiber of f is neat, then X is neat.*

The connection to cardinal functions is made through the related result that the perfect perimage of a compact space is compact. The author observed that in the proof of this result the finiteness was not essential except in actually achieving the finite subcover required of the domain. No use is made in the mapping processes. This led to consideration the Lindelöf degree of a space in its guise as a generalisation of compactness.

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DEFINITION: [1] The *Lindelöf Degree* $L(\mathbb{X})$ of a space $\mathbb{X}$ is $\inf\{K\colon$ every open cover of X has a subcover of cardinality $\leqslant K\} + \omega$.

It seems natural to ask if there is a result for arbitrary Lindelöf degree which is analogous to the perfect preimage theorem for compactness. Using Sakai's strategy of weakening the requirements on the fibers of the mapping, a sharp bound can be found for the Lindelöf degree of the domain of a closed map. This result is preceded by notational convenience and an easy lemma.

DEFINITION: A collection of sets $\mathcal{A}$ is K-*complete* if every nonempty subcollection of cardinality less than K has nonempty intersection which belongs to $\mathcal{A}$. A K-*completion* of a collection of sets $\mathcal{A}$ with the $< K$ intersection property is the set $\{\cap\beta \neq \emptyset : \mathcal{B} \subseteq \mathcal{A}$ and $|\mathcal{A}| < K\}$.

LEMMA 1. *$\mathbb{X}$ has Lindelöf degree $\leqslant K$ if and only if every K^+-complete closed collection has nonempty intersection.*

PROOF: If $\mathcal{F}$ is a closed K^+-complete collection, let $\mathcal{U} = \{\mathbb{X} \setminus F\colon F \in \mathcal{F}\}$. Should $\mathcal{F}$ have empty intersection, $\mathcal{U}$ would cover $\mathbb{X}$ and so there would be a $\mathcal{U}' \subsetneq \mathcal{U}$ with $|\mathcal{U}'| \leqslant K$ such that $\mathcal{U}'$ covers $\mathbb{X}$. Since $\mathcal{U}'$ covers $\mathbb{X}$, $\mathcal{F}' = \{\mathbb{X} \setminus U\colon U \in \mathcal{U}'\}$ has empty intersection. This is a contradiction, since $\mathcal{F}$ is K^+-complete.

Now let $\mathcal{U}$ be an open cover of $\mathbb{X}$ with no subcover of cardinality $\leqslant K$ and $\mathcal{F}$ be the K^+-completion of $\{\mathbb{X} \setminus U\colon U \in \mathcal{U}\}$. Then for all nonempty $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \leqslant K$, $\cap\mathcal{F}' \neq \emptyset$ and $\cap\mathcal{F}' \in \mathcal{F}$; therefore $\mathcal{F}$ is K^+-complete, and we have that $\cap\mathcal{F} \neq \emptyset$. This is a contradiction, so $\mathcal{U}$ has a subcover of cardinality $\leqslant K$. ∎

THEOREM 2. *If $f\colon \mathbb{X} \to \mathbb{Y}$ is a closed map, then*

$$L(\mathbb{X}) = \sup\{L(\mathbb{Y}) \cdot L(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\}.$$

