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A new formula for the determinant and bounds on its tensor and Waring ranks

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Abstract

We present a new explicit formula for the determinant that contains superexponentially fewer terms than the usual Leibniz formula. As an immediate corollary of our formula, we show that the tensor rank of the $n \times n$ determinant tensor is no larger than the n-th Bell number, which is much smaller than the previously best-known upper bounds when $n \geq 4$. Over fields of non-zero characteristic we obtain even tighter upper bounds, and we also slightly improve the known lower bounds. In particular, we show that the 4×4 determinant over $\mathbb{F}_2$ has tensor rank exactly equal to 12. Our results also improve upon the best-known upper bound for the Waring rank of the determinant when $n \geq 17$, and lead to a new family of axis-aligned polytopes that tile $\mathbb{R}^n$.

1. Introduction

The determinant and permanent of an $n \times n$ matrix A are defined by

$$\det(A) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)} \right) \quad \text{and} \quad \text{per}(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^n a_{i,\sigma(i)} \right), \qquad (1)$$

respectively, where S_n is the symmetric group over the set $[n] = \{1, 2, \ldots, n\}$. There are numerous other explicit formulas for the permanent of a matrix, such as Ryser's formula [1]

$$\text{per}(A) = \sum_{S \subseteq [n]} \left(\text{sgn}(S) \prod_{i=1}^n \sum_{j \in S} a_{i,j} \right), \qquad (2)$$

where $\text{sgn}(S) = (-1)^{|S|+n}$, as well as Glynn's formula [6]

$$\text{per}(A) = \frac{1}{2^{n-1}} \sum_{\delta} \left(\text{sgn}(\delta) \prod_{i=1}^n \sum_{j=1}^n \delta_i a_{i,j} \right), \qquad (3)$$

where $\text{sgn}(\delta) = \prod_{k=1}^n \delta_k$ and the outer sum ranges over all vectors $\delta \in \{-1, 1\}^n$ with $\delta_1 = 1$.

These alternative formulas for the permanent, despite looking more complicated than the defining formula of equation (1), have a very similar form: they each consist of a sum of products of n factors, with each factor in the product being a linear combination of entries from a

single row of A. One of the advantages of the formulas in equations (2) and (3) is that there are fewer terms in the outer sum ($2^n - 1$ and 2^{n-1}, respectively, instead of the $n!$ terms of the defining equation (1)), so they can be implemented via fewer multiplications.

On the other hand, while there are numerous known methods of computing the determinant of a matrix, most of them do not provide an explicit formula with fewer terms being summed than the defining formula of equation (1). For example, cofactor expansions are just factored forms of equation (1) that still consist of a sum of $n!$ terms, and most numerical methods (e.g., those based on Gaussian elimination or matrix decompositions) are iterative (see [13], for example) and/or require division by entries of the matrix. The only progress in the direction of finding more efficient explicit formulas for the determinant that we are aware of comes from the fact that, when $n = 3$, there are several ways that are known to write the determinant as a sum of just 5 terms instead of $3! = 6$ (see [4, 7, 10], and the references therein, for example), such as

$$
\begin{aligned}
\det(A) = & (a_{1,2} + a_{1,3})(a_{2,1} + a_{2,3})(a_{3,1} + a_{3,2}) \\
& - a_{1,2}a_{2,1}(a_{3,1} + a_{3,2} + a_{3,3}) \\
& - a_{1,3}(a_{2,1} + a_{2,2} + a_{2,3})a_{3,1} \\
& - (a_{1,1} + a_{1,2} + a_{1,3})a_{2,3}a_{3,2} \\
& + a_{1,1}a_{2,2}a_{3,3}.
\end{aligned}
\tag{4}
$$

Derksen noticed that, when combined with cofactor expansions, formulas like equation (4) generalise to give explicit formulas for the $n \times n$ determinant that consist of a sum of $(5/6)^{\lfloor n/3 \rfloor}n!$ terms [4]. This formula has the fewest terms in the sum known until now, except over fields of characteristic 2, where $\det(A) = \mathrm{per}(A)$ (since $-1 = 1$) and Ryser's formula of equation (2) is a sum of just $2^n - 1$ terms.[1]

Our main contribution is to present a new explicit formula for the determinant (Theorem 2) that improves upon both of these bounds. Our formula reduces to exactly the 5-term formula of equation (4) when $n = 3$, and in general it consists of a sum of exactly B_n terms, where B_n denotes the n-th Bell number (i.e., the number of partitions of $[n]$). Since

$$
\frac{B_n}{(5/6)^{\lfloor n/3 \rfloor}n!} \leq \frac{(4n/5)^n}{(\ln(n+1))^n(5/6)^{\lfloor n/3 \rfloor}n!} \leq \frac{n^n}{(\ln(n+1))^n n!} \leq \frac{1}{e}\left(\frac{e}{\ln(n+1)}\right)^n,
$$

for all $n \geq 1$ (the first inequality above uses the bound $B_n \leq (4n/5)^n/(\ln(n+1))^n$ from [2]), our formula has superexponentially fewer terms than the previously best-known formula. When working over fields of non-zero characteristic, our formula simplifies even further (Corollary 11), to the point of giving a $(2^n - n)$-term formula when the characteristic is 2 (Corollary 13), narrowly surpassing Ryser's $(2^n - 1)$-term formula.

1.1 Tensor rank

We can regard the $n \times n$ determinant over a field $\mathbb{F}$ as a tensor living in $(\mathbb{F}^n)^{\otimes n}$, and we can then ask questions of it like we ask of any tensor. In particular, we can ask what its *tensor rank* is [11]. That is, if we use $\det_{\mathbb{F}}^n \in (\mathbb{F}^n)^{\otimes n}$ to denote the $n \times n$ determinant tensor over the field $\mathbb{F}$, what is the least integer r for which there exist $\{\mathbf{v_{j,k}}\} \subset \mathbb{F}^n$ with

$$
\det_{\mathbb{F}}^n = \sum_{k=1}^{r} \mathbf{v_{1,k}} \otimes \mathbf{v_{2,k}} \otimes \cdots \otimes \mathbf{v_{n,k}}?
\tag{5}
$$

We denote tensor rank (i.e., minimal r) by Trank, so the tensor rank of the determinant is denoted by $\mathrm{Trank}(\det_{\mathbb{F}}^n)$.

Our interest in the tensor rank comes from the fact that every determinant formula present in this paper corresponds to a tensor decomposition of the form of equation (5) by replacing each

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[1] Glynn's 2^{n-1}-term formula of equation (3) does not apply in this setting, since we cannot divide by 2 in characteristic 2.

occurrence of $a_{i,j}$ in the formula by $\mathbf{e_j}$ (the j-th standard basis vector of $\mathbb{F}^n$) in the i-th tensor factor of the tensor decomposition. For example, the defining formula (1) corresponds to the tensor decomposition

$$\det_{\mathbb{F}}^n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \mathbf{e}_{\sigma(1)} \otimes \mathbf{e}_{\sigma(2)} \otimes \cdots \otimes \mathbf{e}_{\sigma(n)},$$

which shows that $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq |S_n| = n!$. Similarly, the formula for the 3×3 determinant from equation (4) immediately gives us the following tensor decomposition of $\det_{\mathbb{F}}^3$, which demonstrates that $\mathrm{Trank}(\det_{\mathbb{F}}^3) \leq 5$:

$$\begin{aligned}
\det_{\mathbb{F}}^3 = & (\mathbf{e_2} + \mathbf{e_3}) \otimes (\mathbf{e_1} + \mathbf{e_3}) \otimes (\mathbf{e_1} + \mathbf{e_2}) \\
& - \mathbf{e_2} \otimes \mathbf{e_1} \otimes (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) \\
& - \mathbf{e_3} \otimes (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) \otimes \mathbf{e_1} \\
& - (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) \otimes \mathbf{e_3} \otimes \mathbf{e_2} \\
& + \mathbf{e_1} \otimes \mathbf{e_2} \otimes \mathbf{e_3}.
\end{aligned}$$

In fact, it is known that $\mathrm{Trank}(\det_{\mathbb{F}}^3) = 5$ over every field [10], so this decomposition is optimal. When $n \geq 4$, the exact value of $\mathrm{Trank}(\det_{\mathbb{F}}^n)$ is not known, and until now the best-known upper bounds on it were exactly the bounds that we discussed earlier:[2]

$$\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq (5/6)^{\lfloor n/3 \rfloor} n! \qquad \text{for all fields } \mathbb{F}, \text{ and}$$
$$\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq 2^n - 1 \qquad \text{if } \mathbb{F} \text{ has characteristic 2.}$$

Our formula improves upon these bounds by showing that $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq B_n$ regardless of the field $\mathbb{F}$ (Corollary 3), and $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq 2^n - n$ if $\mathbb{F}$ has characteristic 2 (Corollary 13). These upper bounds are already known to be tight when $n = 3$, and we show that they are also tight when $n = 4$ and the ground field has two elements (i.e., the tensor rank is exactly $2^n - n = 12$ in this case; see Theorem 17). We also obtain some other (tighter than B_n) upper bounds when $\mathbb{F}$ has any non-zero characteristic (Corollaries 11 and 12).

1.2 Waring rank

Another notion of the rank of the determinant comes from thinking of it as a homogeneous polynomial in the n^2 entries of the matrix on which it acts. That is, we can think of the determinant as the following degree-n polynomial in the n^2 variables $\{x_{i,j}\}$ (we abuse notation slightly and use $\det_{\mathbb{F}}^n$ to refer to both this polynomial, as well as the tensor from equation (5), but which one we mean will always be clear from context):

$$\det_{\mathbb{F}}^n(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^n x_{i,\sigma(i)} \right). \tag{6}$$

We are interested in the *Waring rank* of this polynomial. That is, what is the least integer r for which there exist linear forms $\{\ell_k\} \subset \mathrm{Hom}(\mathbb{F}^{n \times n}, \mathbb{F})$ and scalars $\{c_k\} \subset \mathbb{F}$ with

$$\det_{\mathbb{F}}^n = \sum_{k=1}^r c_k \ell_k^n? \tag{7}$$

We denote the Waring rank of this determinant polynomial by $\mathrm{Wrank}(\det_{\mathbb{F}}^n)$.

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[2] It was mentioned in ref. [10] that $\mathrm{Trank}(\det_{\mathbb{F}}^5) \leq 20$ and $\mathrm{Trank}(\det_{\mathbb{F}}^7) \leq 100$, but these are typographical errors; the authors meant $\mathrm{Trank}(\det_{\mathbb{F}}^4) \leq 20$ and $\mathrm{Trank}(\det_{\mathbb{F}}^5) \leq 100$, which come from the formula $(5/6)^{\lfloor n/3 \rfloor} n!$.

For example, if $\mathbb{F}$ has characteristic not equal to 2 then the Waring rank of the degree-2 polynomial xy is 2, since

$$xy = \frac{1}{4}\left((x+y)^2 - (x-y)^2\right),$$

is a linear combination of 2 squares of linear forms, and it is not possible to write xy as a linear combination (i.e., scalar multiple) of the square of just a single linear form. Similarly, it is well-known that if the characteristic of $\mathbb{F}$ is 0 or strictly larger than n then the Waring rank of the n-variable polynomial $x_1 x_2 \cdots x_n$ is exactly 2^{n-1} as shown in ref. [14].[3] By replacing each term in a formula for the determinant with a linear combination of 2^{n-1} different n-th powers of linear forms, we immediately get the following (also well-known) simple relationship between the tensor and Waring ranks of the determinant:

Lemma 1. *Let $\mathbb{F}$ be a field with characteristic p. If $p = 0$ or $p > n$ then*

$$\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq 2^{n-1} \cdot \mathrm{Trank}(\det_{\mathbb{F}}^n).$$

When combined with the bounds on $\mathrm{Trank}(\det_{\mathbb{F}}^n)$ from Section 1.1, this lemma tells us that if $\mathbb{F}$ is a field with characteristic 0 or strictly larger than n, then

$$\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq 2^{n-1}(5/6)^{\lfloor n/3 \rfloor} n!. \tag{8}$$

This upper bound was improved in ref. [9], in the case when $\mathbb{F}$ also contains a primitive root of unity (e.g., if $\mathbb{F} = \mathbb{C}$), to

$$\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq n \cdot n!. \tag{9}$$

Our formula improves upon these bounds by showing that $\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq 2^{n-1} \cdot B_n$, even without the primitive root of unity assumption (Corollary 3). Our bound is strictly better (i.e., smaller) than the one provided by Inequality (8) for all $n \geq 4$, and is better than the one provided by Inequality (9) for all $n \geq 17$.

1.3 Arrangement of the paper

In Section 2, we present our main contribution, which is a new formula for the determinant of a matrix (Theorem 2). As an immediate corollary, we obtain our new field-independent upper bounds on the tensor and Waring ranks of the determinant (Corollary 3).

