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ON TRANSIENT MARKOV PROCESSES OF ORNSTEIN-UHLENBECK TYPE

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Abstract. For Hunt processes on $\mathbf{R}^d$, strong and weak transience is defined by finiteness and infiniteness, respectively, of the expected last exit times from closed balls. Under some condition, which is satisfied by Lévy processes and Ornstein-Uhlenbeck type processes, this definition is expressed in terms of the transition probabilities. A criterion is given for strong and weak transience of Ornstein-Uhlenbeck type processes on $\mathbf{R}^d$, using their Lévy measures and coefficient matrices of linear drift terms. An example is discussed.

§1. Introduction

Let $\mathbf{R}^d$ be the d-dimensional Euclidean space and $\mathcal{B}$ be the Borel σ-algebra in $\mathbf{R}^d$. We denote the inner product and the norm in $\mathbf{R}^d$ by $\langle x, y\rangle = \sum_{j=1}^d x_j y_j$ and $|x| = \sqrt{\langle x, x\rangle}$, respectively, for $x = (x_j)_{1\le j\le d}$, $y = (y_j)_{1\le j\le d}$. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be a Markov process (in the sense of [1]) with state space $(\mathbf{R}^d, \mathcal{B})$, satisfying $P^x(X_0 = x) = 1$. Assume that X is a Feller process. Here we say that X is a Feller process if it satisfies the following properties: for any continuous function f on $\mathbf{R}^d$ vanishing at infinity,

(i) $P_t f$ is continuous for all $t > 0$,

(ii) $P_t f \to f$ uniformly as $t \to 0$,

where P_t is the transition operator of X. This process X is a Hunt process by virtue of Theorem 9.4 in [1] p.46. The process X is called transient if it satisfies

$$P^x\left(\lim_{t\to\infty} |X_t| = \infty\right) = 1$$

for every $x \in \mathbf{R}^d$. Let $L_B = \sup\{t \ge 0 : X_t \in B\}$ for any $B \in \mathcal{B}$, which is called the last exit time from B, where the supremum of the empty set is 0. Let $T_B = \inf\{t > 0 : X_t \in B\}$ for any $B \in \mathcal{B}$. Now we define strong and weak transience.

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DEFINITION 1. Let X be transient. Then X is called strongly transient if it satisfies

$$E^x L_B < \infty$$

for every closed ball B and every x. In the remaining case it is called weakly transient.

Our definition of strong and weak transience is different in appearance from that of Port [3] given for irreducible Markov chains. But in [3] he showed that strong and weak transience is equivalent to finiteness and infiniteness, respectively, of the expectation of the last exit times from finite sets. Hence the definition above is a natural extension.

Now we introduce the following condition:

$$\inf_{x \in K} E^x e^{-L_K} > 0 \qquad \text{for any compact set } K. \tag{1.1}$$

Our first result is the following theorem.

THEOREM 1.1. *Let X be transient and satisfy the condition* (1.1). *Then X is strongly transient if and only if we have, for any x and any closed ball K,*

$$\int_0^\infty dt\; tP^x(X_t \in K) < \infty. \tag{1.2}$$

In particular we shall investigate transient Ornstein-Uhlenbeck type processes. An Ornstein-Uhlenbeck type process (OU type process) X is a Feller process with state space $(\mathbf{R}^d, \mathcal{B})$ such that the process $\{X_t\}$ under the probability measure P^x has the same finite-dimensional distribution as the process $\{\overline{X}_t\}$ defined by

$$\overline{X}_t = e^{-tQ}x + \int_0^t e^{-(t-u)Q} dZ_u,$$

where $\{Z_t\}$ is a Lévy process, and Q is a real $d \times d$-matrix of which all eigenvalues have positive real parts. Here the stochastic integral with respect to the Lévy process is defined by convergence in probability from integrals of simple functions. This process X is called the OU type process associated with $\{Z_t\}$ and Q. The infinitesimal generator of X is given in [6] and [7]. Let the Lévy process $\{Z_t\}$ have the following characteristic function:

$$E \exp(i\langle z, Z_t\rangle) = \exp(t\psi(z)),$$

$$\psi(z) = -2^{-1}\langle z, Bz\rangle + \int_{\mathbf{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|<1\}}(x)\right) \rho(dx) + i\langle b, z\rangle,$$

where B is a symmetric nonnegative-definite constant matrix, ρ is a measure on $\mathbf{R}^d$ with $\rho(\{0\}) = 0$ and $\int(1 \wedge |y|^2)\rho(dy) < \infty$, b is a constant vector, and $1_{\{|x|<1\}}$ is the indicator function of the set $\{|x| < 1\}$. The measure ρ is called Lévy measure.

