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ALGEBRAS INTERTWINING NORMAL AND DECOMPOSABLE OPERATORS

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Introduction. The celebrated result of Lomonosov [**6**] on the existence of invariant subspaces for operators commuting with a compact operator have been generalized in different directions (for example see [**2**], [**7**], [**8**], [**9**]). The main result of [**9**] (see also [**7**]) is: If $\mathfrak{A}$ is a norm closed algebra of (bounded) operators on an infinite dimensional (complex) Banach space $\mathfrak{X}$, if K is a non-zero compact operator on $\mathfrak{X}$, and if $\mathfrak{A}K \subseteq K\mathfrak{A}$, then $\mathfrak{A}$ has a non-trivial (closed) invariant subspace. In [**7**], it is mentioned that the above result holds if instead of compactness for K we assume that K is a non-invertible injective operator with a non-zero eigenvalue belonging to the class of decomposable, hyponormal, or subspectral operators.

Heydar Radjavi (in a private conversation) asked: Can we get the above results if we omit some or all of the conditions (1) "non-invertibility", (2) "injectivity", and (3) "existence of a non-zero eigenvalue" for K? If not in general, can we get it for "good" operators K, say normal operators?

In this paper we will study this question for normal and decomposable operators. We will show that for these operators the condition (3) can be replaced by a much weaker condition, namely, $\sigma(K) \supsetneq \{0\}$, and that the conditions (1) and (2) can be relaxed for some cases of interest. As a result, we will obtain norms of normal spatial automorphisms of (topologically) transitive algebras of operators.

1. Preliminaries. Throughout $\mathfrak{H}$ and $\mathfrak{X}$ will denote a complex Hilbert and Banach space respectively. The symbols $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{X})$ will be used for the algebra of all bounded linear operators on $\mathfrak{H}$ and $\mathfrak{X}$ respectively. If $T \in \mathfrak{B}(\mathfrak{X})$, the spectrum and spectral radius of T will be denoted by $\sigma(T)$ and $r_\sigma(T)$ respectively. By a subspace we always mean a closed linear manifold. If $\{\mathfrak{M}_\lambda\}_{\lambda\in\Lambda}$ is a family of linear manifolds in $\mathfrak{X}$, then the subspace generated by $\{\mathfrak{M}_\lambda\}_{\lambda\in\Lambda}$ will be denoted by $\bigvee_{\lambda\in\Lambda}\mathfrak{M}_\lambda$. The dual space of $\mathfrak{X}$ will be denoted by $\mathfrak{X}^*$ and if $\mathfrak{M} \subset \mathfrak{X}$, then

$$\mathfrak{M}^\perp = \{x^* \in \mathfrak{X}^*: \langle x, x^* \rangle = 0 \quad \forall x \in \mathfrak{M}\}.$$

If $S \subset \mathbf{C}$, then S^0 will denote the interior of S.

An operator $T \in \mathfrak{B}(\mathfrak{X})$ has the *single-valued extension property* if whenever $\Omega \subset \mathbf{C}$ is open and $f: \Omega \rightarrow \mathfrak{X}$ is an analytic function such that $(\lambda - T)f(\lambda) \equiv 0$

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on Ω, then we have $f \equiv 0$ on Ω. (See [**1**] and [**3**].) Let $x \in \mathfrak{X}$ and $T \in \mathfrak{B}(\mathfrak{X})$ have the single-valued extension property, then the *local resolvent of T at x, $\rho_T(x)$*, is defined by

$$\rho_T(x) = \{\lambda \in \mathbf{C}\text{: there exists an analytic } \mathfrak{X} \text{ valued function } u \text{ defined on a neighborhood of } \lambda \text{ satisfying } (\lambda - T)u(\lambda) \equiv x.\}$$

It is clear that there exists a unique analytic function $x(\lambda)$ defined on $\rho_T(x)$ satisfying $(\lambda - T)x(\lambda) \equiv x$ on $\rho_T(x)$. The *local spectrum of T* at x, $\sigma_T(x)$, is defined to be $\mathbf{C}\backslash\rho_T(x)$. If F is a subset of $\mathbf{C}$, define

$$\mathfrak{X}_T(F) = \{x \in \mathfrak{X}\text{: } \sigma_T(x) \subset F\}.$$

It is easy to see that $\mathfrak{X}_T(F)$ is an invariant linear manifold for T.

