Proceedings of the Edinburgh Mathematical Society (2002) **45**, 507–511 ©
DOI:10.1017/S0013091501000050 Printed in the United Kingdom

PRIME RINGS WITH FINITENESS PROPERTIES ON ONE-SIDED IDEALS

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(Received 15 January 2001)

Abstract Let R be a prime ring with extended centroid C, ρ a non-zero right ideal of R and let $f(X_1,\dots,X_t)$ be a polynomial, having no constant term, over C. Suppose that $f(X_1,\dots,X_t)$ is not central-valued on RC. We denote by $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1,\dots,x_t)$ for $x_i \in \rho$. The main goals of this note are to prove two results concerning the extension properties of finiteness conditions as follows.

(I) If $f(\rho)$ spans a non-zero finite-dimensional C-subspace of RC, then $\dim_C RC$ is finite.

(II) If $f(\rho) \neq 0$ and is a finite set, then R itself is a finite ring.

Keywords: prime ring; polynomial identity (PI); generalized polynomial identity (GPI); extended centroid

AMS 2000 *Mathematics subject classification:* Primary 16N60; 16R50

1. Introduction and results

Throughout this note, R always denotes a prime ring with extended centroid C and ρ a non-zero right ideal of R. The goal of this note is to study the extension properties of certain finiteness conditions from one-sided ideals of a prime ring to the whole prime ring. We were motivated by two elementary observations: (I) if $\dim_C \rho C$ is finite, then so is $\dim_C RC$; and (II) if ρ is a finite subset of R, then so is R. For (I) see [**3**, Lemma 1]. For (II), since, by the primeness of R, R can be embedded in $\mathrm{End}(\rho,+,0)$ via right multiplications and $\mathrm{End}(\rho,+,0)$ is a finite set, R itself is a finite ring. In a recent paper Bell proved the following theorem.

Theorem. *Suppose that ρ is of finite index in R and $[\rho,\rho]$ is finite. Then R is either finite or commutative (see [**1**, Theorem 2.2]).*

Since, in Bell's Theorem, $[\rho,\rho]$ contains all elements $xy-yx$ for all $x,y\in\rho$, we want to extend these results above to more generalized forms. In particular, we shall see that, in Bell's Theorem, the assumption that ρ is of finite index in R is superfluous. Our point of view in this note is different from that of [**1**]. To state our results we require some notation. For a polynomial $f(X_1,\dots,X_t)$ over C, where the X_i are non-commuting indeterminates, we denote by $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1,\dots,x_t)$ for $x_i\in\rho$. For $x,y\in R$, set $[x,y]_0=x$, $[x,y]_1=[x,y]=xy-yx$ and $[x,y]_k=[[x,y]_{k-1},y]$ for $k>1$. Also, $[\rho,\rho]_k$ denotes the additive subgroup of R generated by all elements $[x,y]_k$ for $x,y\in\rho$. We are now ready to state our results.

Theorem 1.1. *Let R be a prime ring with extended centroid C, ρ a non-zero right ideal of R and $f(X_1,\dots,X_t)$ a polynomial, having no constant term, over C. Suppose that $f(X_1,\dots,X_t)$ is not central-valued on RC.*

(I) *Suppose that $f(\rho)$ spans a non-zero finite-dimensional C-subspace of RC. Then $\dim_C RC$ is finite.*

(II) *Suppose that $f(\rho)\neq 0$ and is a finite set. Then R itself is a finite ring.*

Corollary 1.2. *Let R be a prime ring with extended centroid C, ρ a non-zero right ideal of R, and k a non-negative integer.*

(I) *Suppose that $[\rho,\rho]_k$ spans a finite-dimensional C-subspace of RC. Then $\dim_C RC$ is finite.*

(II) *Suppose that $[\rho,\rho]_k$ is a finite set. Then R is either commutative or finite.*

Proof. In view of Theorem 1.1, it suffices to prove that $[\rho,\rho]_k\nsubseteq C$ unless R is commutative. Indeed, suppose that $[\rho,\rho]_k\subseteq C$. Then $[x,y]_{k+1}=0$ for all $x,y\in\rho$. In view of [**5**, Lemma 1], R is commutative. This proves the corollary. □

We remark that Corollary 1.2 (II) gives a generalization of Bell's Theorem [**1**, Theorem 2.2].

