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# Low Complexity Solutions of the Allen–Cahn Equation on Three-spheres

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*Abstract.* In this short note, we prove that on the three-sphere with any bumpy metric there exist at least two pairs of solutions of the Allen–Cahn equation with spherical interface and index at most two. The proof combines several recent results from the literature.

## 1 Introduction

In this short note, we consider the Allen–Cahn equation,

$$-\Delta_g u = \frac{1}{\varepsilon^2}(u - u^3), \tag{1.1}$$

on $S^3$ endowed with any bumpy metric $g$. The Allen–Cahn equation has its origin as a model for phase transitions [AC79] and has received a lot of attention, especially due to its close connections with the theory of minimal surfaces, see *e.g.,* the surveys [T08, S09, P12].

In a beautiful recent paper [GG], Gaspar and Guaraco proved that on any closed manifold, the number of solutions of (1.1) goes to infinity as $\varepsilon \to 0$. Most of these solutions naturally have high complexity, *i.e.,* large index and interfaces of high genus. In this note, we address the remaining question of finding solutions with low complexity, *i.e.,* solutions with interface of genus zero and low index. Before stating the theorem, let us clarify that nontrivial solutions always come in pairs; *i.e.,* whenever $u$ is a solution, $-u$ is also a solution. We prove the following theorem.

***Theorem 1.1*** *Consider the Allen–Cahn equation (1.1) on $S^3$ endowed with any bumpy metric $g$. Then for any small enough $\varepsilon > 0$, there exist at least two pairs $\{u_\varepsilon^1, -u_\varepsilon^1\}$, $\{u_\varepsilon^2, -u_\varepsilon^2\}$ of solutions with spherical interface and index at most two. More precisely, exactly one of the following alternatives occurs:*

(i) *at least one pair of stable solutions with spherical interface and at least two pairs of index-one solutions with spherical interface,*

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(ii) *no stable solutions with spherical interface, at least one pair of index-one solutions with spherical interface, and at least one pair of index-two solutions with spherical interface.*

The two cases are illustrated in Figures 1 and 2 for the scenarios that $(S^3, g)$ looks like a dumbbell and an elongated ellipsoid, respectively.

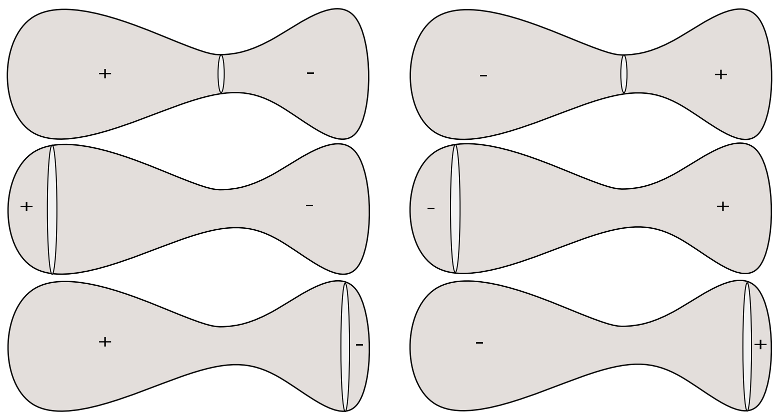


*Figure 1*: Solutions on a dumbbell

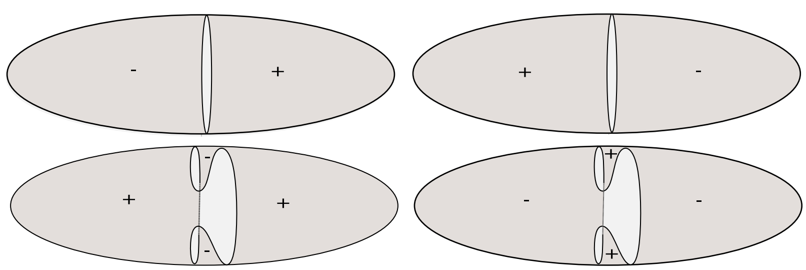


*Figure 2*: Solutions on elongated ellipsoids

The solutions transition from $u_\varepsilon \approx -1$ to $u_\varepsilon \approx +1$ along the spherical interface, and the transition is modelled on the function $\tanh(t/\sqrt{2}\varepsilon)$, where $t$ denotes the signed distance from the interface.

We recall that solutions of (1.1) are critical points of the energy functional

$$E_\varepsilon[u] = \int_M \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2 \, \mathrm{dvol}_g \, .$$

The index of a critical point is defined as the number of directions in which the second variation of the functional is negative.

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We also recall that a Riemannian metric $g$ on a manifold $M$ is called *bumpy* if $(M, g)$ does not contain any immersed closed minimal hypersurface with a non-trivial Jacobi field. By a result of White [W17, W91], bumpy metrics are generic in the sense of Baire.

As an application of Theorem 1.1 we can construct further solutions by gluing together one, two, or three of the solutions from case (i).