Recurrence criterion of OU type processes was found by T. Shiga [9] in one dimension, by K. Sato, T. Watanabe, and M. Yamazato [7] and by K. Sato, T. Watanabe, K. Yamamuro, and M. Yamazato [8] in some multidimensional cases, and finally by T. Watanabe [10] in the general multidimensional case.

Our second result is the following criterion of strong and weak transience for OU type processes.

THEOREM 1.2. *Let X be a transient OU type process. Fix $c > 0$. Then X is strongly transient if and only if*

$$\int_0^\infty t \exp\left[\int_0^t ds \int_{|x|\geq c} \left(e^{-|e^{-sQ}x|} - 1\right) \rho(dx)\right] dt < \infty. \tag{1.3}$$

§2. Proof of Theorem 1

In order to prove Theorem 1, we need the following lemmas. For a nonnegative measurable function f on $\mathbf{R}^d$, define $U^\lambda f$ by

$$U^\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt.$$

Let us denote $U = U^0$.

LEMMA 2.1. *Let X be transient and satisfy the condition* (1.1). *Then $U1_K$ is bounded for any compact set K.*

Proof. In [2] this lemma was proved under some other condition. Let K_n be a sequence of compact sets such that K_n is contained in the interior of K_{n+1} and $\cup_n K_n = \mathbf{R}^d$. Let us denote $g_n(x) = P^x(T_{K_n} < \infty)$. We denote $g_{n,k} = k(g_n - P_{\frac{1}{k}} g_n)$. We notice that $Ug_{n,k} < \infty$. By virtue of Lemma 3.1 in [2] p.403, we have, for any $m > 1$,

$$g_n = \lim_{k\to\infty} \uparrow Ug_{n,m^k}. \tag{2.1.1}$$

Here the right-hand side denotes the increasing limit of Ug_{n,m^k} as $k \to \infty$. Furthermore,

$$1 = \lim_{n\to\infty} \uparrow g_n = \lim_{n\to\infty} \uparrow Ug_{n,m^n}. \tag{2.1.2}$$

Define g^m by

$$g^m = \sum_{n=1}^{\infty} \frac{g_{n,m^n}}{m^n 2^n}.$$

From $g_{n,m^n} \le m^n$ we have $g^m \le 1$. And from (2.1.2) we have $0 < Ug^m \le 1$. Set $h = U^1 g^m$. Then $h \le Ug^m \le 1$. From the resolvent equation we have $Uh \le Ug^m \le 1$. Since $Ug^m > 0$, we get $h > 0$. Furthermore, since $T_{K_n} \circ \theta_t < \infty$ and $L_{K_n} > t$ are equivalent,

$$\begin{aligned}
&U^1 g^m(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^{\infty} e^{-t} \left(P^x \left(T_{K_n} \circ \theta_t < \infty\right) - P^x \left(T_{K_n} \circ \theta_{\frac{1}{m^n}+t} < \infty \right) \right) dt \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \left[E^x \left(1 - e^{-L_{K_n}}\right) - e^{\frac{1}{m^n}} E^x \left(e^{-\frac{1}{m^n}} - e^{-L_{K_n}} : L_{K_n} > \frac{1}{m^n} \right) \right] \\
&\ge \sum_{n=1}^{\infty} \frac{1}{2^n} \left(e^{\frac{1}{m^n}} - 1 \right) E^x \left[e^{-L_{K_n}} : L_{K_n} > \frac{1}{m^n} \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \left(e^{\frac{1}{m^n}} - 1 \right) e^{-\frac{1}{m^n}} E^x \left[E^{X_{\frac{1}{m^n}}} \left[e^{-L_{K_n}} : L_{K_n} > 0 \right] \right].
\end{aligned}$$