2. Proof of Theorem 1.1

To prove Theorem 1.1 we need the following two theorems [**2**, Theorem 1 (II) (i) and Theorem 2]. Recall that a prime ring R is called *centrally closed* if $R=RC$ and that a right ideal of a ring is called a polynomial identity (PI) right ideal if the right ideal is itself a PI-ring. We write $\mathrm{M}_2(\mathrm{GF}(2))$ to stand for the 2 by 2 matrix ring over GF(2), the Galois field of two elements.

Theorem 2.1. *Let R be a centrally closed prime C-algebra and $f(X_1,\dots,X_t)$ a non-zero polynomial over C. Suppose that ρ is a PI right ideal of R such that $f(\rho)\neq 0$. Then there exists an idempotent e in the socle of R such that $\rho=eR$ and $eR(1-e)\subseteq f(\rho)$.*

Theorem 2.2. *Let R be a prime ring with extended centroid C and I a non-zero ideal of R. Suppose that $f(X_1,\dots,X_t)$ is a polynomial over C which is not central-valued on RC. Then $[M,R]\subseteq f(I)$ for some non-zero ideal M of R, except when $R\cong\mathrm{M}_2(\mathrm{GF}(2))$ and $f(R)=\{0,e_{12}+e_{21},1+e_{12},1+e_{21}\}$ or $\{0,1,e_{11}+e_{12}+e_{21},e_{22}+e_{12}+e_{21}\}$.*

https://doi.org/10.1017/S0013091501000050 Published online by Cambridge University Press

From now on, R is always a prime ring with extended centroid C, ρ a non-zero right ideal and $f(X_1,\dots,X_t)$ a non-zero polynomial, having no constant term, over C. We denote by $d_n(X_1,\dots,X_n;Y_1,\dots,Y_{n-1})$ the Capelli polynomial of degree $2n-1$, that is

$$d_n(X_1,\dots,X_n;Y_1,\dots,Y_{n-1})=\sum_{\sigma\in S_n}(-1)^\sigma X_{\sigma(1)}Y_1X_{\sigma(2)}\cdots Y_{n-1}X_{\sigma(n)},$$

where X_i and Y_j are non-commuting indeterminates. To apply Theorem 2.1 to our case, we require the following lemma.

Lemma 2.3. *Suppose that $f(\rho)\neq 0$. Then*

(I) *if the additive subgroup $f(\rho)$ spans a finite-dimensional C-space, then $f(\rho)C=f(\rho C)C$; and*

(II) *if $f(\rho)$ is finite, then so are both C and $f(\rho C)$.*

Proof. (I) Suppose that $\dim_C f(\rho)C=n<\infty$. Then

$$d_{n+1}(f(x_{11},\dots,x_{1t}),\dots,f(x_{n+11},\dots,x_{n+1t});y_1,\dots,y_n)=0 \tag{2.1}$$

for all $x_{ij}\in\rho$ and all $y_i\in R$. Since ρ and ρC satisfy the same generalized polynomial identities (GPIs) [**4**, Lemma 2], equation (2.1) still holds for all $x_{ij}\in\rho C$ and all $y_i\in RC$. In view of [**7**, Theorem 7.6.16], for $x_{ij}\in\rho C$ the $n+1$ elements $f(x_{11},\dots,x_{1t}),\dots,f(x_{n+11},\dots,x_{n+1t})$ are C-dependent and so $\dim_C f(\rho C)C\leqslant n$. But since $f(\rho)C\subseteq f(\rho C)C$, we conclude that $f(\rho)C=f(\rho C)C$ as asserted.

(II) Suppose that $f(\rho)$ is a finite set. Write

$$f(X_1,\dots,X_t)=\sum_{i=1}^{m}f_i(X_1,\dots,X_t), \tag{2.2}$$

where $f_i(X_1,\dots,X_t)$ is the homogeneous part of $f(X_1,\dots,X_t)$ of degree i for $1\leqslant i\leqslant m$. Suppose on the contrary that C is infinite. Choose m distinct elements $\beta_1,\dots,\beta_m$ in C and then a non-zero ideal I of R such that $\beta_iI\subseteq R$. Then $\beta_i\rho I\subseteq\rho$ for each i. Applying a standard determinant argument to equation (2.2) we see that $f_i(\rho I)$ is finite for each i. Since $f(\rho)\neq 0$, we see that $f_j(\rho)\neq 0$ for some j. Thus we may assume from the start that $f(X_1,\dots,X_t)$ is homogeneous of degree $m\geqslant 1$. Suppose that $f(\rho)$ consists of ℓ elements. Since C is an infinite field, we can choose $\mu_1,\dots,\mu_{\ell+1}$ in C such that $\mu_i^m\neq\mu_j^m$ for $i\neq j$. Let J be a non-zero ideal of R such that $\mu_iJ\subseteq R$ for each i. In view of [**4**, Lemma 2], there exist $x_i\in\rho J$, $1\leqslant i\leqslant t$, such that $f(x_1,\dots,x_t)\neq 0$. Now, we see that $\mu_i^mf(x_1,\dots,x_t)=f(\mu_ix_1,\dots,\mu_ix_t)\in f(\rho)$ for each i. This derives a contradiction, as the set $\{\mu_i^mf(x_1,\dots,x_t)\mid 1\leqslant i\leqslant\ell+1\}$ consists of $\ell+1$ elements. Thus C is a finite field. By (I) we see that $f(\rho)C=f(\rho C)C$. But since $f(\rho)$ and C are finite sets, this implies that $f(\rho C)$ is a finite set, proving (II). $\square$