The definition of decomposable operator will not be given here (see [**1**]). Let $T \in \mathfrak{B}(\mathfrak{X})$ be decomposable, then it is well known that T has the single-valued extension property and that $\mathfrak{X}_T(F)$ is closed for every closed set F ([**1**]).

2. Results. Algebras considered will be assumed to contain the identity, although this is not at all essential; the trivial modification necessary for the general case will be obvious to the reader.

We acknowledge that our Theorem 1 and Corollary 3 are strongly inspired from the work of C. Foias [**4**].

THEOREM 1. *Let $\mathfrak{A}$ be a uniformly closed subalgebra of* $\mathfrak{B}(\mathfrak{X})$, *and suppose that* $\mathfrak{A}K \subseteq K\mathfrak{A}$, *for some injective decomposable operator K with $\sigma(K) \supsetneqq \{0\}$. For $\alpha > 0$, let $F_\alpha = \{\lambda \in \mathbf{C}\text{: } |\lambda| \geqq \alpha\}$. Then for every $0 < \alpha < r_\sigma(K)$ the subspace $\bigvee_{T\in\mathfrak{A}} T\mathfrak{X}_k(F_\alpha)$ is non-trivial and invariant under $\mathfrak{A}$.*

Proof. First we will note that for every $\alpha > 0$, $\bigvee_{T\in\mathfrak{A}} T\mathfrak{X}_K(F_\alpha)$ is an invariant subspace for $\mathfrak{A}$, but it may be $\{0\}$ or $\mathfrak{X}$. We will show that there exists a constant $c \geqq 1$ such that for every $\alpha > 0$ we have:

$$(1) \qquad \bigvee_{T\in\mathfrak{A}} T\mathfrak{X}_K(F_\alpha) \subset \mathfrak{X}_K(F_{\alpha/c}).$$

Suppose this is proved, then the proof of the Theorem can be completed as follows: Let $0 < \alpha < r_\sigma(K)$. Then, since K is decomposable, $\{0\} \subsetneqq \sigma(K)$, and $c \geqq 1$ we have

$$(2) \qquad \mathfrak{X}_K(F_{\alpha/c}) \subsetneqq \mathfrak{X} \quad \text{and}$$

$$(3) \qquad \{0\} \subsetneqq \mathfrak{X}_K(F_\alpha).$$

Now since $I \in \mathfrak{A}$, in view of (1), (2) and (3) the subspace $\bigvee_{T\in\mathfrak{A}} T\mathfrak{X}_K(F_\alpha)$ will be non-trivial.

So let us prove the existence of a $c \geqq 1$ for which the relation (1) is true. An application of the Closed Graph Theorem shows that the map $\psi : \mathfrak{A} \to \mathfrak{A}$ defined by

$$\psi(T) = K^{-1}TK$$

is a continuous algebra isomorphism. Now $\|\psi\| \geqq 1$, for $\psi(I) = I$. Let $c = \|\psi\|$. Since $\psi(\mathfrak{A}) \subset \mathfrak{A}$ we can iterate $TK = K\psi(T)$, $T \in \mathfrak{A}$, to get

(4) $\quad TK^n = K^n\psi^n(T)$

where ψ^n is the composition of ψ with itself n times, $n = 1, 2, \ldots$. It follows from (4) that

(5) $\quad K^{*n}T^* = [\psi^n(T)]^*K^{*n}$,

where * denotes the dual operator. Since

$$\|[\psi^n(T)]^*\| = \|\psi^n(T)\| \leqq c^n\|T\|, n = 1, 2, \ldots$$

it follows from (5) that for every $x^* \in \mathfrak{X}^*$ we have:

(6) $\quad \|K^{*n}T^*x^*\| \leqq c^n\|T\| \, \|K^{*n}x^*\|$.