PROOF: Let $K = \sup\{L(\mathbb{Y}) \cdot L(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\}$ and let $\mathcal{F}$ be a closed K^+-complete collection in $\mathbb{X}$. Then $\mathcal{H} = \{f^{\rightarrow}(F)\colon F \in \mathcal{F}\}$ is a closed K^+-complete collection in $\mathbb{Y}$ By the preceding lemma, $\cap\mathcal{H} \neq \emptyset$. Choose any $y \in \cap\mathcal{H}$; let $D = f^{\leftarrow}(\{y\}) \in \mathcal{F}$ and $\mathcal{F}' = \{F \cap D\colon F \in \mathcal{F}\}$. If $\mathcal{F}'$ is not a K^+-complete collection in D, then there is a subcollection $\mathcal{C}'$ of $\mathcal{F}'$ with $|\mathcal{C}'| \leqslant K$ such that $\cap\mathcal{C}' = \emptyset$. Thus there is a $\mathcal{C} \subseteq \mathcal{F}$ with $(\cap\mathcal{C}) \cap D = \emptyset$. Since $\mathcal{F}$ is K^+-complete, $\cap\mathcal{C} \in \mathcal{F}$. But then $y \notin f^{\rightarrow}(\cap\mathcal{C}) \in \mathcal{H}$, a contradiction to $y \in \cap\mathcal{H}$. Thus $\mathcal{F}'$ is a K^+-complete collection in D and lemma 1 proves $\cap\mathcal{F}' \neq \emptyset$. Thus we can conclude that $\mathcal{F}$ has nonempty intersection. The lemma completes the argument that $L(\mathbb{X}) \leqslant K$.

Let $\mathcal{U}$ be an open cover of $\mathbb{Y}$. Then $\mathcal{V} = \{f^{\leftarrow}(U)\colon U \in \mathcal{U}\}$ is an open cover of $\mathbb{X}$, so there is a subcover $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| \leqslant L(\mathbb{X})$. Since $\mathcal{V}$ is comprised of inverse images of elements of $\mathcal{U}$, $\mathcal{U}' = \{f^{\rightarrow}(V)\colon V \in \mathcal{V}'\} \subseteq \mathcal{U}$ and $|\mathcal{U}'| \leqslant L(\mathbb{X})$. Also $L(\mathbb{X})$

bounds the Lindelöf degree of all closed subsets of $\mathbb{X}$; in particular, for any $y \in \mathbb{Y}$, $L(f^{\leftarrow}(\{y\})) \leqslant L(\mathbb{X})$. $\square$

Now that the door has been opened into the realm of cardinal functions, a question presents itself. Is it possible to prove such a theorem for other cardinal functions? The answer is yes in the case of extent, a cardinal function closely related to Lindelöf degree.

DEFINITION: [1] The *extent* $e(\mathbb{X})$ of a space $\mathbb{X}$ is

$$\sup\{|D| : D \text{ is a closed discrete subset of } \mathbb{X}\} + \omega.$$

THEOREM 3. *If $f\colon \mathbb{X} \to \mathbb{Y}$ is a closed map, then*

$$e(\mathbb{X}) = \sup\{e(\mathbb{Y}) \cdot e(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\}.$$

PROOF: Let $K = \sup\{e(\mathbb{Y}) \cdot e(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\}$, and let $D \subseteq \mathbb{X}$ be a closed and discrete subset. Then $f^{\rightarrow}(D)$ is closed. We now show that $f^{\rightarrow}(D)$ is discrete. For $d \in f^{\rightarrow}(D)$, $D \setminus f^{\leftarrow}(\{d\})$ is closed since D is closed and discrete. So $f^{\rightarrow}(D \setminus f^{\leftarrow}(\{d\}))$ is closed and thus $\mathbb{Y} \setminus (f^{\rightarrow}(D \setminus f^{\leftarrow}(\{d\})))$ is open and $f^{\rightarrow}(D) \cap (\mathbb{Y} \setminus (f^{\rightarrow}(D \setminus f^{\leftarrow}(\{d\})))) = \{d\}$. Thus $f^{\rightarrow}(D)$ is discrete, and by assumption $|f^{\rightarrow}(D)| \leqslant K$. Also $D \cap f^{\leftarrow}(\{d\})$ is closed and discrete for every $d \in f^{\rightarrow}(D)$. Thus $|D \cap f^{\leftarrow}(\{d\})| \leqslant K$ for every $d \in f^{\rightarrow}(D)$. Since $D = \bigcup\{D \cap f^{\leftarrow}(\{d\})\colon d \in f^{\rightarrow}(D)\}$, $|D| \leqslant \sup\{e(D) \cdot e(f^{\leftarrow}(\{y\}))\colon y \in D\} \leqslant \sup\{e(\mathbb{Y}) \cdot e(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\}$.

