THE DIVIDED CENTRAL DIFFERENCES OF ZERO

L. CARLITZ AND JOHN RIORDAN

1. Put

$$(1) \qquad S_r(m) = \sum_{n=1}^{m} n^{2r+1}, \qquad T_r(m) = \sum_{n=1}^{m} n^{2r+2}.$$

In a recent paper **(4)**, Lohne showed that

$$(2) \qquad S_r(m) = \sum_{s=1}^{r+1} \frac{K_{rs}}{2s} \frac{(m+s)!}{(m-s)!},$$

$$(3) \qquad T_r(m) = \frac{2m+1}{2} \sum_{s=1}^{r+1} \frac{K_{rs}}{2s+1} \frac{(m+s)!}{(m-s)!},$$

where the coefficients K_{rs} are positive integers and form a numerical triangle defined by

$$(4) \qquad K_{r1} = K_{r,r+1} = 1,$$

$$(5) \qquad K_{r+1,s} = K_{r,s-1} + s^2 K_{rs}.$$

Tambs Lyche **(7)** showed that

$$(6) \qquad K_{rs} = \sum_{t=2}^{s} (-1)^{s+t} \frac{2t^2(t^{2r}-1)}{(s-t)!\,(s+t)!} \qquad (s \geqslant 2).$$

Formulas (2) and (3) closely resemble the well-known expressions for power sums in terms of Stirling numbers of the second kind, namely

$$\sum_{n=1}^{m} n^r = \sum_{s=1}^{r} \frac{1}{s+1} A_{rs} \frac{(m+1)!}{(m-s)!} \qquad (r \geqslant 1),$$

where the Stirling numbers A_{rs} are defined by $A_{00} = 1$ and

$$(7) \qquad A_{r1} = A_{rr} = 1, \qquad A_{r+1,s} = A_{r,s-1} + sA_{rs}.$$

It is accordingly of some interest to see how the coefficients K_{rs} and A_{rs} are related.

From either (2) or (3) we get (Lohne's defining relation)

$$(8) \qquad m^{2r+1} = \sum_{s=1}^{r+1} K_{rs} \frac{(m+s-1)!}{(m-s)!}.$$

This can be rewritten as

$$(9) \qquad m^{2r+2} = \sum_{s=1}^{r+1} K_{rs} m^2(m^2-1^2)\dots(m^2-(s-1)^2)$$
$$= \sum_{s=1} K_{rs} m^{[2s]},$$

Received October 2, 1961. The first author was supported in part by NFS grant G-16485.

the last using the notation of Steffensen **(6)** for central factorials, defined by **(6**, p. 8**)**

$$m^{[n]} = m(m + n/2 - 1)(m + n/2 - 2) \dots (m - n/2 + 1).$$

Equation (9) may be taken as the starting point. Note first its resemblance to

$$m^r = \sum_{s=0}^{r} A_{rs} m(m-1) \dots (m - s + 1).$$

Next, since

$$m^2 m^{[2s]} = m^{[2s+2]} + s^2 m^{[2s]},$$

equation (5) follows at once. For a general expression for K_{rs}, introduce the central difference δ:

$$\delta f(x) = f(x + 1/2) - f(x - 1/2)$$

and note that

$$\delta x^{[n]} = n x^{[n-1]}.$$

Then for any polynomial $f(x)$ **(6**, p. 13**)**

$$f(x) = \sum_{s=0} x^{[s]} \delta^s f(0)/s!\,.$$

Used with equation (9), this shows that

(10) $$K_{rs} = \delta^{2s} 0^{2r+2}/(2s)!\,;$$

in words, the K_{rs} are the divided central differences of zero, of even order. Written out, (10) is the same as

(10a) $$K_{rs} = \frac{1}{(2s)!}\sum_{t=0}^{2s} (-1)^t \binom{2s}{t} (s-t)^{2r+2}$$
$$= \frac{2}{(2s)!}\sum_{t=0}^{s} (-1)^t \binom{2s}{t} (s-t)^{2r+2},$$

the second form by symmetry. The second form is similar to (6) and their equivalence is readily verified. We remark that the Stirling numbers A_{rs} are the divided (ordinary) differences of zero.