Now let $x^* \in \mathfrak{X}^*_{K^*}(D_\alpha)$, where for $\alpha > 0$, $D_\alpha = \{\lambda \in \mathbf{C}: |\lambda| \leqq \alpha\}$. (Note that K^* is decomposable too [**5**].) Then $\mathbf{C}\backslash D_\alpha \subset \rho_{K^*}(x^*)$ and the unique analytic function $x^*(\lambda)$ which satisfies $(\lambda - K^*)x^*(\lambda) = x^*$ for all $\lambda \in \rho_{K^*}(x^*)$, has the power series representation

$$x^*(\lambda) = \textstyle\sum_{n=0}^{\infty} K^{*n}x^*/\lambda^{n+1},$$

which is convergent for all $|\lambda| > \alpha$. Using this and (6) it follows that the series

$$y(\lambda) = \textstyle\sum_{n=0}^{\infty} K^{*n}T^*x^*/\lambda^{n+1}$$

defines an analytic function for $|\lambda| > \alpha c$ which satisfies $(\lambda - K^*)y(\lambda) = T^*x^*$ for $|\lambda| > \alpha c$. Thus $T^*x^* \in \mathfrak{X}^*_{K^*}(D_{\alpha c})$ if $x^* \in \mathfrak{X}^*_{K^*}(D_\alpha)$ and $T \in \mathfrak{A}$, i.e.,

(7) $\quad T^*\mathfrak{X}^*_{K^*}(D_\alpha) \subset \mathfrak{X}^*_{K^*}(D_{\alpha c}),\ T \in \mathfrak{A},\ \alpha > 0.$

But this implies that

(8) $\quad T\mathfrak{X}_K(F_\alpha{}^0) \subset \overline{\mathfrak{X}_K(F_{\alpha/c}{}^0)}$.

To see this let $x \in \mathfrak{X}_K(F_\alpha{}^0)$ and $u^* \in (\mathfrak{X}_K(F_{\alpha/c}{}^0))^\perp$ be arbitrary. If E is a closed subset of $\mathbf{C}$, then in view of [**5**] we have:

$$\mathfrak{X}_{K^*}{}^*(E) = (\mathfrak{X}_K(\mathbf{C}/E))^\perp.$$

Thus $u^* \in \mathfrak{X}^*_{K^*}(D_{\alpha/c})$ and by (7) we have

(9) $\quad T^*u^* \in \mathfrak{X}^*_{K^*}(D_\alpha) = (\mathfrak{X}_K(F_\alpha{}^0))^\perp$.

Now by (9) we have

$$\langle Tx, u^* \rangle = \langle x, T^*u^* \rangle = 0,$$

which proves (8). We need to prove

(10) $\quad T\mathfrak{X}_K(F_\alpha) \subset \mathfrak{X}_K(F_{\alpha/c}),\quad \alpha > 0,\quad T \in \mathfrak{A}.$

To do so we note that $F_\alpha = \bigcap_{\beta<\alpha} F_\beta{}^0$ and using (8) we have

$$T\mathfrak{X}_K(F_\alpha) = T\mathfrak{X}_K(\bigcap_{\beta<\alpha} F_\beta{}^0) = T[\bigcap_{\beta<\alpha} \mathfrak{X}_K(F_\beta{}^0)] \subset \bigcap_{\beta<\alpha} T\mathfrak{X}_K(F_\beta{}^0)$$
$$\subset \bigcap_{\beta<\alpha} \mathfrak{X}_K(F_{\beta/c}{}^0) \subset \bigcap_{\beta<\alpha} \mathfrak{X}_K(F_{\beta/c}) = \mathfrak{X}_K(F_{\alpha/c})$$

which establishes (10). Now (1) follows from (10) immediately, and hence the proof is complete.

COROLLARY 1. *If in Theorem* 1 *we have* $\mathfrak{A}K = K\mathfrak{A}$, *for a (not necessarily injective) decomposable operator with* $\sigma(K) \supsetneqq \{0\}$, *then* $\mathfrak{A}$ *has a non-trivial invariant subspace.*

Proof. If K is not injective, then it follows from $\mathfrak{A}K = K\mathfrak{A}$ that the null-space of K will be invariant under $\mathfrak{A}$. If K is injective, then the result follows from Theorem 1.