Clearly the extent of a space bounds the extent of closed subsets. Let $D \subseteq \mathbb{Y}$ be closed and discrete, and choose any $F \subseteq f^{\leftarrow}(D)$ such that $|F \cap f^{\leftarrow}(\{d\})| = 1$ if and only if $d \in D$. This F is then closed and discrete. For each $x \in \mathbb{X}$, choose $V_x \subseteq \mathbb{Y}$ to be open with $f(x) \in V_x$ and such that $|V_x \cap D| = 1$ if $f(x) \in D$ and $|V_x \cap D| = 0$ if $f(x) \notin D$. For $x \in F$, $|V_x \cap D| = 1$, so $|f^{\leftarrow}(V_x) \cap F| = 1$; F is discrete. For $x \notin f^{\leftarrow}(D)$ we have $|f^{\leftarrow}(V_x) \cap f^{\leftarrow}(D)| = 0$, and hence $f^{\leftarrow}(V_x) \cap F = \emptyset$. Now let $x \in f^{\leftarrow}(D) \setminus F$. Then there is a U open in $\mathbb{X}$ with $x \in U$ and $f^{\leftarrow}(\{f(x)\}) \cap F \cap U = \emptyset$. So $x \in f^{\leftarrow}(V_x) \cap F \cap U$ and $f^{\leftarrow}(V_x) \cap U = f^{\leftarrow}(\{f(x)\}) \cap F \cap U = \emptyset$. Thus F must be closed and $|D| = |F| \leqslant e(\mathbb{X})$, so we have $\sup\{e(\mathbb{Y}) \cdot e(f^{\leftarrow}(\{y\}))\colon y \in \mathbb{Y}\} \leqslant e(\mathbb{X})$. $\square$

The case of open mappings should also be explored. The journey is short, for one example is sufficient to show that theorems for open mappings analogous to those stated above for extent and Lindelöf degree cannot be proved.

EXAMPLE 4. An open finite to one map from a space of uncountable extent and Lindelöf degree onto a compact space.

Let A be any uncountable set, and $p \notin 2 \times A$. Let $\mathbb{X}$ have as underlying set $(2 \times A) \cup \{p\}$ and

$$\{\{x\}\colon x \in 2 \times A\} \cup \{\{p\} \cup ((\{1\} \times A) \setminus F)\colon F \subseteq \{1\} \times A \text{ is finite}\}$$

be a base for the topology on $\mathbb{X}$. Now let $\mathbb{Y}$ have as underlying set $A \cup \{p\}$, and a base for the topology on $\mathbb{Y}$ be the collection $\{\{x\}: x \in A\} \cup \{\mathbb{X} \setminus F: F \in |A|^{<\omega}\}$. Define $f: \mathbb{X} \to \mathbb{Y}$ by $f(p) = p$ and $f(\langle i, x\rangle) = x$. The function is at most 2 to 1 and continuous, since cofinite sets containing p have cofinite sets containing p as inverse images, and inverse images of any set not containing p are open. Similarly, the image of any open set in $\mathbb{X}$ containing p is cofinite, since the original set must be cofinite when intersected with $\{1\} \times A$; any other subset not containing p has open image in $\mathbb{Y}$.

The space $\mathbb{Y}$ is the one point compactification of A with the discrete topology, and consequently is both compact and has countable extent. The space $\mathbb{X}$, however, has a closed discrete set of uncountable cardinality, namely $\{0\} \times A$. □

It was not clear in which direction to turn following the positive result in the case of extent. However, a positive answer was also found in the case of tightness.

DEFINITION: [1] The *tightness* $t(x, \mathbb{X})$ of a point x in a space $\mathbb{X}$ is $\inf\{K: x \in \overline{B} \Rightarrow$ there is an $A \subseteq B$ with $|A| \leqslant K$ and $x \in \overline{A}\}$. The *tightness* $t(\mathbb{X})$ of a space $\mathbb{X}$ is $\sup\{t(x, \mathbb{X}): x \in \mathbb{X}\} + \omega$.