We present two independent proofs of our formula. First, we present a combinatorial proof in Section 3. This proof has the advantage of being rather mechanical (and thus easy to verify), but the disadvantage of not providing much insight into the formula. Second, we present a geometric proof in Section 4 (this section can be read or skipped quite independently of the rest of the paper). This proof has the advantage of providing some insight into *why* the formula works and how it was actually found, but is less direct: it is established for all matrices A in a small open ball in the space of $n \times n$ real matrices, which is sufficient to prove equality of the determinant polynomial and the polynomial in Theorem 2, thereby proving the result in full generality over arbitrary commutative rings. This proof also demonstrates some new axis-aligned polytope tilings of $\mathbb{R}^n$, where the 1-skeleta of the polytopes can be naturally identified with flip graphs for ordered partial partitions.

In Section 5, we investigate what our formula says about the tensor rank of the determinant over fields with non-zero characteristic, and we obtain tighter upper bounds than the field-independent one (Corollaries 11, 12, and 13). Finally, in Section 6, we (very slightly) improve upon the best-known *lower* bound for the tensor rank of the determinant over arbitrary fields

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[3] If the characteristic is between 2 and n (inclusive) then the Waring rank of $x_1 x_2 \cdots x_n$ is infinite: it cannot be written as a linear combination of *any* number of n-th powers of linear forms.

(Theorem 14), we provide a further improvement for the determinant over finite fields (Theorem 15), and we show that the 4×4 determinant over the field with two elements has tensor rank equal to exactly 12 (Theorem 17), demonstrating optimality of our formula in this case.

2. The formula

Before presenting our formula for the determinant, we first need the concept of a partial partition of $[n]$, which is a set of disjoint subsets (called parts) of $[n]$. If the union of a partial partition is $[n]$ then it is a (non-partial) partition. There is a natural bijective correspondence between partial partitions of $[n]$ with no singleton parts and (non-partial) partitions of $[n]$, which works by erasing all singleton sets from a partition or adding singletons of all members that are missing from a partial partition. For example, when $n = 3$, there are 5 partial partitions of $[n]$ with no singletons and also 5 partitions of $[n]$ as follows:

$$
\begin{array}{cc}
\underline{\text{Partitions}} & \underline{\text{Partial partitions with no singletons}} \\
\{\{1, 2, 3\}\} & \{\{1, 2, 3\}\} \\
\{\{1, 2\}, \{3\}\} & \{\{1, 2\}\} \\
\{\{1, 3\}, \{2\}\} & \{\{1, 3\}\} \\
\{\{2, 3\}, \{1\}\} & \{\{2, 3\}\} \\
\{\{1\}, \{2\}, \{3\}\} & \{\}
\end{array}
\tag{10}
$$

We denote the set of partial partitions of $[n]$ by $\mathrm{PP}(n)$. Just like (non-partial) partitions of $[n]$ give rise to equivalence relations on $[n]$, partial partitions give rise to *partial* equivalence relations on $[n]$: relations that are symmetric and transitive, but need not be reflexive. We denote the partial equivalence relation induced by the partial partition P by $\underset{P}{\sim}$. In other words, for $i, j \in [n]$, $i \underset{P}{\sim} j$ means that there is a part in P containing both i and j. Moreover, for $k \in [n]$, $k \underset{P}{\sim} k$ is not guaranteed, since k might not be in any part of P.

With the above preliminaries out of the way, we now present our formula for the determinant, which works over any field:

Theorem 2. *Let A be an $n \times n$ matrix. Then*

$$
\det(A) = \sum_{P \in \mathrm{PP}(n)} \mathrm{sgn}(P) |P|! \prod_{i=1}^{n}
\begin{cases}
\displaystyle\sum_{j \underset{P}{\sim} i, j \neq i} a_{i,j} & \text{if } i \underset{P}{\sim} i; \\
a_{i,i} + \displaystyle\sum_{j \underset{P}{\sim} j} a_{i,j} & \text{if } i \underset{P}{\nsim} i,
\end{cases}
\tag{11}
$$

where $\mathrm{sgn}(P) = \displaystyle\prod_{S \in P} (-1)^{|S|+1}$.

We note that the quantity $\mathrm{sgn}(P)$ is equal to the sign of a permutation with cycle type $\{|S| : S \in P\}$, which seems like a quite natural notion for the "sign" of a partial partition (e.g., the partial partition $\{\{1, 2, 3\}, \{4, 5\}\}$ has the same sign as the permutation $(1, 2, 3)(4, 5)$). While the sum described by Theorem 2 is over all of $P \in \mathrm{PP}(n)$, if P contains a singleton part $\{i\}$ then that term in the sum equals 0 since

$$
\sum_{j \underset{P}{\sim} i, j \neq i} a_{i,j}
$$

is an empty sum. The formula (11) can thus be rewritten as a sum over the $P \in \mathrm{PP}(n)$ with no singleton parts. For example, if $n = 2$ then there are two partial partitions of $[n]$ with no singletons: $P_1 = \{\{1, 2\}\}$ and $P_2 = \{\}$. We can compute $\mathrm{sgn}(P_1) = -1$, $\mathrm{sgn}(P_2) = 1$, $|P_1|! = 1! = 1$ and $|P_2|! = 0! = 1$, so Theorem 2 says that

$$\det(A) = -a_{1,2}a_{2,1} + a_{1,1}a_{2,2},$$

which is of course the same as the usual formula for the determinant from equation (1). When $n = 3$, the 5 partial partitions with no singletons from equation (10) result in exactly the 5-term formula for the determinant that we saw in equation (4). When $n = 4$, Theorem 2 gives the following 15-term formula for the determinant (surpassing the prior state of the art formula, which has 20 terms):

$$
\begin{aligned}
\det(A) = \ &a_{1,1}a_{2,2}a_{3,3}a_{4,4} \\
&- (a_{1,2} + a_{1,3} + a_{1,4})(a_{2,1} + a_{2,3} + a_{2,4})(a_{3,1} + a_{3,2} + a_{3,4})(a_{4,1} + a_{4,2} + a_{4,3}) \\
&+ (a_{1,1} + a_{1,2} + a_{1,3} + a_{1,4})(a_{2,3} + a_{2,4})(a_{3,2} + a_{3,4})(a_{4,2} + a_{4,3}) \\
&+ (a_{1,3} + a_{1,4})(a_{2,1} + a_{2,2} + a_{2,3} + a_{2,4})(a_{3,1} + a_{3,4})(a_{4,1} + a_{4,3}) \\
&+ (a_{1,2} + a_{1,4})(a_{2,1} + a_{2,4})(a_{3,1} + a_{3,2} + a_{3,3} + a_{3,4})(a_{4,1} + a_{4,2}) \\
&+ (a_{1,2} + a_{1,3})(a_{2,1} + a_{2,3})(a_{3,1} + a_{3,2})(a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4}) \\
&- a_{1,2}a_{2,1}(a_{3,1} + a_{3,2} + a_{3,3})(a_{4,1} + a_{4,2} + a_{4,4}) \\
&- a_{1,3}(a_{2,1} + a_{2,2} + a_{2,3})a_{3,1}(a_{4,1} + a_{4,3} + a_{4,4}) \\
&- a_{1,4}(a_{2,1} + a_{2,2} + a_{2,4})(a_{3,1} + a_{3,3} + a_{3,4})a_{4,1} \\
&- (a_{1,1} + a_{1,2} + a_{1,3})a_{2,3}a_{3,2}(a_{4,2} + a_{4,3} + a_{4,4}) \\
&- (a_{1,1} + a_{1,2} + a_{1,4})a_{2,4}(a_{3,2} + a_{3,3} + a_{3,4})a_{4,2} \\
&- (a_{1,1} + a_{1,3} + a_{1,4})(a_{2,2} + a_{2,3} + a_{2,4})a_{3,4}a_{4,3} \\
&+ 2a_{1,2}a_{2,1}a_{3,4}a_{4,3} \\
&+ 2a_{1,3}a_{2,4}a_{3,1}a_{4,2} \\
&+ 2a_{1,4}a_{2,3}a_{3,2}a_{4,1}.
\end{aligned}
\tag{12}
$$

In general, since there are B_n partitions of $[n]$, there are also B_n partial partitions of $[n]$ with no singleton parts, and thus B_n (potentially) non-zero terms in the formula (11). This demonstrates part (a) of the following corollary (part (b) then follows from Lemma 1):

Corollary 3. *Let $\mathbb{F}$ be a field. Then*

(a) $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq B_n$, *and*

(b) *if $\mathbb{F}$ has characteristic 0 or strictly larger than n then $\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq 2^{n-1} \cdot B_n$.*

3. Combinatorial proof

We now present a combinatorial proof of Theorem 2. This proof works by just brute-force showing that the formula (11), when expanded as a linear combination of monomials, gives the exact same quantity as the defining formula (1). More precisely, let $f : [n] \to [n]$ be a function (not necessarily a permutation). Our goal is to show that the coefficient of $a_{1,f(1)}a_{2,f(2)} \cdots a_{n,f(n)}$ is the same in equation (11) as it is in the defining formula for the determinant (1).

To this end we say that a partial partition $P \in \mathrm{PP}(n)$ is *algebraically compatible* with f if, for all $i \in [n]$, we have the following two properties:

(α) If $i \underset{P}{\sim} i$ then $f(i) \neq i$ and $f(i) \underset{P}{\sim} i$, and

(β) If $i \underset{P}{\not\sim} i$ then $f(i) = i$ or $f(i) \underset{P}{\sim} f(i)$.

If we let ACPP(f) denote the set of partial partitions that are algebraically compatible with f, then equation (11) says exactly that the coefficient c_f of the coefficient of $a_{1,f(1)}a_{2,f(2)}\cdots a_{n,f(n)}$ in an expansion of the determinant is equal to

$$c_f = \sum_{P\in\mathrm{ACPP}(f)} \mathrm{sgn}(P)|P|!. \tag{13}$$

Lemma 4. *Let c_f be the coefficient of $a_{1,f(1)}a_{2,f(2)}\cdots a_{n,f(n)}$ after expanding the polynomial in the right-hand side of equation (11). Then $c_f = \mathrm{sgn}(f)$ if f is a permutation and $c_f = 0$ otherwise.*

Proof. To prove Lemma 4 (and thus Theorem 2), we now split into two cases.

Case 1: f is not a permutation.

As f is not surjective, there exists $i \notin \mathrm{range}(f)$. Let $j = f(i)$ and observe that $j \neq i$. Take an arbitrary $P \in \mathrm{ACPP}(f)$. Note that $j \sim j$ and, if $i \sim i$, we have $i \sim j$. If P' is obtained by removing i from the part of j (if $i \sim i$) or introducing i into the part of j (otherwise), then $P' \in \mathrm{ACPP}(f)$ and has the opposite sign to P. This defines an involution on $\mathrm{ACPP}(f)$ mapping each algebraically compatible partial partition to one of opposite sign, so $c_f = 0$.

Case 2: f is a permutation.

We can write f as a product of disjoint cycles of length at least 2: $f = \sigma_1\sigma_2\cdots\sigma_k$. Each cycle $\sigma = (i_1\ i_2\ \ldots\ i_\ell)$ corresponds naturally to a subset $S_\sigma := \{i_1, i_2, \ldots, i_\ell\} \subseteq [n]$ (though this correspondence is many-to-one since the order of the entries σ matters, whereas it does not matter in S_σ). Similarly, from f we can build the partial partition $P_f := \{S_{\sigma_1}, S_{\sigma_2}, \ldots, S_{\sigma_k}\}$.

If i_1 and i_2 are in the same cycle of f then, for any $P \in \mathrm{ACPP}(f)$ we have $i_1 \underset{P}{\sim} i_2$. It follows that $P \in \mathrm{ACPP}(f)$ if and only if $P = P_f$ or P can be obtained from P_f by unioning together some of its parts. In other words, there exists a (non-partial) partition $K = \{K_1, \ldots, K_m\}$ of $[k]$ such that

$$P = \left\{ \bigcup_{i\in K_j} S_{\sigma_i} : j \in [m] \right\}. \tag{14}$$

If $\left\{ {k \atop m} \right\}$ denotes the (k, m)-th Stirling number of the 2nd kind, then there are $\left\{ {k \atop m} \right\}$ partitions of $[k]$ with exactly m parts, so there are $\left\{ {k \atop m} \right\}$ partial partitions P of the form (14) with exactly m parts. Each one has $|P| = m$ and $\mathrm{sgn}(P) = (-1)^{k+m}\mathrm{sgn}(f)$, so equation (13) can be written more explicitly as

$$
\begin{aligned}
c_f &= \sum_{P\in\mathrm{ACPP}(f)} \mathrm{sgn}(P)|P|! \\
&= \sum_{m=1}^{k} (-1)^{k+m}\,\mathrm{sgn}(f) \left\{ {k \atop m} \right\} m! \\
&= (-1)^k\,\mathrm{sgn}(f)\left(\sum_{m=1}^{k} \left\{ {k \atop m} \right\} (-1)^m m! \right).
\end{aligned}
\tag{15}
$$

We now plug $x = -1$ into the well-known formula

$$\sum_{m=1}^{k} \left\{ {k \atop m} \right\} x(x-1)(x-2)\cdots(x-m+1) = x^k$$

to see that

$$\sum_{m=1}^{k} \left\{ {k \atop m} \right\} (-1)^m m! = (-1)^k. \tag{16}$$

Substituting equation (16) into the bottom line of equation (15) shows that $c_f = (-1)^{2k} \operatorname{sgn}(f) = \operatorname{sgn}(f)$, which completes the proof. □

4. Geometric interpretation and proof

We now present an alternate proof of Theorem 2 that perhaps provides a bit more insight into why the formula (11) works. We take the base field to be $\mathbb{R}$ throughout this section, but remark that this does not lose any generality: proving Theorem 2 over $\mathbb{R}$ establishes equality between two polynomials in the ring $\mathbb{Z}[a_{1,1}, a_{1,2}, \ldots, a_{n,n}]$, which must therefore also hold over any commutative ring.