COROLLARY 2. *Let* $\mathfrak{A}$ *be a uniformly closed subalgebra of* $\mathfrak{B}(\mathfrak{H})$, *and suppose that* $\mathfrak{A}K \subseteq K\mathfrak{A}$, *for some non-invertible and non-zero normal or scalar operator* K. *Then* $\mathfrak{A}$ *will have a non-trivial invariant subspace.*

Proof. First suppose that K is normal. Then if K is injective, the result follows from Theorem 1. If K is not injective then $\mathfrak{R}(K)$ (the range of K) will not be dense in $\mathfrak{H}$. But $\mathfrak{A}K \subseteq K\mathfrak{A}$ implies that $\mathfrak{R}(K)$ is invariant under $\mathfrak{A}$, and hence $\overline{\mathfrak{R}(K)}$ will be a non-trivial invariant subspace of $\mathfrak{A}$. If K is a scalar operator (in the sense of N. Dunford [**3**]) then $K = S^{-1}NS$ for some normal operator N and an invertible operator S. Let $\mathfrak{B} = S\mathfrak{A}S^{-1}$, then $\mathfrak{B}N \subseteq N\mathfrak{B}$ and by the first part of the proof $\mathfrak{B}$, and hence $\mathfrak{A}$, will have a non-trivial invariant subspace.

COROLLARY 3. *Let* $\mathfrak{A}$ *and* K *be as in Theorem* 1. *If* 0 *is an accumulation point of* $\sigma(K)$, *then* $\mathfrak{A}$ *has an infinite ascending chain of invariant subspaces.*

Proof. Let c be as in the proof of Theorem 1 and choose a sequence $\alpha_n \in \sigma(K)$ such that

(i) $\alpha_n \rightarrow 0$,

(ii) $|\alpha_{n+1}| < |\alpha_n|/c$, and

(iii) $\Delta_n = \sigma(K) \cap \{\lambda \in \mathbf{C}: |\alpha_{n+1}| < |\lambda| < |\alpha_n|/c\} \neq \emptyset$.

Consider the subspaces

$$\mathfrak{M}_n = \bigvee_{T\in\mathfrak{A}} T\mathfrak{X}_K(F_{|\alpha_n|}),\ n = 1, 2, \ldots$$

Since $\{|\alpha_n|\}$ is a decreasing sequence, it follows that $\{\mathfrak{M}_n\}_{n\in\mathbf{N}}$ are ascending. By Theorem 1 they are invariant under $\mathfrak{A}$. We have:

$$\mathfrak{M}_n \subsetneqq \mathfrak{M}_{n+1},\ n = 1, 2, \ldots$$

To see this we note that $\mathfrak{X}_K(\Delta_n) \neq \{0\}$ (this follows from the properties of decomposable operators) and hence

$$\mathfrak{M}_n \subset \mathfrak{X}_K(F_{|\alpha_n|/c}) \subsetneqq \mathfrak{X}_K(F_{|\alpha_{n+1}|}) \subset \mathfrak{M}_{n+1}.$$

(The last inclusion follows from the fact that $I \in \mathfrak{A}$). This finishes the proof.

Let us now consider the question: where was the condition $0 \in \sigma(K)$ used in the proof of Theorem 1? A careful checking shows that it was actually used in the derivation of the relation (2), which was in turn used in establishing that $\bigvee_{T \in \mathfrak{A}} T\mathfrak{X}_K(F_\alpha)$ is a proper subspace of $\mathfrak{X}$. Now suppose that in the statement of Theorem 1 the operator K is just an invertible decomposable operator. If α, $0 < \alpha < r_\sigma(K)$, can be chosen so that

$$[r_\sigma(K^{-1})]^{-1} = \inf \{|\lambda| : \lambda \in \sigma(K)\} < \alpha/\|\psi\|,$$

which is possible if and only if

(11) $\|\psi\| < r_\sigma(K^{-1}) \cdot r_\sigma(K),$

then again the relation (2) holds and Theorem 1 will be true. For a normal operator $K \in \mathfrak{B}(\mathfrak{H})$ the condition (11) becomes:

(12) $\|\psi\| < \|K^{-1}\| \, \|K\|.$

Note that the right hand side of the inequality (12) is the norm of the spatial automorphism of $\mathfrak{B}(\mathfrak{H})$ defined by

$$T \mapsto K^{-1}TK.$$

(The inequalities (11) and (12) somehow involve the "smallness" of $\mathfrak{A}$ and the "thickness" of the smallest annulus with center at 0 and containing $\sigma(K)$).