Finally, the following results are immediate consequences of (9):

$$\sum_{s=1}^{r+1} (-1)^s((s-1)!)^2 K_{rs} = 0,$$

$$\sum_{s=1}^{r+1} (-1)^{s-1}(1 + 1^2)(1 + 2^2) \dots (1 + (s-1)^2) K_{rs} = (-1)^r.$$

2. Turn now to generating functions. First, using (10a),

$$\sum_{r=0}^{\infty} K_{rs} \frac{x^{2r+2}}{(2r+2)!} = \frac{1}{(2s)!}\sum_{t=0}^{2s} (-1)^t \binom{2s}{t} \left(\frac{e^{x(s-t)} + e^{-x(s-t)}}{2} - 1\right)$$

so that

$$(11) \qquad \sum_{r=0}^{\infty} K_{rs} \frac{x^{2r+2}}{(2r+2)!} = \frac{1}{(2s)!} (e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})^{2s} \qquad (s \geqslant 1).$$

Then from (11)

$$(12) \qquad \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} K_{rs} \frac{x^{2r+2} y^{2s}}{(2r+2)!} = \cosh(2y \sinh \tfrac{1}{2}x) - 1.$$

These results may be compared with

$$(13) \quad \sum_{r=0}^{\infty} A_{rs} x^r / r! = (e^x - 1)^s / s!, \qquad \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} A_{rs} x^r y^s / r! = \exp(y(e^x - 1)).$$

Again, returning to (4) and (5), and writing

$$K_s(x) = \sum_{r=0}^{\infty} K_{rs} x^r,$$

it is evident that

$$(1 - s^2 x) K_s(x) = x K_{s-1}(x),$$
$$K_1(x) = (1 - x)^{-1},$$

so that

$$(14) \qquad K_s(x) = x^{s-1} / (1 - x)(1 - 2^2 x) \ldots (1 - s^2 x),$$

which may be compared with (**3**, p. 175; **5**, p. 43)

$$(15) \qquad \sum_{r=0}^{\infty} A_{rs} x^r = x^s / (1 - x)(1 - 2x) \ldots (1 - sx).$$

The generating functions lead to relations of the numbers as follows. First, by (11) and (13),

$$\sum_{r=1}^{\infty} K_{r-1,s} \frac{x^{2r}}{(2r)!} = e^{-sx} \sum_{r=0}^{\infty} A_{r,2s} \frac{x^r}{r!},$$

which implies

$$(16) \qquad K_{r-1,s} = \sum_{j=0}^{2r} (-1)^j \binom{2r}{j} s^j A_{2r-j,2s},$$

$$(17) \qquad A_{r,2s} = \sum_{2j \leqslant r} \binom{r}{2j} s^{r-2j} K_{j-1,s},$$

and, incidentally,

$$0 = \sum_{j=0}^{2r+1} (-1)^j \binom{2r+1}{j} s^j A_{2r+1-j,2s}.$$

Next, by (14) and (15),

$$\sum_{r=0}^{\infty} K_{rs} x^{2r+2} = (-1)^s \sum_{r=0}^{\infty} A_{rs} x^r \sum_{t=0}^{\infty} A_{ts} (-x)^t,$$

so that

https://doi.org/10.4153/CJM-1963-010-8 Published online by Cambridge University Press

(18) $$K_{r-1,s} = \sum_{t=s}^{2r-s} (-1)^{s+t} A_{ts} A_{2r-t,s}.$$

3. For relations with Bernoulli numbers, we recall that

$$\sum_{n=1}^{m-1} n^r = (B_{r+1}(m) - B_{r+1})/(r+1),$$

where $B_k(x)$ is the Bernoulli polynomial of degree k ($B_k(0) = B_k$). Thus by (1)

$$S_r(m) = m^{2r+1} + (B_{2r+2}(m) - B_{2r+2})/(2r+2).$$

Comparing this with (2) we get

$$\sum_{s=1}^{r+1} \frac{K_{rs}}{2s} (m^2 - 1) \ldots (m^2 - (s-1)^2)(m+s) = m^{2r} + \frac{B_{2r+2}(m) - B_{2r+2}}{(2r+2)m}.$$

Since

$$B_k(x) = \sum_{s=0}^{k} \binom{k}{s} B_{k-s} x^s,$$

we find, equating coefficients of m,

(19) $$(2r+1)B_{2r} = \sum_{s=1}^{r+1} (-1)^{s-1}((s-1)!)^2 s^{-1} K_{rs},$$

which may be compared with the corresponding representation in Stirling numbers (**5**, p. 45)

$$B_r = \sum_{s=0}^{r} (-1)^s s!\,(s+1)^{-1} A_{rs}.$$

From (19) and (10a) we get the explicit formula

(20) $$(2r+1)B_{2r} = \sum_{s=1}^{r+1} s^{-1} \sum_{t=-s}^{s} (-1)^{t+1} \frac{(s-1)!\,(s-1)!}{(s-t)!\,(s+t)!} t^{2r+2}.$$