Throughout the rest of this section, we use 3D terminology (e.g., "volume" and "parallelepiped") if the dimension n is unknown or greater than 2. The n standard basis vectors are denoted $\mathbf{e_1}, \ldots, \mathbf{e_n}$. It is notationally convenient to also define $\mathbf{e_0}$ to be the negated sum of the standard basis vectors, so that $\mathbf{e_0} + \mathbf{e_1} + \cdots + \mathbf{e_n} = 0$.

4.1 Tilings in general

We consider the matrix A as defining a lattice in $\mathbb{R}^n$, where the lattice points consist of integer linear combinations of the columns of the matrix. More specifically, this lattice is

$$\Lambda_A := \{A\mathbf{v} : \mathbf{v} \in \mathbb{Z}^n\}.$$

The usual connection between the determinant and this lattice is the fact that a parallelepiped with side vectors equal to the columns of A tiles $\mathbb{R}^n$ by translates in the lattice Λ_A and has signed volume equal to $\det(A)$ (see Fig. 1a). We will prove Theorem 2 by constructing a polytope whose volume is given by the formula in equation (11), but that also tiles $\mathbb{R}^n$ by translates in the lattice Λ_A and thus must also have signed volume equal to $\det(A)$ (see Fig. 1b).[4]

For now, instead of placing the usual parallelepiped at each lattice point, we place a cuboid with dimensions $a_{1,1} \times a_{2,2} \times \cdots \times a_{n,n}$, extending in the positive direction. That is, we define the cuboid C_A to be the Cartesian product of closed intervals of lengths given by the diagonal entries of A:

$$C_A := \prod_{i=1}^{n} [0, a_{i,i}],$$

and we consider the set of cuboids $\{C_A + \mathbf{z} : \mathbf{z} \in \Lambda_A\}$. Depending on the values of the off-diagonal entries of A, these cuboids may overlap and/or there may be gaps between them (see Fig. 2).

Since these cuboids may overlap and/or have gaps between them, they do not typically form a valid tiling of $\mathbb{R}^n$. In order to "fix" the fact that these cuboids can overlap, we define F_A to

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[4]This technique has been used in the past to create notched cube tilings of $\mathbb{R}^n$ [16], for example; our tilings will also be axis-aligned polytopes, but will otherwise be slightly more complicated.
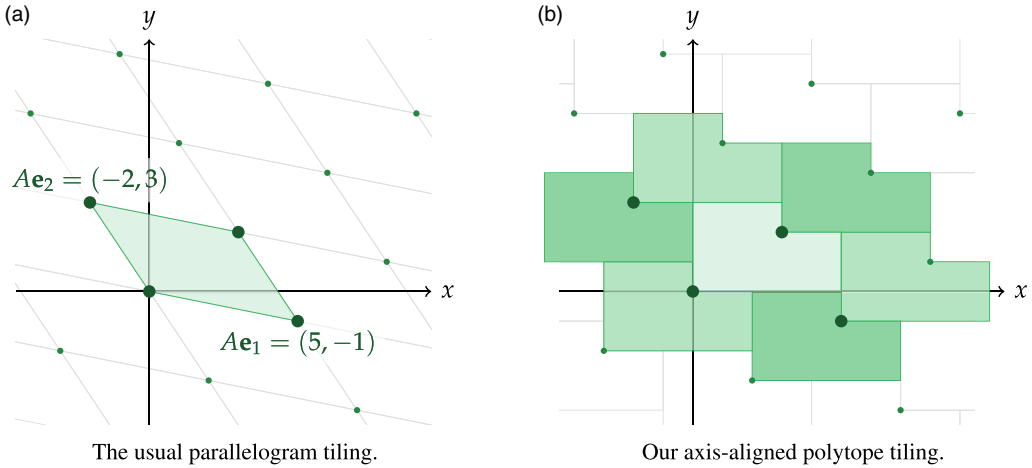
(a) $A\mathbf{e}_2 = (-2,3)$ $A\mathbf{e}_1 = (5,-1)$

The usual parallelogram tiling.

(b)

Our axis-aligned polytope tiling.

Figure 1. Two tilings of $\mathbb{R}^2$ on the same lattice in which the tiles have area equal to $\det\left(\left[\begin{smallmatrix} 5 & -2 \\ -1 & 3 \end{smallmatrix}\right]\right) = 13$.

(a) $A\mathbf{e}_2 = (2,3)$ $A\mathbf{e}_1 = (5,-4)$

A rectangular packing (with gaps) of $\mathbb{R}^2$ corresponding the matrix $\left[\begin{smallmatrix} 5 & 2 \\ -4 & 3 \end{smallmatrix}\right]$.

(b) $A\mathbf{e}_2 = (-2,3)$ $A\mathbf{e}_1 = (5,-1)$

A rectangular covering (with overlaps) of $\mathbb{R}^2$ corresponding the matrix $\left[\begin{smallmatrix} 5 & -2 \\ -1 & 3 \end{smallmatrix}\right]$.

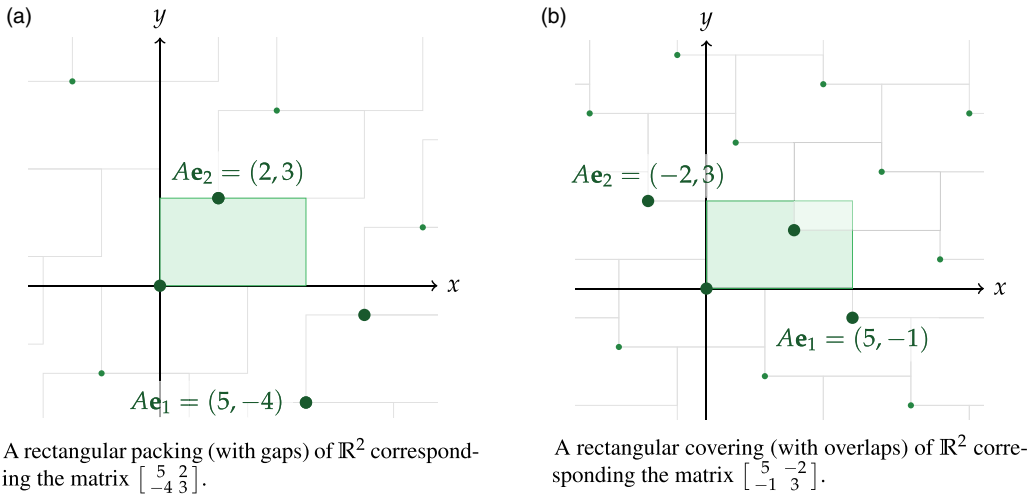**Figure 2.** Two 5×3 rectangular not-quite-tilings of $\mathbb{R}^2$ coming from matrices with diagonal entries 5 and 3. The shaded rectangle is C_A, while the other rectangles are its translates on the lattice Λ_A.

be the axis-aligned polytope obtained from C_A by removing copies of C_A translated by sums of non-empty subsets of the columns of A.[5] We take the closure so that F_A contains its boundary:

$$F_A := \overline{C_A \setminus \bigcup_{\varnothing \neq S \subseteq [n]} \left\{ x + \sum_{i \in S} A\mathbf{e_i} : x \in C_A \right\}}. \tag{17}$$

In order to similarly "fix" the fact that there may be gaps between the cuboids, for the remainder of the proof we only consider matrices A with the following properties:

---

[5]It may be tempting to remove *all* translates of C_A from C_A, but look at Fig. 1b to see why this does not work; we only want to remove the overlap at the top-right vertex of C_A *or* bottom-left of C_A, not both.

(a) Each diagonal entry $a_{i,i}$ is strictly positive;

(b) Each off-diagonal entry, $a_{i,j}$ with $i \neq j$, is strictly negative; and

(c) Each row has strictly positive sum, i.e., A is strictly diagonally dominant.

Doing so simplifies the rest of the proof considerably (since the determinant for these matrices is strictly positive, so it equals a volume instead of a signed volume, for example), and it does not result in any loss of generality. Indeed, if two polynomials agree on some open set then they must agree everywhere; the definitional formula of the determinant from equation (1) and the formula described by Theorem 2 are both polynomials in the n^2 entries of the matrix A, so if they agree on some open set (like the set of matrices described by conditions (a)–(c), or even just the smaller set described by the upcoming Lemma 5) then they must agree everywhere.

In the next section we establish that, subject to these conditions (a)–(c), the translates of F_A by vectors in Λ_A tile the ambient space $\mathbb{R}^n$: their union covers all of space, and any two distinct translates intersect on a set of measure zero (i.e., on their boundary or not at all). We also show that two tiles $F_A + A\mathbf{u}$ and $F_A + A\mathbf{v}$ share a common boundary if and only if the difference $\mathbf{u} - \mathbf{v}$ is the sum of some non-empty proper subset of the vectors $\{\mathbf{e_0}, \mathbf{e_1}, \dots, \mathbf{e_n}\}$, in which case we describe these two translates as "neighbours". This is equivalent to $\mathbf{u} - \mathbf{v}$ being a non-zero vector with all entries in $\{0, 1\}$ or all entries in $\{0, -1\}$. Each tile has exactly $2(2^n - 1)$ neighbours.

This tiling admits a proper $(n + 1)$-colouring, where $F_A + A\mathbf{v}$ is coloured according to the sum of the coordinates of $\mathbf{v}$ modulo $n + 1$. Observe that if two tiles $F_A + A\mathbf{u}$ and $F_A + A\mathbf{v}$ are neighbours, then the coordinate sum of $\mathbf{u} - \mathbf{v}$ is in $[-n, -1] \cup [1, n]$ and is therefore non-zero modulo $n + 1$, so the tiles are assigned distinct colours. Figure 1b shows F_A and its neighbours coloured in this manner.

Moreover, the tiling is homeomorphic to the standard permutohedral tiling of $\mathbb{R}^n$ obtained by taking the Voronoi tessellation of the lattice $A_n^\star$ defined in [3]. For $n = 2$, this is the familiar hexagonal tessellation; for $n = 3$, this is the tessellation by truncated octahedra whose centres form the body-centred cubic lattice.

4.2 The proof itself

For an arbitrary $n \times n$ matrix A, we shall consider five different propositions (each of which may be true or false, depending on A):

- $P_1(A)$: $\det(A)$ is equal to the formula described by equation (11).
- $P_2(A)$: the translates $\{F_A + \mathbf{z} : \mathbf{z} \in \Lambda_A\}$ have pairwise measure-zero intersections.
- $P_3(A)$: the translates $\{F_A + \mathbf{z} : \mathbf{z} \in \Lambda_A\}$ cover all of $\mathbb{R}^n$.
- $P_4(A)$: the volume of F_A is equal to the formula described by equation (11).
- $P_5(A)$: the volume of F_A is equal to $\det(A)$.

Any two of $P_2(A), P_3(A), P_5(A)$ together imply the third (being equivalent to the claim that the translates of F_A by the vectors in the lattice Λ_A tile space). Also, any two of $P_1(A), P_4(A), P_5(A)$ together imply the third (by transitivity of equality). We summarise these relationships in Fig. 3.

Theorem 2 is equivalent to $P_1(A)$ being true for all matrices A. The proof proceeds by showing that $P_3(A)$ and $P_4(A)$ are true for all matrices A satisfying the three conditions (a)–(c), and that $P_2(A)$ is true for all matrices A in a small open neighbourhood of a canonical matrix B. We deduce that $P_1(A)$ holds on a non-empty open set, and therefore (being an equality between two polynomials) holds in general, completing the proof of Theorem 2. Note that, by traversing the implications in the other direction, it follows that $P_5(A)$ and $P_2(A)$ are true for all matrices A satisfying conditions (a)–(c), and not just those in the small neighbourhood for which we prove $P_2(A)$ directly.
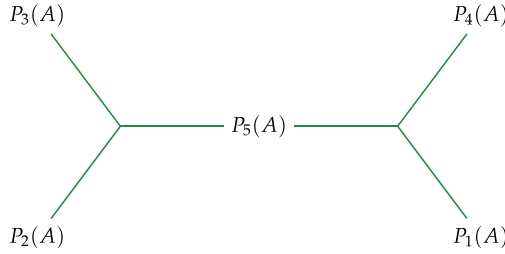
Figure 3. A summary of the relationship between the properties $P_1(A)$–$P_5(A)$. Any two of the properties in the left Y-shape imply the third, and any two of the properties in the right Y-shape imply the third.