THEOREM 5. *If $f: \mathbb{X} \to \mathbb{Y}$ is a closed map with regular domain $\mathbb{X}$, then*

$$t(\mathbb{X}) = \sup\{t(\mathbb{Y}) \cdot t(f^{\leftarrow}(\{y\})): y \in \mathbb{Y}\}.$$

PROOF: Let $A \subseteq \mathbb{X}$, and fix $x \in \overline{A}$. Then $f(x) \in \overline{f^{\rightarrow}(A)} = f^{\rightarrow}(\overline{A})$. The collection $\mathcal{D} = \{D \subseteq f^{\rightarrow}(A): |D| \leqslant t(\mathbb{Y})$ and $f(x) \in \overline{D}\}$ is nonempty. In $\mathbb{X}$, let $\mathcal{B} = \{B \subseteq A: f^{\rightarrow}(B) = D$ for some $D \in \mathcal{D}$ and $|f^{\leftarrow}(f^{\rightarrow}(\{a\})) \cap B| \leqslant 1$ for all $a \in \mathbb{X}\}$. For $B \in \mathcal{B}$, if $\overline{B} \cap f^{\leftarrow}(f^{\rightarrow}(\{x\})) = \emptyset$, then $f^{\rightarrow}(\overline{B}) = \overline{f^{\rightarrow}(B)} = \overline{D}$ for some D, and cannot contain $f(x)$, a clear contradiction. So for all $B \in \mathcal{B}$, $\overline{B} \cap f^{\leftarrow}(f^{\rightarrow}(\{x\})) \neq \emptyset$. We show that x is an element of the closure of $C = \{p \in f^{\leftarrow}(f^{\rightarrow}(\{x\})): p \in \overline{B}$ for some $B \in \mathcal{B}\}$. Assume $x \notin \overline{C}$. Then there is an open set $U \subseteq \mathbb{X}$ with $x \in U$ and $\overline{U} \cap \overline{C} = \emptyset$. Thus $x \in \overline{U \cap A}$ and $f(x) \in \overline{f^{\rightarrow}(U \cap A)}$. Hence there is $D \in \mathcal{D}$ with $D \subseteq f^{\rightarrow}(U \cap A)$, so there is $B \in \mathcal{B}$ with $B \subseteq U \cap A \subseteq A$ and $\overline{B} \subseteq \overline{U}$, $\overline{U} \cap \overline{C} = \emptyset$; but $\overline{B} \cap C \neq \emptyset$, a contradiction.

Thus we have $H \subseteq C$, with $|H| \leqslant t(f^{\leftarrow}(f^{\rightarrow}(\{x\})))$ such that $x \in \overline{H}$. For each $h \in H$, there is $B_h \in \mathcal{B}$ with $h \in \overline{B}_h$. Let $M = \bigcup_{h \in H} B_h$. Then $|M| \leqslant t(\mathbb{Y}) \cdot \sup\{t(f^{\leftarrow}(f^{\rightarrow}(\{y\}))): y \in \mathbb{X}\}$. Let U be an open set, $x \in U$. Then there is $h \in H$ with $h \in U$. However, $h \in \overline{B}_h$, so $U \cap B_h \neq \emptyset$. Thus $U \cap M \neq \emptyset$. We conclude that $x \in \overline{M}$, yielding the inequality $t(\mathbb{X}) \leqslant \sup\{t(\mathbb{Y}) \cdot t(f^{\leftarrow}(\{y\})): y \in \mathbb{Y}\}$.