Now there is a compact set K_{n-2} such that $K \subset K_{n-2}$. We can choose a nonnegative continuous function f such that the support of f is K_{n-1}, $f \le 1$, and $f = 1$ on K_{n-2}. Then, for sufficiently large m,

$$\begin{aligned}
&E^x \left[E^{X_{\frac{1}{m^n}}} \left[e^{-L_{K_n}} : L_{K_n} > 0 \right] \right] \\
&\ge E^x \left[f(X_{\frac{1}{m^n}}) \right] \inf_{y \in K_{n-1}} E^y \left[e^{-L_{K_n}} : L_{K_n} > 0 \right] \\
&\ge \left(f(x) - \frac{1}{2} \right) \inf_{y \in K_{n-1}} E^y e^{-L_{K_n}}.
\end{aligned}$$

Here we got the last inequality by using the Feller property. Hence, using the condition (1.1), we have $\inf_{x \in K} h(x) > 0$ from $f = 1$ on K_{n-2}. From this fact we get

$$U1_K \le \frac{Uh}{\inf_{x\in K} h(x)} \le \frac{1}{\inf_{x\in K} h(x)},$$

concluding the proof.

LEMMA 2.2. *Let X be transient and satisfy the condition* (1.1). *If B and C are compact sets such that the interior of B contains C, then there are positive constants α and β such that*

$$\alpha P^x(T_C < \infty) \le U1_B(x) \le \beta P^x(T_B < \infty) \tag{2.2.1}$$

for every x.

Proof. Using the strong Markov property, we have

$$\begin{aligned} U1_B(x) &\ge E^x\left[E^{X_{T_C}}\left[\int_0^\infty 1_B(X_t)dt\right] : T_C < \infty\right] \\ &\ge P^x\left(T_C < \infty\right) \inf_{x\in C} U^\lambda 1_B(x), \end{aligned}$$

where $0 < \lambda < \infty$. Since X is a Feller process, we have $\inf_{x\in C} U^\lambda 1_B(x) > 0$. Hence we obtain the first inequality in (2.2.1). Next we shall prove the second inequality. We have

$$\begin{aligned} U1_B(x) &= E^x\left[E^{X_{T_B}}\left[\int_0^\infty 1_B(X_t)dt\right] : T_B < \infty\right] \\ &\le P^x\left(T_B < \infty\right) \sup_{x\in B} U1_B(x). \end{aligned}$$

Using Lemma 2.1, we have $\sup_{x\in B} U1_B(x) < \infty$. This shows the Lemma.

Proof of Theorem 1. We have

$$E^x L_B = \int_0^\infty P^x\left(T_B \circ \theta_t < \infty\right) dt.$$

Let A be a closed ball such that B is contained in the interior of A. By virtue of Lemma 2.2, we get

$$\begin{aligned} \frac{1}{\alpha}\int_0^\infty tP_t(x,A)dt &= \frac{1}{\alpha}\int_0^\infty P_tU1_A(x)dt \\ &\ge E^x L_B \\ &\ge \frac{1}{\beta}\int_0^\infty P_tU1_B(x)dt = \frac{1}{\beta}\int_0^\infty tP_t(x,B)dt. \end{aligned}$$

Hence,

$$E^x L_B < \infty \qquad \text{for any closed ball } B$$

if and only if

$$\int_0^\infty tP_t(x,B)dt < \infty \qquad \text{for any closed ball } B.$$

We have thus proved the theorem.

Remark 2.3. If X is a transient OU type process or a transient Lévy process, it satisfies the condition (1.1). Because, for any compact set K, there is a compact set C such that $E^x e^{-L_K} \geq E^0 e^{-L_C}$ for all $x \in K$. Further (1.1) holds in case of transient strong Feller processes. Here we say that X is a strong Feller process if it is a Feller process and if (i) in the Feller property holds for any bounded measurable function f. In fact, we have $E^x \exp\{-L_K\} \geq E^x \exp\{-(L_K \circ \theta_t + t)\} = e^{-t} P_t E^x \exp\{-L_K\}$, and the last expression is continuous in x by virtue of the strong Feller property. Since $E^x \exp\{-L_K\} > 0$ for all $x \in K$, we have the condition (1.1).