Lemma 5. *Consider the $n \times n$ matrix*

$$B := (n+1)I - J, \tag{18}$$

where J is the matrix with all entries equal to 1. *There is some ε-neighbourhood N_ε of B with the property that, for all $A \in N_\varepsilon$ and all $\mathbf{y} \neq \mathbf{z} \in \Lambda_A$, it is the case that $(F_A + \mathbf{y}) \cap (F_A + \mathbf{z})$ has measure zero (i.e., any two tiles in the tiling overlap only on their boundary or not at all).*

Before proving the above lemma, we note that we will choose the ε-neighbourhood N_ε to be small enough that every $A \in N_\varepsilon$ satisfies the three conditions (a)–(c) that we described earlier (which is possible since B satisfies those three conditions and they define an open set).

B is a positive-definite symmetric matrix with eigenvalues 1 (with multiplicity 1) and $n+1$ (with multiplicity $n-1$). The singular values of B are equal to its eigenvalues, so the minimum singular value of B is also equal to 1. By choosing ε small enough, we can ensure that every $A \in N_\varepsilon$ has minimum singular value greater than $\frac{1}{2}$.

For concreteness, let $\varepsilon_0 > 0$ be a constant (depending only on n) that ensures that every $A \in N_\varepsilon$ satisfies conditions (a)–(c) and has minimum singular value greater than $\frac{1}{2}$.

Proof of Lemma 5. It suffices to prove that $F_A \cap (F_A + \mathbf{z})$ has measure zero whenever $\mathbf{z} \in \Lambda_A$ is non-zero. Since A is strictly diagonally dominant and thus invertible, $\mathbf{z} \in \Lambda_A$ being non-zero is equivalent to $\mathbf{z} = A\mathbf{v}$ for some non-zero $\mathbf{v} \in \mathbb{Z}^n$.

If $\mathbf{v}$ is non-zero and has all entries in $\{0, 1\}$ or all entries in $\{0, -1\}$ then, by the construction of F_A given in equation (17), we know that $F_A \cap (F_A + A\mathbf{v})$ has measure zero (we call these $2(2^n - 1)$ tiles $F_A + A\mathbf{v}$ the "neighbours" of F_A; when $n = 2$ they are exactly the 6 tiles that touch the central shaded tile in Fig. 1b). Our goal now is to show that every non-neighbour of F_A has empty intersection with F_A.

To this end, first consider the matrix B from equation (18). For this matrix, the polytope under consideration is

$$F_B = \left\{ (x_1, x_2, \ldots, x_n) : 0 \leq x_i^{\uparrow} \leq i \text{ for all } i \in [n] \right\},$$

where $x_i^{\uparrow}$ is the i-th smallest entry of $(x_1, x_2, \ldots, x_n)$ (see Fig. 4). Equivalently, F_B is the union of the $n!$ images of the cuboid $[0, 1] \times [0, 2] \times \cdots \times [0, n]$ under arbitrary permutations of the coordinates. We claim that F_B does not intersect $F_B + B\mathbf{v}$ if $F_B + B\mathbf{v}$ is not a neighbour of F_B.

To prove this claim, suppose that $F_B + B\mathbf{v}$ is not a neighbour of F_B.

Firstly, consider the case where there exist indices i and j such that $\mathbf{v_i} - \mathbf{v_j} \geq 2$. Then $(B\mathbf{v})_i - (B\mathbf{v})_j = (n+1)(\mathbf{v_i} - \mathbf{v_j}) \geq 2(n+1)$. Consequently, one of $(B\mathbf{v})_i$ and $(B\mathbf{v})_j$ has absolute value $\geq n+1$, which means that the bounding cubes of $F_B + B\mathbf{v}$ and F_B (which each have sidelength n) are disjoint.

$$V = a_{1,1}a_{2,2}a_{3,3} = 27 \qquad V = -(a_{1,2} + a_{1,3})(a_{2,1} + a_{2,3})(a_{3,1} + a_{3,2})$$

$$= 8$$

$$V = a_{1,2}a_{2,1}(a_{3,1} + a_{3,2} + a_{3,3})$$
$$+ a_{1,3}(a_{2,1} + a_{2,2} + a_{2,3})a_{3,1}$$
$$+ (a_{1,1} + a_{1,2} + a_{1,3})a_{2,3}a_{3,2}$$
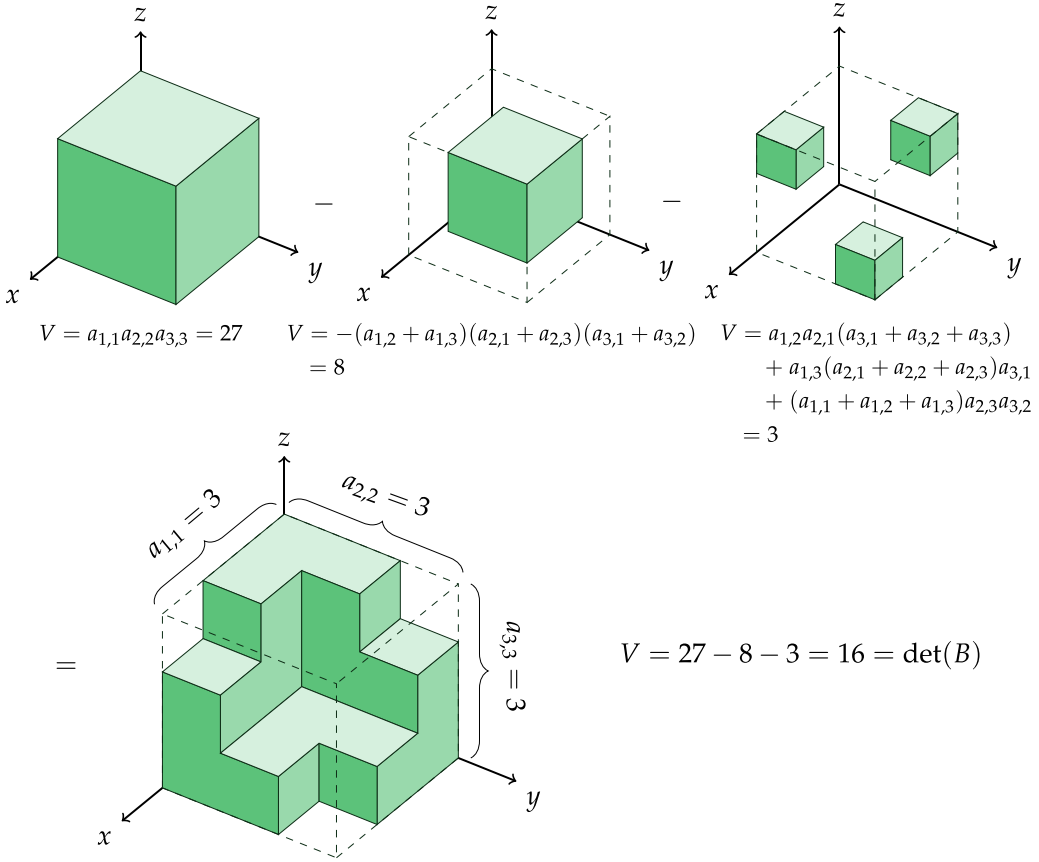$$= 3$$

$$V = 27 - 8 - 3 = 16 = \det(B)$$

Figure 4. The polytope F_B for the matrix $B = (n + 1)I - J$, with $n = 3$. Its volume is equal to the sum and difference of the volumes of 5 different cubes, corresponding to the 5-term formula for the determinant (4).

This leaves the case where all entries of **v** differ by at most 1. In other words, there exists $c \in \mathbb{Z}$ such that for all i, we have $\mathbf{v_i} \in \{c - 1, c\}$, and moreover there exists at least one such j such that $\mathbf{v_j} = c$.

We can assume without loss of generality that c is positive (because $F_B + B\mathbf{v}$ is disjoint from F_B if and only if $F_B - B\mathbf{v}$ is disjoint from F_B). Moreover, if $c \in \{0, 1\}$ then the polytopes are neighbours, so we have $c \geq 2$.

By permuting the coordinates, we can assume that $\mathbf{v_1}, \ldots, \mathbf{v_k} = c$ and $\mathbf{v_{k+1}}, \ldots, \mathbf{v_n} = c - 1$. Then we can compute that:

$$(B\mathbf{v})_1 = \cdots = (B\mathbf{v})_k = n - k + c$$

and, because $c \geq 2$, this means that at least k of the entries $(B\mathbf{v})_i \geq n - k + 2$. As such, $B\mathbf{v} \notin F_B$, because every point $x \in F_B$ has at least $n - k + 1$ entries $x_i \leq n - k + 1$. Moreover, because F_B is contained in the positive orthant, we also have $x + B\mathbf{v} \notin F_B$ for all $x \in F_B$, establishing that the two polytopes are disjoint.

Since F_B has empty intersection with $F_B + B\mathbf{v}$ whenever they are non-neighbours, and B has integer entries, the distance between non-neighbours F_B and $F_B + B\mathbf{v}$ must be at least 1. Since the coordinates of F_A and $F_A + A\mathbf{v}$ are continuous in the entries of A, we conclude that there is some ε-neighbourhood N_ε of B with the property that $F_A \cap (F_A + A\mathbf{v}) = \emptyset$ whenever $A \in N_\varepsilon$ and $F_A + A\mathbf{v}$ is not a neighbour of F_A.

However, the particular choice of ε can depend on the vector $\mathbf{v}$, so we denote it by $\varepsilon_{\mathbf{v}}$. To fix this problem, note that if $\|\mathbf{v}\| \geq 2K$ then $\|A\mathbf{v}\| > K$ (since the minimum singular value of A is at least $\frac{1}{2}$), so there are only finitely many $\mathbf{v} \in \mathbb{Z}^n$ for which $F_A \cap (F_A + A\mathbf{v})$ is potentially non-empty: choose K to be an upper bound on the diameter of F_A; the only $\mathbf{v} \in \mathbb{Z}^n$ that need to be considered are those in the ball of radius $2K$. Defining ε to be the minimum of ε_0 (defined at the beginning of this proof) and these finitely many $\varepsilon_{\mathbf{v}}$'s completes the proof. $\qquad\square$

Lemma 6. *If A satisfies conditions (a)–(c) above then $F_A + \Lambda_A = \mathbb{R}^n$ (i.e., there are no gaps in the tiling).*

Proof. We first note that it suffices to show that, for each $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{z} \in \Lambda_A$ such that $\mathbf{x} + \mathbf{z} \in C_A$. To see this, recall from equation (17) that F_A is constructed from C_A by removing its immediate neighbours in the positive direction, so if $\mathbf{x} + \mathbf{z} \in C_A$ then $\mathbf{x} \in (F_A + A\mathbf{b}) - \mathbf{z}$ for some binary vector $\mathbf{b} \in \{0, 1\}^n$. Since $A\mathbf{b} - \mathbf{z} \in \Lambda_A$, this implies $\mathbf{x} \in F_A + \Lambda_A$.

To find $\mathbf{z}$, first pick some $\mathbf{z}^{(0)} \in \Lambda_A$ such that each entry of $\mathbf{y}^{(0)} := \mathbf{x} + \mathbf{z}^{(0)}$ is non-negative (such a $\mathbf{z}^{(0)}$ exists since A is strictly diagonally dominant and thus invertible). Set $k = 0$ and proceed inductively as follows:

(i) If $\mathbf{y}_i^{(\mathbf{k})} \leq a_{i,i}$ for all $1 \leq i \leq n$ then $\mathbf{y}^{(\mathbf{k})} \in C_A$, so we can choose $\mathbf{z} = \mathbf{z}^{(\mathbf{k})}$ and be done.

(ii) Otherwise, pick an index $1 \leq i \leq n$ for which $\mathbf{y}_i^{(\mathbf{k})} > a_{i,i}$ and set $\mathbf{y}_i^{(\mathbf{k+1})} = \mathbf{y}_i^{(\mathbf{k})} - A\mathbf{e_i}$ and $\mathbf{z}_i^{(\mathbf{k+1})} = \mathbf{z}_i^{(\mathbf{k})} - A\mathbf{e_i}$. Increase k by 1 and then repeat these bullet points.