Since for $r > 0$, both $B_{2r+1} = 0$ and $\delta^{2s}0^{2r+1}/(2s)! = 0$, it follows that

(21) $$(r+1)B_r = \sum_{0<2s\leqslant r+2} s^{-1} \sum_{t=-s}^{s} (-1)^t \frac{(s-1)!\,(s-1)!}{(s-t)!\,(s+t)!} t^{r+2}.$$

4. The Stirling number polynomials

$$a_r(y) = \sum_{s=0}^{r} A_{rs} y^s$$

are familiar. They are defined effectively by the second generating function of (13), or by one of its consequences

(22) $$a_{r+1}(y) = y[a(y) + 1]^r = \sum_{s=0}^{r} \binom{r}{s} y a_s(y).$$

Put $a_r(1) = a_r$.

https://doi.org/10.4153/CJM-1963-010-8 Published online by Cambridge University Press

Analogously we define

$$k_{2r+2}(y) = \sum_{s=1}^{r+1} K_{rs} y^{2s},$$

whose generating function by (12) is

$$\cosh(2y \sinh \tfrac{1}{2}x) = \sum_{r=0}^{\infty} k_{2r}(y) x^{2r}/(2r)! \qquad (k_0(y) = 1). \tag{23}$$

It is also convenient to define $k_{2r+1}(y)$ by

$$\sinh(2y \sinh \tfrac{1}{2}x) = \sum_{r=0}^{\infty} k_{2r+1}(y) x^{2r+1}/(2r+1)! \tag{24}$$

so that

$$\exp(2y \sinh \tfrac{1}{2}x) = \sum_{r=0}^{\infty} k_r(y) x^r/r!\,. \tag{25}$$

Since, with $D = d/dx$,

$$\begin{aligned} y \sinh(2y \sinh \tfrac{1}{2}x) &= \operatorname{sech} \tfrac{1}{2}x \, D \cosh(2y \sinh \tfrac{1}{2}x) \\ &= \operatorname{sech} \tfrac{1}{2}x \sum_{r=0}^{\infty} k_{2r+2}(y) \frac{x^{2r+1}}{(2r+1)!} \end{aligned}$$

and

$$\operatorname{sech} \tfrac{1}{2}x = \sum_{r=0}^{\infty} 2^{-2r} E_{2r} x^{2r}/(2r)!\,,$$

where the E_{2r} are the Euler numbers in the even suffix notation, it follows that

$$y k_{2r+1}(y) = \sum_{s=0}^{r} \binom{2r+1}{2s} 2^{-2s} E_{2s} k_{2r-2s+2}(y). \tag{26}$$

Differentiation of (25) gives

$$y \cosh \tfrac{1}{2}x \exp(2y \sinh \tfrac{1}{2}x) = \sum_{r=0}^{\infty} k_{r+1}(y) x^r/r!$$

and therefore

$$k_{r+1}(y) = y \sum_{2s \leqslant r} \binom{r}{2s} 2^{-2s} k_{r-2s}(y), \tag{27}$$

which may be compared with (22).

The first few values of $k_r(y)$ are

$$\begin{aligned} &k_0(y) = 1, \qquad k_2(y) = y^2, \qquad k_4(y) = y^2 + y^4, \\ &k_1(y) = y, \qquad k_3(y) = \tfrac{1}{4}y + y^3, \qquad k_5(y) = \frac{1}{16}y + \frac{5}{2}y^3 + y^5. \end{aligned}$$

Now let p be a fixed odd prime. Differentiating (25) p times we get

$$D^p \exp(2y \sinh \tfrac{1}{2}x) = \sum_{r=0}^{\infty} k_{r+p}(y) x^r/r!\,. \tag{28}$$

https://doi.org/10.4153/CJM-1963-010-8 Published online by Cambridge University Press