We will summarize the above discussion:

THEOREM 2. *Let $\mathfrak{A}$ be a uniformly closed subalgebra of $\mathfrak{B}(\mathfrak{X})$ $(\mathfrak{B}(\mathfrak{H}))$ and suppose that $\mathfrak{A}K \subseteq K\mathfrak{A}$, where K is an invertible decomposable (respectively, normal) operator for which the norm of the spatial automorphism ψ of $\mathfrak{A}$ defined by*

$$\psi : A \mapsto K^{-1}AK$$

satisfies the inequality (11) (*respectively* (12)), *then $\mathfrak{A}$ has a non-trivial invariant subspace.*

Let us call an identity containing uniformly closed subalgebra of $\mathfrak{B}(\mathfrak{X})$ (topologically) *transitive* if it has no non-trivial invariant subspace. As an immediate corollary of Theorem 2 we obtain the following result.

COROLLARY. *Let $\mathfrak{A}$ be transitive and K be an invertible decomposable (normal) operator for which $\mathfrak{A}K \subseteq K\mathfrak{A}$. Then the norm of the spatial automorphism ψ of $\mathfrak{A}$ defined by*

$$\psi(A) = K^{-1}AK, \; A \in \mathfrak{A}$$

is at least $r_\sigma(K^{-1}) \cdot r_\sigma(K)$ (respectively, equal to $\|K^{-1}\| \cdot \|K\|$.)

The following examples show that if $\|\psi\| = \|K^{-1}\| \, \|K\|$, then $\mathfrak{A}$ can have no or many non-trivial invariant subspaces.

Example 1. If $\mathfrak{A} = \mathfrak{B}(\mathfrak{H})$ and K is any invertible normal operator on $\mathfrak{H}$, then $\|\psi\| = \|K^{-1}\| \, \|K\|$ and obviously $\mathfrak{A}$ does not have a non-trivial invariant subspace. In this example the smallest annulus with center at 0 containing $\sigma(K)$ can be as "thick" as we please, but the algebra $\mathfrak{A}$ is very "big".

Example 2. Let $\mathfrak{A}$ be the algebra of all compact operators on $\mathfrak{H}$ and K be any unitary operator on $\mathfrak{H}$. Then obviously $\mathfrak{A}K = K\mathfrak{A}$, $\|\psi\| = \|K^{-1}\| \cdot \|K\| = 1$, and $\mathfrak{A}$ has no non-trivial invariant subspace. Here $\mathfrak{A}$ is "small", but $\sigma(K)$ is very "thin".

Example 3. Let $\mathfrak{H} = \mathfrak{L}^2(0, 1)$ and

$$\mathfrak{A} = \{M_\phi : \phi \in \mathfrak{L}^\infty(0, 1)\}$$

where M_ϕ: $\mathfrak{L}^2(0, 1) \rightarrow \mathfrak{L}^2(0, 1)$ is the multiplication operator $M_\phi(f) = \phi \cdot f$, $f \in \mathfrak{L}^2(0, 1)$. Let θ: $[0, 1] \rightarrow [0, 1]$ be defined by $\theta(x) = 1 - x$. Then θ is a bijective Lebesgue measurable function (in fact, continuous) which preserves the Lebesgue measure on $[0, 1]$. Let U: $\mathfrak{H} \rightarrow \mathfrak{H}$ be the unitary operator defined by $U(f) = f \circ \theta$. Let $\phi \in \mathfrak{L}^\infty(0, 1)$, then

$$(U^{-1}M_\phi U)f = (U^{-1}M_\phi)(f \circ \theta) = (U^{-1})(\phi \cdot f \circ \theta) = (\phi \circ \theta^{-1}) \cdot f = (M_{\phi \circ \theta^{-1}})f$$

and hence $U^{-1}M_\phi U = M_{\phi \circ \theta^{-1}}$. This shows that $\mathfrak{A}U = U\mathfrak{A}$. Here the norm of the algebra automorphism ψ: $\mathfrak{A} \rightarrow \mathfrak{A}$ defined by $\psi(A) = U^{-1}AU$ is 1, which is equal to $\|U^{-1}\| \, \|U\|$, and the algebra $\mathfrak{A}$ has many non-trivial invariant (in fact reducing) subspaces.

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