Since the diagonal entries of A are strictly positive, it is clear that $0 \leq \mathbf{y}_i^{(\mathbf{k+1})} < \mathbf{y}_i^{(\mathbf{k})}$. However, decreasing the i-th entry like this comes at the expense of increasing the other entries (since the off-diagonal entries of A are negative). It is thus not obvious that the inductive procedure described above terminates. To see that it does, we demonstrate the existence of a vector $\mathbf{v} \in \mathbb{R}^n$ and a scalar $0 < d \in \mathbb{R}$ with the properties that $\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k})} \geq 0$ for all k and $(\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k})}) - (\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k+1})}) \geq d$ for all k, implying that the procedure terminates for some $k \leq (\mathbf{v} \cdot \mathbf{y}^{(\mathbf{0})})/d$.

To construct such a $\mathbf{v}$ and d, let $c \in \mathbb{R}$ be large enough that $cI - A^T$ has all entries strictly positive. The Perron–Frobenius theorem tells us that there is a strictly positive real eigenvalue λ with a corresponding entrywise strictly positive eigenvector $\mathbf{v}$ such that $(cI - A^T)\mathbf{v} = \lambda\mathbf{v}$. Since $\mathbf{v}$ and $\mathbf{y}^{(\mathbf{k})}$ are both entrywise non-negative, we have $\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k})} \geq 0$ for all k. Furthermore, since $(c - \lambda, \mathbf{v})$ is also an eigenvalue-eigenvector pair of A^T, and A is strictly diagonally dominant (and c and λ are both real), we know that $c - \lambda > 0$. It follows that

$$\mathbf{v} \cdot (A\mathbf{e_i}) = (A^T\mathbf{v}) \cdot \mathbf{e_i} = (c - \lambda)\mathbf{v} \cdot \mathbf{e_i} = (c - \lambda)v_i > 0 \quad \text{for all} \quad 1 \leq i \leq n.$$

If we choose $d := (c - \lambda) \min_i\{v_i\} > 0$ then it follows that $(\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k})}) - (\mathbf{v} \cdot \mathbf{y}^{(\mathbf{k+1})}) = \mathbf{v} \cdot (A\mathbf{e_i}) \geq d$, which completes the proof. $\qquad\square$

Lemma 7. *If A satisfies conditions (a)–(c) above then the volume of F_A is given by the expression in Equation (11).*

Proof. The volume of the cuboid C_A is clearly equal to $a_{1,1}a_{2,2} \cdots a_{n,n}$, which is one of the terms in the sum (11) (it is the term corresponding to the empty partial partition). We now use inclusion-exclusion to show that the rest of the terms in that sum correspond to volumes that were removed by translations of C_A when creating F_A as in equation (17).

For a non-empty $S \subseteq [n]$, the set

$$C_A \cap \left(C_A + \sum_{i \in S} A\mathbf{e_i} \right) \tag{19}$$

is a cuboid with its i-th side length ℓ_i equal to

$$
\ell_i = \begin{cases} a_{i,i} - \sum\limits_{j \in S} a_{i,j}, & \text{if } i \in S \\ a_{i,i} + \sum\limits_{j \in S} a_{i,j}, & \text{if } i \notin S \end{cases} = \begin{cases} - \sum\limits_{j \in S, j \neq i} a_{i,j}, & \text{if } i \in S \\ a_{i,i} + \sum\limits_{j \in S} a_{i,j}, & \text{if } i \notin S. \end{cases}
$$

Multiplying these side lengths together gives us the volume of the cuboid (19). Subtracting this quantity for all non-empty $S \subseteq [n]$ (i.e., subtracting the volume of all of these cuboids that are removed from C_A to create F_A) results in the following (not yet correct) formula for the volume of F_A:

$$
a_{1,1} a_{2,2} \cdots a_{n,n} - \sum_{\varnothing \neq S \subseteq [n]} \prod_{i=1}^{n} \ell_i = a_{1,1} a_{2,2} \cdots a_{n,n} - \sum_{\varnothing \neq S \subseteq [n]} \prod_{i=1}^{n} \begin{cases} - \sum\limits_{j \in S, j \neq i} a_{i,j}, & \text{if } i \in S \\ a_{i,i} + \sum\limits_{j \in S} a_{i,j}, & \text{if } i \notin S. \end{cases} \tag{20}
$$

When $n \leq 3$ the formula (20) is the same as the formula (11) and indeed equals the volume of F_A. In particular, if $n = 2$ then it expresses the area of the shaded tile F_A in Fig. 1b as the area $a_{1,1} a_{2,2}$ of the rectangle C_A from Fig. 2b minus the area of the rectangular overlapping region at its top-right vertex, and if $n = 3$ then it expresses the volume of F_A as the volume $a_{1,1} a_{2,2} a_{3,3}$ of the cuboid C_A minus the volumes of 4 other cuboids (these cuboids are depicted in the case of the matrix $(n+1)I - J$ in Fig. 4, and the picture for other matrices satisfying (a)–(c) is similar).

When $n \geq 4$, however, the formula (20) is not quite correct, since there are overlaps-of-overlaps of the translates of C_A, so some volume that is removed from C_A in equation (20) is removed multiple times. To correct this mistake, we proceed via inclusion-exclusion: we add back the volumes that were subtracted too many times, then we subtract volumes that were added back too many times, and so on.

For example, when $n = 4$, the cuboids corresponding to the subsets $S_1 = \{1, 2\}$, and $S_2 = \{1, 2, 3, 4\}$ (i.e., the cuboids $C_A + A(\mathbf{e_1} + \mathbf{e_2})$ and $C_A + A(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3} + \mathbf{e_4})$) overlap with each other, so formula (20) is too small: it subtracts off the volume of the cuboid

$$
(C_A + A(\mathbf{e_1} + \mathbf{e_2})) \cap (C_A + A(\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3} + \mathbf{e_4}))
$$

twice (this volume equals $a_{1,2} a_{2,1} a_{3,4} a_{4,3}$). In fact, there are exactly six cuboids whose volumes were subtracted off twice by formula (20), given by $S_2 = \{1, 2, 3, 4\}$ and S_1 being any 2-element subset of S_2. To correct this mistake, we simply add the volumes of these cuboids back in. The cuboids corresponding to $S_1 = \{1, 2\}$ and $S_1 = \{3, 4\}$ have the same volume as each other, as do the cuboids corresponding to $S_1 = \{1, 3\}$ and $S_1 = \{2, 4\}$, as do the cuboids corresponding to $S_1 = \{1, 4\}$ and $S_1 = \{2, 3\}$, thus giving the final three terms (each with a coefficient of 2) in equation (12).

In general, k pairwise distinct (but not necessarily disjoint) subsets $S_1, S_2, \ldots, S_k \subseteq [n]$ are such that

$$
\bigcap_{j=1}^{k} \left(C_A + \sum_{i \in S_j} A \mathbf{e_i} \right) \tag{21}
$$

has non-zero volume if and only if there is a chain of inclusions among them, which we will assume is $S_1 \subset S_2 \subset \cdots \subset S_k \subseteq [n]$ without loss of generality, and $|S_\ell \setminus S_{\ell-1}| \geq 2$ for all $1 \leq \ell \leq k$ (we define $S_0 = \varnothing$ for convenience). Indeed, if $|S_\ell \setminus S_{\ell-1}| = 0$ then $S_\ell = S_{\ell-1}$, and if $|S_\ell \setminus S_{\ell-1}| = 1$ then the two cuboids corresponding to $j = \ell - 1$ and $j = \ell$ in equation (21) intersect only on their boundary ($S_\ell \setminus S_{\ell-1} = \{i\}$ for some $i \in [n]$, so the cuboids are offset from each other by $a_{i,i}$ in the direction of the i-th coordinate axis, which is also their width in that direction).

The volume of the cuboid described by the intersection in equation (21) is the product of its side lengths, which equals

$$\prod_{i=1}^{n} \begin{cases} -\sum\limits_{j \in S_\ell \setminus S_{\ell-1}, j \neq i} a_{i,j}, & \text{if } i \in S_\ell \setminus S_{\ell-1} (1 \leq \ell \leq k) \\ a_{i,i} + \sum\limits_{j \in S_k} a_{i,j}, & \text{if } i \notin S_k. \end{cases} \tag{22}$$

The result now follows from using inclusion-exclusion, associating the chain $S_1 \subset S_2 \subset \cdots \subset S_k \subseteq [n]$ with the partial partition

$$P = \{S_1, S_2 \setminus S_1, S_3 \setminus S_2, \ldots, S_k \setminus S_{k-1}\},$$

and noting that there are exactly $k!$ different chains $S_1 \subset S_2 \subset \cdots \subset S_k \subseteq [n]$ that give rise to this particular partial partition (since the order of the parts in P does not matter).[6] This completes the proof of Theorem 2. $\square$

4.3 The Polytope F_A

For an $n \times n$ matrix A satisfying the conditions (a)–(c), we defined F_A in equation (17) by subtracting translated copies of the cuboid C_A from the original cuboid C_A. In this subsection, we study the polytope F_A and its 1-skeleton (i.e., its vertices and 1-dimensional edges between them; see Fig. 5), obtaining a description of it as a flip graph associated with a family of combinatorial objects.

To start, we describe the combinatorial objects that will be the vertices of the graph. An *ordered partial partition* (OPP) on $[n]$ is defined to be a relation $\preccurlyeq$ with the following properties:

  (i)   Transitive: if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.
  (ii) Weakly reflexive: if $x \preccurlyeq y$, then $x \preccurlyeq x$ and $y \preccurlyeq y$.
  (iii) Weakly connected: if $x \preccurlyeq x$ and $y \preccurlyeq y$, then either $x \preccurlyeq y$ or $y \preccurlyeq x$ or both.

The set of all ordered partial partitions on $[n]$ is denoted by $\mathrm{OPP}(n)$ and the number of them is enumerated in ref. [15].

The name "ordered partial partition" refers to the fact that these relations on $[n]$ correspond bijectively with (ordered, possibly empty) tuples of pairwise disjoint non-empty subsets of $[n]$ (i.e., partial partitions of $[n]$ in which we care about the order of the parts):

$$(X_1, X_2, \ldots, X_k),$$

where $X_i \cap X_j = \emptyset$ whenever $i \neq j$. This tuple of sets corresponds to the original relation in the following way: $x \preccurlyeq y$ if and only if there exist integers $1 \leq i \leq j$ such that $x \in X_i$ and $y \in X_j$.

For each $z \in [n]$, we define a function $f_z : \mathrm{OPP}(n) \to \mathrm{OPP}(n)$ as follows:

  (1)   If $X_1 = \{z\}$ then discard X_1, obtaining the tuple $(X_2, \ldots, X_k)$.
  (2)   If $X_j = \{z\}$ for some $j \geq 2$, then merge it into the immediately preceding part, obtaining the tuple

$$(X_1, \ldots, X_{j-1} \cup \{z\}, X_{j+1}, \ldots, X_k).$$

---

[6] For example, this partial partition has $|P| = k$ and $\mathrm{sgn}(P) = (-1)^{|S_k|+k}$, with the $(-1)^k$ coming from the fact that k specifies which level of the inclusion-exclusion we are at, and $(-1)^{|S_k|}$ coming from the fact that $|S_k|$ specifies how many terms in the product (22) are in the top branch (with a negative sign). If $S_k = [n]$ then it is a (non-partial) partition, and the partial partition has no singletons because $|S_\ell \setminus S_{\ell-1}| \geq 2$ for all $1 \leq \ell \leq k$.

(a)

(b)
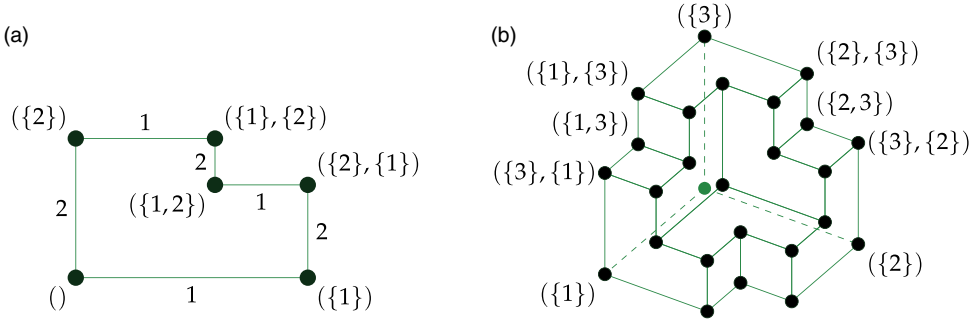
The 1-skeleton of the 2D polytope F_A from Figure 1b, or equivalently the flip graph of OPP(2). Edges are labelled by the z for which the involution f_z transforms the OPPs that it connects to each other.

The 1-skeleton of the 3D polytope F_B from Figure 4, or equivalently the flip graph of OPP(3). Not all vertices are labelled; the central vertex at the front is $(\{1, 2, 3\})$ while the central vertex at the back is $()$.