By the formula for derivatives of a composite function **(5**, p. 35**)** we find that

$$D^p \exp(2y \sinh \tfrac{1}{2}x) = Y_p(g_1, \ldots, g_p) \exp(2y \sinh \tfrac{1}{2}x)$$

with Y_p the Bell multivariable polynomial, and

$$g_k = D^k (2y \sinh \tfrac{1}{2}x).$$

But, see **(1)**,

$$Y_p(g_1, \ldots, g_p) \equiv g_p + g_1{}^p \qquad (\text{mod } p)$$

and

$$g_p = y\, 2^{1-p} \cosh \tfrac{1}{2}x \equiv y \cosh \tfrac{1}{2}x \qquad (\text{mod } p),$$
$$g_1{}^p = y^p (\cosh \tfrac{1}{2}x)^p \equiv y^p \qquad (\text{mod } p).$$

Hence

$$D^p \exp(2y \sinh \tfrac{1}{2}x) \equiv (D + y^p) \exp(2y \sinh \tfrac{1}{2}x) \qquad (\text{mod } p)$$

and by (28) we get

(29) $$k_{r+p}(y) \equiv k_{r+1}(y) + y^p k_r(y) \qquad (\text{mod } p).$$

This congruence is of precisely the same form as the congruence satisfied by the $a_r(y)$ defined above, namely, see **(8)**,

$$a_{r+p}(y) \equiv a_{r+1}(y) + y^p a_r(y) \qquad (\text{mod } p).$$

It follows that the numbers $k_r \equiv k_r(1)$ have the same period as the a_r; that is, by the result given in **(2)**

(30) $$k_{r+P} \equiv k_r \qquad (\text{mod } p)$$

where

$$P = (p^p - 1)(p - 1)^{-1}.$$

Some further properties of the numbers k_r are as follows. First k_{2r} is integral, while k_{2r+1} has the denominator 2^{2r}. Indeed it follows at once from (27) that

$$2^{2r} k_{2r+1} \equiv 1 \qquad (\text{mod } 4);$$

more precisely

$$2^{2r} k_{2r+1} \equiv 1 + 4r(2r - 1) \qquad (\text{mod } 16).$$

To find the residue (mod 4) of k_{2r}, it is convenient to define

(31) $$k'_{2r+2} = \sum_{\substack{s=1 \\ s \text{ odd}}}^{r+1} K_{rs}.$$

Then by (5)

$$K_{rs} \equiv K_{r-1,s-1} + K_{r-1,s} \qquad (\text{mod } 4) \qquad (s \text{ odd}).$$

Summing this over odd s gives

(32) $$k'_{2r+2} \equiv k_{2r} \qquad (\text{mod } 4).$$

https://doi.org/10.4153/CJM-1963-010-8 Published online by Cambridge University Press

On the other hand

$$K_{rs} \equiv K_{r-1,s-1} \qquad (\text{mod } 4) \qquad (s \text{ even})$$

so that

$$k_{2r+2} - k'_{2r+2} \equiv k'_{2r} \qquad (\text{mod } 4).$$

Using (32) this becomes

$$k_{2r+2} \equiv k_{2r} + k_{2r-2} \qquad (\text{mod } 4). \tag{33}$$

Iteration of (33) shows that

$$k_{2r+12} \equiv k_{2r} \qquad (\text{mod } 4). \tag{34}$$

Thus the period (mod 4) is 12, and

$$\left.\begin{array}{lll} k_{12r} \equiv 1, & k_{12r+4} \equiv 2, & k_{12r+8} \equiv 1, \\ k_{12r+2} \equiv 1, & k_{12r+6} \equiv 3, & k_{12r+10} \equiv 0. \end{array}\right\} \qquad (\text{mod } 4)$$

This shows that the only even k_{2r} are those of the form k_{6r+4}.

References

1. E. T. Bell, *Exponential polynomials*, Ann. Math., *35* (1934), 258–277.
2. M. Hall, *Arithmetic properties of a partition function*, Bull. Amer. Math. Soc., *40* (1934), abstract 200.
3. C. Jordan, *Calculus of finite differences* (New York, 1947).
4. J. Lohne, *Potenssummer av de naturlige tall* (Power sums of natural numbers), Nord. Mat. Tidskrift, *6* (1958), 155–158.
5. J. Riordan, *An introduction to combinatorial analysis* (New York, 1958).
6. J. F. Steffensen, *Interpolation* (Baltimore, 1927).
7. R. Tambs Lyche, *Tillegg til for anstaende artikkel* (Supplement to the preceding article), Nord. Mat. Tidskrift, *6* (1958), 159–161.
8. J. Touchard, *Propriétés de certains nombres recurrents*, Ann. Soc. Sc. Bruxelles, *A53* (1953), 21–31.

Duke University
Bell Telephone Laboratories