***Corollary 1.2*** *If the Allen–Cahn equation (1.1) on $S^3$ endowed with any bumpy metric $g$ has a stable solution with spherical interface, then there are at least seven pairs of solutions with index at most two.*

***Remark 1.3*** In case (ii), the two spherical interfaces always intersect (*cf.* [HK, Lemma 3.4]), and thus the solutions of case (ii) cannot be glued together smoothly to construct further low index solutions.

To prove Theorem 1.1, we combine several results from the literature, including in particular, the gluing construction from Pacard–Ritore [PR03], the existence result for minimal two-spheres from Haslhofer–Ketover [HK], and the index estimate from Chodosh–Mantoulidis [CM]. An interesting feature is that the proof relies on some methods from other fields, in particular, mean curvature flow with surgery, that have not been used before in the analysis of the Allen–Cahn equation.

## 2 The Proof

**Proof of Theorem 1.1** We first consider the case that there exists at least one solution $u_\varepsilon$ of (1.1) that is stable and has spherical interface (for every small enough $\varepsilon$). Having a spherical interface means that for $\varepsilon \to 0$ (after passing to a subsequence), the associated varifolds

$$V_\varepsilon(\phi) = \frac{3}{4} \int \varepsilon |\nabla u_\varepsilon(x)|^2 \phi\big(x, T_x\{u_\varepsilon = u_\varepsilon(x)\}\big)\, \mathrm{dvol}_g(x)$$

converge to an embedded minimal two-sphere $\Sigma$, possibly with multiplicity. It follows from classical theory [TW12, T95] (see also [H]) that $\Sigma$ is a stable critical point of the area functional. Thus, by a result of Ketover–Liokumovich [KL, Thm. 1.6] each of the two balls $B^{\ell/r}$ in $S^3$ bounded by $\Sigma$ contains in its interior an index-one minimal two-sphere $\Sigma^{\ell/r}$. We can now apply the main theorem of Pacard-Ritore [PR03] (see also [P12]) to get solutions $u_\varepsilon^{\ell/r}$ with spherical interface $\Sigma^{\ell/r}$. Since for the Pacard–Ritore solutions $u_\varepsilon^{\ell/r}$ the associated varifolds $V_\varepsilon^{\ell/r}$ converge to $\Sigma^{\ell/r}$ with multiplicity one, it follows from Chodosh–Mantoulidis [CM, §6] (see also [dPKW13, §9] for closely related computations) that the index is actually continuous under the limit; *i.e.,* for $\varepsilon$ small enough, we get

$$\mathrm{ind}(u_\varepsilon^{\ell/r}) = \mathrm{ind}(\Sigma^{\ell/r}) = 1,$$

where the possibility of nullity is ruled out by the bumpiness assumption. Together with the obvious fact that whenever $u$ is a solution of (1.1), $-u$ also is a solution, this finishes the proof in case (i).

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Consider now the case that (1.1) admits no stable solutions with spherical interface. Suppose towards a contradiction that $(S^3, g)$ contains a stable embedded minimal two-sphere $\Sigma$. Then, arguing as above, for $\varepsilon$ small enough, the associated Pacard–Ritore solutions would satisfy

$$\operatorname{ind}(u_\varepsilon) = \operatorname{ind}(\Sigma) = 0,$$

contradicting our assumption that (1.1) has no stable solutions with spherical interface. Hence, by a classical result of Simon–Smith [SS82] (see also [MN12, Lem. 3.5] and [HK, Lem. 3.2]) there exists an embedded minimal two-sphere $\Sigma^1 \subset (S^3, g)$, which is obtained by one parameter min-max, realizes the 1-width, and has index one (since it cannot be stable). By a recent result from Haslhofer-Ketover [HK, Thm. 1.4], which has been established using combined efforts from mean curvature flow with surgery [BH, HK17, BHH] and min-max theory [KMN], there also exists an embedded minimal two-sphere $\Sigma^2 \subset (S^3, g)$, which has index two. Arguing as above, for $\varepsilon$ small enough the associated Pacard–Ritore solutions satisfy

$$\operatorname{ind}(\pm u_\varepsilon^1) = \operatorname{ind}(\Sigma^1) = 1 \quad \text{and} \quad \operatorname{ind}(\pm u_\varepsilon^2) = \operatorname{ind}(\Sigma^2) = 2.$$

This establishes case (ii), and thus finishes the proof of the theorem. ■

**Proof of Corollary 1.2** By Theorem 1.1 we have one spherical interface $\Sigma$ of index zero, and two spherical interfaces $\Sigma^{\ell/r}$ of index one. In addition to the three pairs of solutions from Theorem 1.1, we can construct four further pairs of solutions by taking the Pacard–Ritore solutions associated with $\Sigma \cup \Sigma^\ell$, $\Sigma \cup \Sigma^r$, $\Sigma^\ell \cup \Sigma^r$, and $\Sigma \cup \Sigma^\ell \cup \Sigma^r$, respectively. It is easy to see that the index is additive, and thus these four additional pairs of solutions have index at most two. ■

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