Figure 5. The 1-skeleton of a polytope is the set of its vertices and 1-dimensional edges between them. Propositions 8 and 9 show that the 1-skeleton of the polytope F_A is isomorphic to a graph whose vertices are the ordered partial partitions on $[n]$ and whose edges are described by the involutions f_z.

(3) If $z \in X_j \neq \{z\}$ for some $j \geq 1$, then split it off into its own immediately next part, obtaining the tuple

$$(X_1, \ldots, X_j \setminus \{z\}, \{z\}, X_{j+1}, \ldots, X_k).$$

(4) Otherwise, if $z \notin \bigcup_{j=1}^k X_j$, then make it into its own first part, obtaining the tuple

$$(\{z\}, X_1, \ldots, X_k).$$

For each $z \in [n]$, this function f_z is an involution on OPP(n) with no fixed points: if $f_z(\preccurlyeq) = \preccurlyeq'$, then $f_z(\preccurlyeq') = \preccurlyeq$. Moreover, if $x, y \in [n]$ with $x \neq z$, then $x \preccurlyeq y$ if and only if $x \preccurlyeq' y$.

We form a graph (called a *flip graph*) by letting its vertices be the elements of OPP(n) and its edges be between ordered partial partitions that are exchanged by an involution f_z for some $z \in [n]$. The resulting flip graph is n-regular, since each vertex gets one edge for each $z \in [n]$; we claim that this graph is isomorphic to the 1-skeleton of the polytope F_A (see Fig. 5).

Proposition 8. *Suppose that A is an $n \times n$ matrix satisfying conditions (a)–(c). Then the vertices of F_A are in bijective correspondence with the elements of* OPP(n). *In particular, the coordinates of the vertex* **v** *corresponding to the ordered partial partition $\preccurlyeq$ are given by*

$$v_i = \sum_{j \,:\, i \preccurlyeq j} a_{i,j} \quad \text{for all} \quad 1 \leq i \leq n.$$

Proof. Recall that we originally defined F_A by subtracting translated copies of the cuboid C_A:

$$F_A := C_A \setminus \overline{\bigcup_{\varnothing \neq S \subseteq [n]} \left\{ x + \sum_{i \in S} A\mathbf{e_i} : x \in C_A \right\}}. \tag{23}$$

However, C_A can itself be written as

$$C_A = \overline{P \setminus \bigcup_{i \in [n]} \left\{ x + A\mathbf{e_i} : x \in P \right\}},$$

where P is the positive orthant, consisting of all vectors where every coordinate is non-negative. As such, the definition (23) of F_A still works even if we replace each instance of C_A with P.

Given a vertex $\mathbf{v}$ of F_A, observe that it is the intersection of n facets (codimension-1 faces) $f_1, f_2, \ldots, f_n$ of F_A, where the standard basis vector $\mathbf{e_i}$ is perpendicular to the facet f_i. The facet f_i must be contained in a facet of one of the translates of P used to construct F_A; the vertex $\mathbf{v}$ is then the vertex of the intersection of these n (not necessarily distinct) translates of the positive orthant P.

Each vertex of F_A that arises therefore corresponds to the vertex of an intersection of translated positive orthants, which is the lowest (in terms of each coordinate) vertex of the corresponding intersection of translated copies of C_A. We already characterised these intersections in the proof of Lemma 7 when we performed the inclusion-exclusion calculation of the volume of F_A, identifying them with chains of subsets of $[n]$, which are in turn naturally identified with ordered partial partitions.[7] □

Proposition 9. *Let A be an $n \times n$ matrix satisfying conditions (a)–(c) and let $1 \le i \le n$. Then two vertices $\mathbf{v}$ and $\mathbf{w}$ of F_A are connected by an edge parallel to the standard basis vector $\mathbf{e_i}$ if and only if the corresponding (in the sense of Proposition 8) elements of* OPP(n) *are swapped by the involution f_i.*

Proof. Combinatorially, the polytope F_A does not depend on the choice of matrix A (besides the fact that A satisfies conditions (a)–(c)), so suppose without loss of generality that the entries are linearly independent over $\mathbb{Q}$.

Then we have that $v_k = w_k$ if and only if $\sum_{j:k\preccurlyeq j} a_{k,j} = \sum_{j:k\preccurlyeq'j} a_{k,j}$ if and only if

$$\{j : k \preccurlyeq j\} = \{j : k \preccurlyeq 'j\}. \tag{24}$$

Consequently, two vertices lie on a line parallel to the basis vector $\mathbf{e_i}$ if and only if equation (24) is true for all $k \ne i$.

This implies that the ordered partial partitions induced on $[n] \setminus \{i\}$ are equal, so the original ordered partial partitions can only differ in terms of where i is located. It also constrains the position of i relative to the other parts, namely

$$\{k \ne i : k \preccurlyeq i\} = \{k \ne i : k \preccurlyeq 'i\}. \tag{25}$$

If we remove and reinsert i, there are two possibilities: it is either introduced as its own singleton part, or it is not (either by being introduced into an existing part, or being deleted completely). In each of these two cases, the position of i is determined by equation (25). One of these cases corresponds to $\preccurlyeq'=\preccurlyeq$, while the other corresponds to $\preccurlyeq'=f_i(\preccurlyeq)$.

Given that F_A is an orthogonal polytope, it follows that there are at most n other vertices that can possibly be connected to a given vertex $\mathbf{v}$ by an edge, namely the vertices obtained by applying f_i to the corresponding element of OPP(n). Moreover, every vertex in an orthogonal polytope must have degree n, so all of these edges exist. The result follows. □

We can label the edges $\{\preccurlyeq, \preccurlyeq'\}$ of the flip graph, as in Fig. 5a, with the value of $i \in [n]$ for which $\preccurlyeq'=f_i(\preccurlyeq)$. The higher-dimensional faces of F_A then have concise descriptions in terms of this graph: the r-dimensional faces parallel to the linear subspace generated by the basis vectors $\{\mathbf{e_i} : i \in S\}$ (where $S \subseteq [n]$ has $|S| = r$) are precisely the connected components of the subgraph obtained by taking only the edges whose labels are in S.

This description of F_A by specifying its vertex coordinates is more general than the original definition in terms of subtracting translated copies of C_A: it generalises to arbitrary square matrices A, instead of requiring that A satisfy the conditions (a)–(c). Note, however, that the resulting polytope may self-intersect and thus care is required to define "volume" in such a way that it is equal to $\det(A)$.

---

[7] In the proof of Lemma 7, we ignored the partial partitions containing singletons $\{i\}$, because they give measure-zero intersections. Here, however, they do still need to be included, because they correspond to vertices on a boundary face of the cuboid C_A (specifically those vertices where $v_i = a_{i,i}$).

5. Fields of non-zero characteristic

For fields $\mathbb{F}$ of characteristic $p > 0$, many of the terms in the formula (11) are multiples of p and thus equal to zero. As a result, the formula simplifies and we get an even tighter upper bound on the tensor rank of $\det_{\mathbb{F}}^n$. In particular, we have the following variant of Theorem 2:

Theorem 10. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$ with characteristic $p > 0$. Then*

$$\det(A) = \sum_{\substack{P \in \mathrm{PP}(n) \\ |P| \leq p-1}} \mathrm{sgn}(P)|P|! \prod_{i=1}^{n} \begin{cases} \displaystyle\sum_{\substack{j \underset{P}{\sim} i, j \neq i}} a_{i,j} & \text{if } i \underset{P}{\sim} i; \\ a_{i,i} + \displaystyle\sum_{j \underset{P}{\sim} j} a_{i,j} & \text{if } i \underset{P}{\nsim} i, \end{cases} \tag{26}$$

where $\mathrm{sgn}(P) = \prod_{S \in P} (-1)^{|S|+1}$.

Proof. This formula is identical to the one provided by Theorem 2, except the partial partitions P that we sum over here are restricted to have $|P| \leq p - 1$. Since any term with $|P| \geq p$ has a coefficient of $|P|!$, which is a multiple of p, which equals 0 in $\mathbb{F}$, this restriction on $|P|$ does not change the value of the sum. □

In order to count the number of non-zero terms in the sum (26), and thus obtain a tighter an upper bound on $\mathrm{Trank}(\det_{\mathbb{F}}^n)$ when $\mathbb{F}$ has characteristic $p > 0$, we need to introduce another combinatorial quantity. Let $B_{n,k}$ be the number of partial partitions of $[n]$ that contain exactly k parts and no singletons, or equivalently the number of partitions of $[n]$ that contain exactly k parts all of which have size strictly greater than 1. The sum in equation (26) contains exactly $\sum_{k=0}^{p-1} B_{n,k}$ (potentially) non-zero terms, so we immediately get the following corollary:

Corollary 11. *Let $\mathbb{F}$ be a field with characteristic $p > 0$. Then* $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq \displaystyle\sum_{k=0}^{p-1} B_{n,k}$.

By making use of Lemma 1, we could also say that if $\mathbb{F}$ has characteristic $p > n$ then $\mathrm{Wrank}(\det_{\mathbb{F}}^n) \leq 2^{n-1} \sum_{k=0}^{p-1} B_{n,k}$. However, this provides no improvement over Corollary 3, since $\sum_{k=0}^{p-1} B_{n,k} = B_n$ whenever $p > n$ (in fact, whenever $p > \lfloor n/2 \rfloor + 1$, since no partial partition can contain more than $\lfloor n/2 \rfloor$ parts unless it has some singletons).

Numerous properties and formulas for $B_{n,k}$ are known (see [5, 12] and the references therein). We summarise those that are most relevant for us here:

(i) $B_{n,0} = 1$.

(ii) $B_{n,1} = \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1$, since $\binom{n}{k}$ counts the number of partial partitions of $[n]$ with one part of size k. It follows that if $\mathbb{F}$ has characteristic 2 then $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq B_{n,0} + B_{n,1} = 2^n - n$ (we return to this special case in more detail in Section 5.1).

(iii) The recurrence relation $B_{n,k} = (k+1)B_{n-1,k} + (n-1)B_{n-2,k-1}$ holds whenever $n \geq 2k \geq 2$. When combined with the facts that $B_{n,0} = 1$ for all $n \geq 0$ and $B_{n,k} = 0$ whenever $n < 2k$, this recurrence relation can be used to compute $B_{n,k}$ for all n and k.

(iv) In particular, for small values of k we have the following additional explicit formulas:

$$B_{n,2} = \frac{1}{2}\left(3^n - (n+2)2^n + (n^2 + n + 1)\right),$$

$$B_{n,3} = \frac{1}{6}\left(4^n - (n+3)3^n + 3(n^2 + 3n + 4)2^{n-2} - (n^3 + 2n + 1)\right), \quad \text{and}$$

$$B_{n,4} = \frac{1}{24}\left(5^n - (n+4)4^n + 2(n^2 + 5n + 9)3^{n-1}\right.$$
$$\left. - (n^3 + 3n^2 + 8n + 8)2^{n-1} + (n^4 - 2n^3 + 5n^2 + 1)\right).$$

(v) $B_{n,k} \leq (k+1)^n/k!$ (furthermore, for fixed k we have $B_{n,k} \sim (k+1)^n/k!$, so this inequality is not too lossy). To verify this inequality, consider the following function f from the set of $P \in \mathrm{OPP}(n)$[8] in which there are exactly k parts and no singletons to the set $\{0, 1, 2, \ldots, k\}^n$:

$$f(P) := (f_1(P), f_2(P), \ldots, f_n(P)), \quad \text{where}$$

$$f_i(P) = \begin{cases} 0 & \text{if } i \underset{P}{\not\sim} i, \\ j & \text{if } i \text{ is in the } j-\text{th part of } P. \end{cases}$$

Since $f(P)$ completely specifies the ordered partial partition P, f is injective, so there are no more than $|\{0, 1, 2, \ldots, k\}^n| = (k+1)^n$ different $P \in \mathrm{OPP}(n)$ with exactly k parts and no singletons. If we then forget about the ordering of those k parts, we see that there are no more than $(k+1)^n/k!$ members of $\mathrm{PP}(n)$ with exactly k parts and no singletons, so $B_{n,k} \leq (k+1)^n/k!$.