Lemma 2.4. *Let R be a centrally closed prime C-algebra with non-zero socle H. Suppose that e is a non-trivial idempotent in H.*

(I) *If $\dim_C eR(1-e) < \infty$, then R is finite dimensional over C.*

(II) *If $eR(1-e)$ is a finite set, then R is a finite ring.*

Proof. (I) Suppose that $\dim_C eR(1-e) = n < \infty$. Note that e and $1-e$ are C-independent. Thus we see that

$$S_{n+1}(eX_1(1-e)Y, \dots, eX_{n+1}(1-e)Y)$$

is a non-trivial GPI for R, where $S_{n+1}(X_1, \dots, X_{n+1})$ is the standard polynomial of degree $n+1$. Since R is centrally closed, it follows from Martindale's Theorem [**6**] that R is a strongly primitive ring. By assumption, $e \in H$ and so $\dim_C eRe < \infty$. Thus $\dim_C eR = \dim_C eRe + \dim_C eR(1-e) < \infty$. That is, R contains a non-zero right ideal eR, which is finite dimensional over C. In view of [**3**, Lemma 1], $\dim_C R < \infty$ follows. This proves (I).

(II) Since $eR(1-e) \neq 0$, we choose $x_0 \in R$ such that $ex_0(1-e) \neq 0$. But $Cex_0(1-e) \subseteq eR(1-e)$, so the finiteness of $eR(1-e)$ implies that C is a finite field. By (I), R is finite dimensional over C and so R is a finite ring, proving (II). □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (I) Suppose that $f(\rho)$ spans a non-zero finite-dimensional C-subspace of RC. It follows from Lemma 2.3 (I) that $0 \neq f(\rho C)$ is finite dimensional over C. Clearly, ρC is a PI right ideal of RC. Note that RC is a centrally closed prime C-algebra. In view of Theorem 2.1, there exists an idempotent e in the socle of RC such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$ as $f(\rho C) \neq 0$. If $e \neq 1$, applying Lemma 2.4 we conclude that $\dim_C RC$ is finite. Suppose next that $e = 1$. Then $\rho C = RC$. Since $f(X_1, \dots, X_t)$ is not central-valued on RC, by Theorem 2.2 there exists a non-zero ideal I of RC such that $[I, RC] \subseteq f(\rho C)$ unless $R = RC \cong \mathrm{M}_2(\mathrm{GF}(2))$. There is nothing to prove for the exceptional case. Therefore, we may assume that $[I, RC] \subseteq f(\rho C)$ and so $\dim_C [I, RC] < \infty$, implying that R itself is a PI-ring. Applying Posner's Theorem yields that $\dim_C RC$ is finite.

(II) Suppose that $f(\rho) \neq 0$ and is a finite set. By Lemma 2.3, $f(\rho C)$ is a finite set. Clearly, in this case ρC must be a PI right ideal of R. In view of Theorem 2.1, there exists an idempotent e in the socle of RC such that $\rho C = eRC$ and $eRC(1-e) \subseteq f(\rho C)$. If $e \neq 1$, Lemma 2.4 implies that R is a finite ring. Otherwise, $e = 1$ follows and so $\rho C = RC$. As in the argument given in (I), we may assume that $R \ncong \mathrm{M}_2(\mathrm{GF}(2))$. By Theorem 2.2 there exists a non-zero ideal I of RC such that $[I, RC] \subseteq f(\rho C)$ and so $[I, RC]$ is a finite set. But since R is not commutative, $[I, RC] \neq 0$ and so C must be finite. Now, RC is a PI-ring with C a finite field. It is now clear that R itself is a finite ring, proving (II). □

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