Now let $y \in Y$ and $B \subseteq Y$ with $y \in \overline{B}$. Let $x \in f^{\leftarrow}(\{y\})$; then $x \in f^{\leftarrow}(\overline{B}) = \overline{f^{\leftarrow}(B)}$. Let A be a subset of $f^{\leftarrow}(B)$ with $x \in \overline{A}$. Then $y \in f^{\rightarrow}(\overline{A}) = \overline{f^{\rightarrow}(A)}$ since f is closed, and $f^{\rightarrow}(A) \subseteq B$. Since $|f^{\rightarrow}(A)| \leqslant |A|$, we have that $t(y, Y) \leqslant t(x, X) \leqslant t(X)$. Therefore, $t(Y) \leqslant t(X)$, proving equality for $t(\mathbb{X}) = \sup\{t(\mathbb{Y}) \cdot t(f^{\leftarrow}(\{y\})): y \in \mathbb{Y}\}$. □

Also following the pattern established earlier, a negative result is found in the case of an open mapping.

EXAMPLE 6. An open mapping with first countable fibers from a non-countably tight space onto a first countable space.

Let A be an uncountable set, B be a countably infinite set, and $\{p\} \notin A \times B$. Define $\mathbb{X}$ to be the space $(A \times B) \cup \{p\}$ whose basic open sets are $\{\{x\} : x \in A \times B\} \cup \{\{p\} \cup (M \times N) : M$ is cocountable in A and N is cofinite in $B\}$. Let $\mathbb{Y}$ be the one point compactification of B with the discrete topology, naming p as the added point. The mapping defined in a manner similar to the function in Example 6.7 is easily seen to be open and continuous.

The fibers are clearly first countable. $\mathbb{Y}$ is first countable since there are only countably many cofinite sets in a countable set. $\mathbb{X}$ is not countably tight. Let M be any countable subset of $A \times B$. The point p is not a member of $\overline{M}$, since there is a basic open (cocountable with respect to A) subset of $\mathbb{X}$ containing p which misses M. ◻

Two cardinal functions which are not so well behaved in the closed preimage are the related ideas of spread and cellularity. Indeed, strong examples abound which smash any hope of proving a theorem like Theorems 2, 3, and 5 for spread. Even strengthening the mappings to projections in the square of a compact space is not enough.

DEFINITION: [1] The *spread* $s(\mathbb{X})$ of a space $\mathbb{X}$ is

$$\sup\{|D| : D \text{ is a discrete subset of } \mathbb{X}\} + \omega.$$

EXAMPLE 7. A compact space of countable spread whose product with itself has uncountable spread.

Let $\mathbb{X}$ be $2 \times \mathbb{I}$ with the lexicographic order topology. $\mathbb{X}$ is the union of $[0, 1]$ with the Sorgenfrey topology and $[0, 1]$ with the reverse Sorgenfrey topology, and thus has countable spread. However, the product of the Sorgenfrey line with itself has an uncountable discrete set, namely $\{\langle x, y\rangle : x + y = 1$ and $x, y \neq 1\}$. $\mathbb{X}^2$ contains a copy of this set, and so has uncountable spread. The projection is an open, closed, compact mapping whose fibers have countable spread. This precludes any theorems of the type listed above for the spread of a space. ◻

DEFINITION: [1] The *cellularity* $c(\mathbb{X})$ of a space $\mathbb{X}$ is

$$\sup\{|\mathcal{C}| : \mathcal{C} \text{ is an open disjoint collection in } \mathbb{X}\} + \omega.$$

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REMARK. It is independent of the usual axioms whether the product of two compact spaces of countable cellularity has countable cellularity. Let $\mathbb{X}$ be a Suslin Continuum, the existence of which is guaranteed by axiom $\diamondsuit$. It is known that the product of a Suslin space with itself is not *ccc*; however, assuming $MA + \neg CH$ it is true that the product of two *ccc* spaces is *ccc*. For complete details, see [**2**].

Cellularity, too, is very badly behaved. Todorčević has shown that even under the best of circumstances, namely in the square of a compact space, cellularity is not preserved.

THEOREM 8. [4] *There is a compact Hausdorff space* $\mathbb{X}$ *such that* $c(\mathbb{X}^2) > c(\mathbb{X})$.

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