In summary, all of the above observations and formulas, when combined with Corollary 11, lead to the following (slightly weaker when $p \geq 7$, but much easier to evaluate and work with) corollary:

Corollary 12. *Let $\mathbb{F}$ be a field with characteristic $p > 0$.*

(a)   *If $p = 2$ then $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq 2^n - n$.*

(b)   *If $p = 3$ then $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq \dfrac{1}{2}(3^n - n2^n + (n^2 - n + 1))$.*

(c)   *If $p = 5$ then $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq \dfrac{1}{24}\big(5^n - n4^n + 2(n^2 - n + 9)3^{n-1} - (n^3 - 3n^2 + 14n$*
$-16)2^{n-1} + (n^4 - 6n^3 + 17n^2 - 20n + 9)\big).$

(d)   *In general, $\mathrm{Trank}(\det_{\mathbb{F}}^n) \leq \displaystyle\sum_{k=1}^{p} \dfrac{k^n}{(k-1)!}$. In particular, $\mathrm{Trank}(\det_{\mathbb{F}}^n) \in O(p^n)$.*

5.1 Rank of the permanent tensor

Glynn's formula (3) shows that the tensor rank of the permanent tensor is at most 2^{n-1}, as long as the field does not have characteristic 2. When the characteristic is 2, the permanent and determinant tensors are the same, and the best-known upper bound on their rank is now $2^n - n$, as described by Corollary 12 (surpassing the $2^n - 1$ that comes from Ryser's formula (2)). The following corollary clarifies slightly what this $2^n - n$ term formula for the determinant and permanent looks like in characteristic 2. We note that the "$\ominus$" in the following corollary is the symmetric difference operation on sets, and $\mathrm{per}_{\mathbb{F}}^n$ refers to the determinant tensor $\mathrm{per}_{\mathbb{F}}^n = \sum_{\sigma \in S_n} \mathbf{e}_{\sigma(1)} \otimes \mathbf{e}_{\sigma(2)} \otimes \cdots \otimes \mathbf{e}_{\sigma(n)}$:

Corollary 13. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$ with characteristic 2. Then*

$$\mathrm{per}(A) = \det(A) = \sum_{S \subseteq [n]} \left(\prod_{i=1}^{n} \sum_{j \in S \ominus \{i\}} a_{i,j} \right). \tag{27}$$

In particular, whenever S is a singleton the inner sum is empty, so $\mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n) = \mathrm{Trank}(\det_{\mathbb{F}}^n) \leq 2^n - n$.

---

[8] Recall that $\mathrm{OPP}(n)$ is the set of ordered partial partitions, defined in Section 4.3.

The above corollary comes directly from plugging $p = 2$ into Theorem 10 and discarding all terms coming from partial partitions that have two or more parts. The partial partitions with just zero or one part are in bijection with the subsets S of $[n]$, giving the result.

For $n \leq 3$, the formula provided by Corollary 12 is the same as the field-independent formula that we have seen already (except with signs ignored since $-1 = 1$ in characteristic 2). The first non-trivial case comes when $n = 4$. In this case, the resulting formula has 12 terms; it is the same as the formula displayed in equation (12), but without the final 3 terms (i.e., the terms with a coefficient of 2).

6. Tensor rank lower bounds

All of the results that we have presented so far bound $\mathrm{Trank}(\det_{\mathbb{F}}^n)$ and $\mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n)$ from above. In the other direction, the best-known lower bounds on $\mathrm{Trank}(\det_{\mathbb{F}}^n)$ and $\mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n)$ when $n \geq 4$ are due to Derksen [4], who showed that

$$\mathrm{Trank}(\det_{\mathbb{F}}^n) \geq \binom{n}{\lfloor n/2 \rfloor} \quad \text{and} \quad \mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n) \geq \binom{n}{\lfloor n/2 \rfloor} \tag{28}$$

(he only stated these bounds for $\mathbb{F} = \mathbb{C}$, but his proof works over any field).[9]

The known upper bounds show that this lower bound on the permanent cannot be improved by more than a factor of $\Theta(\sqrt{n})$ over any field. In particular, if $\mathbb{F}$ has characteristic not equal to 2 then Glynn's formula (3) implies

$$2^n \sqrt{\frac{2}{\pi n}} \sim \binom{n}{\lfloor n/2 \rfloor} \leq \mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n) \leq 2^{n-1}.$$

Similarly, Corollary 13 tells us that if $\mathbb{F}$ has characteristic 2 then we have

$$2^n \sqrt{\frac{2}{\pi n}} \sim \binom{n}{\lfloor n/2 \rfloor} \leq \mathrm{Trank}(\det_{\mathbb{F}}^n) = \mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n) \leq 2^n - n.$$

We can, however, make some small improvements. In particular, we refine Derksen's analysis slightly to improve these lower bounds by 1:

Theorem 14. *Let $\mathbb{F}$ be any field. Then* $\mathrm{Trank}(\det_{\mathbb{F}}^n) \geq \binom{n}{\lfloor n/2 \rfloor} + 1$ *and* $\mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n) \geq \binom{n}{\lfloor n/2 \rfloor} + 1$.

Proof. We begin by recalling the definition of the antisymmetric subspace $\mathcal{A}$ of $(\mathbb{F}^n)^{\otimes s}$:[10]

$$\mathcal{A} := \mathrm{span}\left\{ \sum_{\sigma \in S_s} \mathrm{sgn}(\sigma) \mathbf{e}_{i_{\sigma(1)}} \otimes \mathbf{e}_{i_{\sigma(2)}} \otimes \cdots \otimes \mathbf{e}_{i_{\sigma(s)}} : 1 \leq i_1 < i_2 < \cdots < i_s \leq n \right\}. \tag{29}$$

We just prove the inequality in the statement of the theorem that involves $\mathrm{Trank}(\det_{\mathbb{F}}^n)$; the bound on $\mathrm{Trank}(\mathrm{per}_{\mathbb{F}}^n)$ follows similarly by just ignoring signs throughout the proof and omitting $\mathrm{sgn}(\sigma)$ from the definition (29) of the 'antisymmetric subspace' $\mathcal{A}$.[11]

---

[9] There are some better lower bounds when $n \in \{5, 7\}$ in ref. [10].

[10] There are other definitions of the antisymmetric subspace (see [[8], Section 3.1.3], for example) that are equivalent over fields of characteristic not equal to 2, but in characteristic 2 it is important that we choose this definition in this proof.

[11] For example, if $n = s = 2$ then this means that $\mathcal{A} = \mathrm{span}\{\mathbf{e_1} \otimes \mathbf{e_2} + \mathbf{e_2} \otimes \mathbf{e_1}\}$; it does *not* mean that $\mathcal{A}$ is the symmetric subspace (it does not contain $\mathbf{e_1} \otimes \mathbf{e_1}$ or $\mathbf{e_2} \otimes \mathbf{e_2}$).

It is well-known that if T is any matrix flattening of $\det_{\mathbb{F}}^n$ then $\mathrm{Trank}(\det_{\mathbb{F}}^n) \geq \mathrm{rank}(T)$. We choose T to be the flattening obtained by partitioning $(\mathbb{F}^n)^{\otimes n}$ as $(\mathbb{F}^n)^{\otimes \lfloor n/2 \rfloor} \otimes (\mathbb{F}^n)^{\otimes \lceil n/2 \rceil}$, which we think of as an $n^{\lfloor n/2 \rfloor} \times n^{\lceil n/2 \rceil}$ matrix. The columns of this matrix T span the antisymmetric subspace $\mathcal{A}$ of $(\mathbb{F}^n)^{\otimes \lfloor n/2 \rfloor}$, which has dimension $\binom{n}{\lfloor n/2 \rfloor}$, so

$$\mathrm{Trank}(\det_{\mathbb{F}}^n) \geq \mathrm{rank}(T) = \binom{n}{\lfloor n/2 \rfloor}, \tag{30}$$

recovering Derksen's lower bound.

To show that Inequality (30) is actually strict (and thus complete the proof), note that for any tensor decomposition of the form

$$\det_{\mathbb{F}}^n = \sum_{k=1}^{r} \mathbf{v_{1,k}} \otimes \mathbf{v_{2,k}} \otimes \cdots \otimes \mathbf{v_{n,k}},$$

we have

$$T = \sum_{k=1}^{r} \left(\mathbf{v_{1,k}} \otimes \cdots \otimes \mathbf{v_{\lfloor n/2 \rfloor, k}} \right) \left(\mathbf{v_{\lfloor n/2 \rfloor + 1, k}} \otimes \cdots \otimes \mathbf{v_{n,k}} \right)^T. \tag{31}$$

In particular, this implies that the range of T (i.e., the antisymmetric subspace of $(\mathbb{F}^n)^{\otimes \lfloor n/2 \rfloor}$) is contained in the span of the r elementary tensors $\mathbf{v_{1,k}} \otimes \cdots \otimes \mathbf{v_{\lfloor n/2 \rfloor, k}}$ ($1 \leq k \leq r$). If $r \leq \binom{n}{\lfloor n/2 \rfloor}$ (the dimension of the antisymmetric subspace) then this implies that each of these elementary tensors are in the antisymmetric subspace, contradicting the fact that *no* non-zero elementary tensors are in the antisymmetric subspace.[12] We thus conclude that $r > \binom{n}{\lfloor n/2 \rfloor}$, which completes the proof. □

6.1 Better lower bounds over finite fields

In the case when $\mathbb{F} = \mathbb{F}_q$ is the field with q elements, we can refine the argument of Theorem 14 even further to get even better lower bounds on $\mathrm{Trank}(\det_{\mathbb{F}_q}^n)$. In particular, we have the following:

Theorem 15. *Let $q \geq 2$ be a prime power, let $n \geq 5$, and let x be the (unique) positive real solution to the equation $\log_q (x+1) = x - \binom{n}{\lfloor n/2 \rfloor}$. Then*

$$\mathrm{Trank}(\det_{\mathbb{F}_q}^n) \geq \lceil x \rceil.$$

In particular, $\mathrm{Trank}(\det_{\mathbb{F}_q}^n) \geq \binom{n}{\lfloor n/2 \rfloor} + \log_q \left(\binom{n}{\lfloor n/2 \rfloor} \right)$.

Before we can prove this, we begin by establishing a useful lemma about the tensor rank of members of the antisymmetric subspace:[13]

Lemma 16. *Suppose that $\mathbb{F}$ is a field, n, s are positive integers, and T is a non-zero element of the antisymmetric subspace of $(\mathbb{F}^n)^{\otimes s}$. Then $\mathrm{Trank}(T) \geq s$.*

Proof. If $s = 1$, the result is immediate, because every non-zero tensor has positive tensor rank. We shall henceforth assume that $s \geq 2$.

We view the tensor T as a multilinear form from $(\mathbb{F}^n)^s$ to $\mathbb{F}$. By assumption that it is non-zero, there must exist vectors $\mathbf{x_1}, \ldots, \mathbf{x_s} \in \mathbb{F}^n$ such that $T(\mathbf{x_1}, \ldots, \mathbf{x_s})$ is non-zero. By multiplying one of these vectors by an appropriate scalar, we can assume without loss of generality that $T(\mathbf{x_1}, \ldots, \mathbf{x_s}) = 1$.

---

[12] In the case of the permanent, we can see that $\mathcal{A}$ contains no non-zero elementary tensors $\mathbf{v} \otimes \cdots \otimes \mathbf{v}$ by noting that such a tensor has a non-zero "diagonal" entry (i.e., there exists some i for which $(\mathbf{e_i} \otimes \cdots \otimes \mathbf{e_i})^*(\mathbf{v} \otimes \cdots \otimes \mathbf{v}) \neq 0$), but every member of the antisymmetric subspace $\mathcal{A}$ has all diagonal entries equal to 0.

[13] See footnote [10].

Note that $\{\mathbf{x_1}, \dots, \mathbf{x_s}\}$ must be linearly independent. Otherwise, we could write one of the vectors as a linear combination of the others, which we will assume without loss of generality is $\mathbf{x_1}$. That is, $\mathbf{x_1} = \alpha_2 \mathbf{x_2} + \cdots + \alpha_s \mathbf{x_s}$, and then we would have the following by linearity in the first argument:

$$T(\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_s}) = \sum_{i=2}^{s} \alpha_i T(\mathbf{x_i}, \mathbf{x_2}, \dots, \mathbf{x_s}).$$

By antisymmetry, each term on the right-hand side vanishes, and the left-hand side equals 1, so we obtain a contradiction. As such, it follows that $\{\mathbf{x_1}, \dots, \mathbf{x_s}\}$ is indeed linearly independent.

For a fixed $1 \leq i \leq s$, we define a covector $\mathbf{y_i} : \mathbb{F}^n \to \mathbb{F}$ by contracting T on its first $s-1$ arguments with the elements of $\{\mathbf{x_1}, \dots, \mathbf{x_s}\} \setminus \{\mathbf{x_i}\}$ and applying an appropriate sign change:

$$\mathbf{y_i}(\mathbf{v}) := (-1)^{s-i} T(\mathbf{x_1}, \dots, \mathbf{x_{i-1}}, \mathbf{x_{i+1}}, \dots, \mathbf{x_s}, \mathbf{v})$$

Observe that the antisymmetry properties of T imply the following:

$$\mathbf{y_i}(\mathbf{x_j}) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Equivalently, if we consider the space V spanned by $\mathbf{x_1}, \dots, \mathbf{x_s}$, then the covectors $\mathbf{y_1}, \dots, \mathbf{y_s}$ form a basis for $V^\star$, specifically the dual basis of $\mathbf{x_1}, \dots, \mathbf{x_s}$.

Consequently, the matrix obtained from T by the flattening $(\mathbb{F}^n)^{\otimes s} = (\mathbb{F}^n)^{\otimes(s-1)} \otimes (\mathbb{F}^n)$ has tensor rank at least s, and thus $\mathrm{Trank}(T) \geq s$. $\qquad\square$

Proof of Theorem 15. We showed in the proof of Theorem 14 that if $1 \leq s \leq n$ and

$$\det_{\mathbb{F}_q}^n = \sum_{k=1}^{r} \mathbf{v_{1,k}} \otimes \mathbf{v_{2,k}} \otimes \cdots \otimes \mathbf{v_{n,k}}$$

then the antisymmetric subspace of $(\mathbb{F}_q^n)^{\otimes s}$ must be contained in the span of the r elementary tensors $\mathbf{v_{1,k}} \otimes \cdots \otimes \mathbf{v_{s,k}}$ $(1 \leq k \leq r)$. We will use this fact with $s = \lceil n/2 \rceil$ (in the proof of Theorem 14 we instead used $s = \lfloor n/2 \rfloor$).

Let $P : (\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil} \to (\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil} / \mathcal{A}$ be a projection onto the quotient of the antisymmetric subspace $\mathcal{A}$ inside of $(\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil}$. If the antisymmetric subspace of $(\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil}$ is contained in the span of r elementary tensors $\mathbf{v_{1,j}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, j}}$ $(1 \leq j \leq r)$, then the span of the set

$$B := \left\{ P(\mathbf{v_{1,j}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, j}}) : 1 \leq j \leq r \right\} \tag{32}$$

must have dimension at most $r - \binom{n}{\lfloor n/2 \rfloor}$. Since we are working over $\mathbb{F}_q$, there are thus at most

$$q^{r - \binom{n}{\lfloor n/2 \rfloor}} - 1$$

non-zero vectors in B.

Next, we note that there are exactly r members in B (i.e., the vectors $P(\mathbf{v_{1,i}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, i}})$ and $P(\mathbf{v_{1,j}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, j}})$ are distinct whenever $i \neq j$). To see why this is the case, we apply Lemma 16: given any tensor T in the antisymmetric subspace of $(\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil}$, we have $\mathrm{Trank}(T) \geq \lceil n/2 \rceil \geq 3$, which implies that $P(\mathbf{v_{1,i}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, i}} - \mathbf{v_{1,j}} \otimes \cdots \otimes \mathbf{v_{\lceil n/2 \rceil, j}}) = \mathbf{0}$ never occurs.

It follows that the set B consists of exactly r non-zero vectors, but also no more than $q^{r - \binom{n}{\lfloor n/2 \rfloor}} - 1$ non-zero vectors, so we must have $r \leq q^{r - \binom{n}{\lfloor n/2 \rfloor}} - 1$. Rearranging shows that $\log_q (r+1) \leq r - \binom{n}{\lfloor n/2 \rfloor}$. Since the function

$$f(x) := \log_q (x+1) - x + \binom{n}{\lfloor n/2 \rfloor}$$

is monotonically decreasing on $(0, \infty)$, has $f(1) > 0$, and $\lim_{x \to \infty} f(x) = -\infty$, we conclude that there is exactly one positive $\tilde{x} \in \mathbb{R}$ for which $f(\tilde{x}) = 0$. The least integer $r \geq 1$ for which $\log_q (r + 1) \leq r - \binom{n}{\lfloor n/2 \rfloor}$ is $r = \lceil \tilde{x} \rceil$. The "in particular" statement of the theorem comes from the fact that $f(x) > 0$ when $x = \binom{n}{\lfloor n/2 \rfloor} + \log_q \left(\binom{n}{\lfloor n/2 \rfloor} \right)$ too. $\qquad \square$

When $n = 4$, if we apply the same reasoning as in the proof of Theorem 15 then we have to be slightly careful, since in this case $\lceil n/2 \rceil = 2$ so there are antisymmetric tensors in $(\mathbb{F}_q^n)^{\otimes \lceil n/2 \rceil}$ with rank 2, leading to fewer than r members of the set B from equation (32). Instead of the inequality $r \leq q^{r - \binom{n}{\lfloor n/2 \rfloor}} - 1$, we obtain the weaker inequality $r/2 \leq q^{r - \binom{n}{\lfloor n/2 \rfloor}} - 1$, which leads to the bound

$$\text{Trank}(\det_{\mathbb{F}_q}^4) \geq \begin{cases} 9 & \text{if } q = 2; \\ 8 & \text{if } q \in \{3, 4\}; \\ 7 & \text{if } q \geq 5 \text{is any other prime power.} \end{cases} \qquad (33)$$

We can actually prove a much stronger lower bound in the $n = 4$, $q = 2$ case, which matches our upper bound from Theorem 10. In particular, we have the following theorem, whose proof is computer-assisted and described in the next subsection:

Theorem 17. *Over the field of two elements, the tensor rank of the determinant and permanent of a 4×4 matrix is exactly 12:*

$$\text{Trank}(\det_{\mathbb{F}_2}^4) = \text{Trank}(\text{per}_{\mathbb{F}_2}^4) = 12.$$

6.2 Proof that the tensor rank of $\det_4^{\mathbb{F}_2}$ is 12

The formula in Theorem 10 shows that the tensor rank is at most 12, and we have already shown a lower bound of 9. It suffices, therefore, to show that it is impossible to express the determinant tensor as the sum of $r \in \{9, 10, 11\}$ rank-1 tensors.

Suppose otherwise. By considering the flattening $(\mathbb{F}_2^4)^{\otimes 4} = (\mathbb{F}_2^4)^{\otimes 2} \otimes (\mathbb{F}_2^4)^{\otimes 2}$, we can write

$$\det_4^{\mathbb{F}_2} = \sum_{i=1}^{r} A_i \otimes B_i$$

where each A_i and B_i is a 4×4 rank-1 matrix. Recall that the span of $\{A_i : 1 \leq i \leq r\}$ must contain the (6-dimensional) antisymmetric subspace of $(\mathbb{F}_2^4)^{\otimes 2}$.

Without loss of generality, we can assume that $A_1 \leq A_2 \leq \cdots \leq A_r$, where $\leq$ denotes the lexicographical order on the space of 4×4 matrices over $\mathbb{F}_2$. Moreover, given any such tensor decomposition of $\det_4^{\mathbb{F}_2}$ and an invertible matrix L, observe that the determinant is unaffected by the change of basis specified by L, so we also have

$$\det_4^{\mathbb{F}_2} = \sum_{i=1}^{r} L A_i L^T \otimes L B_i L^T.$$

By applying a suitable change of basis L, we can without loss of generality assume that at least one of the matrices – necessarily the lexicographically first matrix, and therefore A_1 by definition – has a single 1 in either the last or penultimate entry of the matrix, and zeroes in all other entries:

$$A_1 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0 \end{bmatrix}$$

We performed a backtracking search[14] for candidate r-tuples $(A_1, \ldots, A_r)$ of rank-1 matrices subject to these constraints. If the intersection of the 6-dimensional antisymmetric subspace and the span of the initial segment $(A_1, \ldots, A_s)$ has rank less than $6 - r + s$, then it is impossible for the span of $(A_1, \ldots, A_r)$ to contain the antisymmetric subspace, and therefore we can eliminate that entire branch of the search tree.

The largest search ($r = 11$) consumed 357 core-hours of CPU time, or 4 hours of wall-clock time when parallelised on an AWS r5a.24xlarge instance. The total CPU time grows rapidly as a function of r:

r	CPU time
9	7 seconds
10	22 minutes
11	357 hours

It transpires that, when $r \geq 9$, there do exist r-tuples of rank-1 matrices $A_1, \ldots, A_r$ which contain the antisymmetric subspace in their linear span, so additional ideas are necessary to eliminate these candidates.

Lemma 18. *Suppose that* $\det_4^{\mathbb{F}_2} = \sum_{i=1}^r A_i \otimes B_i$ *and r is minimal. Let $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^4$ be a pair of (not necessarily distinct) non-zero vectors. Then the size of the set of indices $\{i : \mathbf{u}^T A_i \mathbf{v} = 1\}$ is not equal to 1.*

Proof. Suppose otherwise, namely that there is a unique index i for which $\mathbf{u}^T A_i \mathbf{v} = 1$.

If we contract both sides of the equation $\det_4^{\mathbb{F}_2} = \sum_{i=1}^r A_i \otimes B_i$ with $\mathbf{u}$ and $\mathbf{v}$ on the first tensor factor, all but the i-th term of the sum will vanish and we obtain (in index notation)

$$u^j v^k \det_4^{j,k,\ell,m} = (B_i)^{\ell,m}.$$

If $\mathbf{u} = \mathbf{v}$, then the left-hand-side is the all-zeroes matrix, and therefore so is B_i. But that means that we have a tensor decomposition of rank $r - 1$, contradicting minimality. Otherwise, $\mathbf{u} \neq \mathbf{v}$ and we can apply a suitable change of basis so that, without loss of generality, $\mathbf{u} = \mathbf{e_1}$ and $\mathbf{v} = \mathbf{e_2}$. Then the left-hand-side of the equation is the rank-2 matrix $\mathbf{e_3} \otimes \mathbf{e_4} + \mathbf{e_4} \otimes \mathbf{e_3}$, but the right-hand-side has rank 1, again obtaining a contradiction. $\square$

This lemma eliminates all candidate 9-tuples and 10-tuples, establishing a tensor rank lower bound of 11, and leaves a single candidate 11-tuple up to change of basis and up to transposing the matrices A_i:

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[14] Source code is available at: https://gitlab.com/apgoucher/det4f2

$$A_1 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 1\ 0\ 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 1\ 0\ 1 \\ 0\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0\ 0\ 0\ 0 \\ 0\ 1\ \mathbf{1}\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 1\ 1\ 0 \end{bmatrix}, A_6 = \begin{bmatrix} 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0 \end{bmatrix}, A_7 = \begin{bmatrix} 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix}, A_8 = \begin{bmatrix} 1\ 0\ \mathbf{1}\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 1\ 0\ 1\ 0 \end{bmatrix}, \quad (34)$$

$$A_9 = \begin{bmatrix} 1\ 1\ 0\ 0 \\ 1\ 1\ 0\ 0 \\ 1\ 1\ 0\ 0 \\ 1\ 1\ 0\ 0 \end{bmatrix}, A_{10} = \begin{bmatrix} 1\ 1\ 0\ 1 \\ 1\ 1\ 0\ 1 \\ 1\ 1\ 0\ 1 \\ 0\ 0\ 0\ 0 \end{bmatrix}, \quad \text{and} \quad A_{11} = \begin{bmatrix} 1\ 1\ \mathbf{1}\ 0 \\ 1\ 1\ \mathbf{1}\ 0 \\ 0\ 0\ 0\ 0 \\ 1\ 1\ 1\ 0 \end{bmatrix}.$$

The only matrices in this set to have a non-zero $(1, 3)$ entry are A_8 and A_{11}; the only matrices to have a non-zero $(2, 3)$ entry are A_5 and A_{11}. By contracting both sides of the equation $\det_4^{\mathbb{F}_2} = \sum_{i=1}^r A_i \otimes B_i$ with $\mathbf{e_1} \otimes \mathbf{e_3}$, we get $B_8 + B_{11} = \mathbf{e_2} \otimes \mathbf{e_4} + \mathbf{e_4} \otimes \mathbf{e_2}$. Similarly, by contracting both sides of that equation by $\mathbf{e_2} \otimes \mathbf{e_3}$, we get $B_5 + B_{11} = \mathbf{e_1} \otimes \mathbf{e_4} + \mathbf{e_4} \otimes \mathbf{e_1}$.

There are only six different ways to write each of these rank-2 matrices as the sum of two rank-1 matrices. In particular, we must have

$$B_8, B_{11} \in \{\mathbf{e_2} \otimes \mathbf{e_4}, \mathbf{e_4} \otimes \mathbf{e_2}, \mathbf{e_2} \otimes (\mathbf{e_2} + \mathbf{e_4}), (\mathbf{e_2} + \mathbf{e_4}) \otimes \mathbf{e_2}, (\mathbf{e_2} + \mathbf{e_4}) \otimes \mathbf{e_4}, \mathbf{e_4} \otimes (\mathbf{e_2} + \mathbf{e_4})\}$$

and

$$B_5, B_{11} \in \{\mathbf{e_1} \otimes \mathbf{e_4}, \mathbf{e_4} \otimes \mathbf{e_1}, \mathbf{e_1} \otimes (\mathbf{e_1} + \mathbf{e_4}), (\mathbf{e_1} + \mathbf{e_4}) \otimes \mathbf{e_1}, (\mathbf{e_1} + \mathbf{e_4}) \otimes \mathbf{e_4}, \mathbf{e_4} \otimes (\mathbf{e_1} + \mathbf{e_4})\}.$$

Observe that these two sets are disjoint, so B_{11} cannot be in both of them. This is a contradiction that eliminates this single candidate solution for $r = 11$, completing the proof of Theorem 17.

Acknowledgements

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