Remark 2.4. Let

$$r(t) = \int_t^\infty ds \int dx h(x) \int P^x \left(X_s \in dy\right) h(y),$$

where h is a nonnegative continuous function on $\mathbf{R}^d$ with compact support, not identically zero. In [4] the definition of strong and weak transience for Lévy processes is given by $\int_0^\infty r(t)dt < \infty$ and $= \infty$, respectively. In case of an OU type process, since the condition of Theorem 1 is satisfied by virtue of Remark 2.3, we see that it is strongly transient if and only if $\int_0^\infty r(t)dt < \infty$.

§3. Proof of Theorm 2

In this section let X be an OU type process. First we shall state a fundamental lemma discovered by T. Watanabe [10] on boundedness of some integrals. This will play an important role in the proof of Theorem 2. Let n and m be positive integers. The number $\bar{z}$ signifies the complex conjugate of z. Let $\gamma_j (1 \leq j \leq n)$ be complex numbers. Let $P_j(s)$, $1 \leq j \leq n$, be polynomials with degrees being at most m. We assume that $P_j(s)$ has real coefficients if γ_j is real, and that $\gamma_j = \overline{\gamma_k}$ and $P_j(s) = \overline{P_k(s)}$ for some k if γ_j is not real. Define a function $f(s)$ on $\mathbf{R}^1$ as

$$f(s) = \sum_{j=1}^n e^{\gamma_j s} P_j(s).$$

Further let $I(x)$ be a real bounded measurable function on $\mathbf{R}^1$ and let $J(x) = \int_0^x I(u)du$.

LEMMA 3.1. (T. Watanabe [10]) *Suppose that*

$$\sup_{x\in\mathbf{R}^1} |I(x)| \leq 1 \quad \textit{and} \quad \sup_{x\in\mathbf{R}^1} |J(x)| \leq 1.$$

(i) *We obtain*

$$\left|\int_0^N I(f(s))ds\right| \leq K_1 + K_2 \log\left(\frac{1}{|f(0)|} \vee 1\right) \qquad \textit{for every } N > 0.$$

(ii) *In addition, suppose that*

$$\sup_{x\neq 0} \frac{|I(x)|}{|x|} \leq 1.$$

Then we obtain

$$\left|\int_M^N I(f(s))ds\right| \leq K_3 \quad \textit{for every } M \textit{ and } N \textit{ with } M < N.$$

Here K_1, K_2, and K_3 are positive constants depending only on m, n, and $\{\gamma_j\}_{j=1}^n$.

For $c > 0$ we denote the restriction of ρ to the set $\{x : |x| \geq c\}$ by ρ_c. Let $\{Z_t^c\}$ be a compound Poisson process with Lévy measure ρ_c. The process X^c denotes the OU type process associated with $\{Z_t^c\}$ and Q.

LEMMA 3.2. *Let X be transient. If X^c is weakly transient for some $c > 0$, then X is weakly transient.*

Proof. Choose a Lévy process $\{W_t\}$ independent of $\{Z_t^c\}$ such that $\{W_t + Z_t^c\}$ and $\{Z_t\}$ have common finite-dimensional distributions. Let Y be the OU type process associated with $\{W_t\}$ and Q. Then, identifying X^c and Y with $\overline{X^c}_t$ and $\overline{Y}_t$ defined similarly to $\overline{X}_t$ in section 1, we see that the processes X^c and Y are independent. We notice that Y has a limit distribution μ, because the Lévy measure $\rho - \rho_c$ satisfies $\int_{|x|\geq 1} \log|x|(\rho - \rho_c)(dx) < \infty$ (see [6]). By Theorem 1 and Remark 2.3, we can choose closed balls E_1 and E_2 such that $\int_0^\infty tP^x(X_t^c \in E_1)dt = \infty$ for some x and E_2 is a

μ-continuity set with $\mu(E_2) > 0$. Now let a closed ball E contain $E_1 + E_2$. Let t be large enough. Then

$$\begin{aligned} P^x(X_t \in E) &= P^0(X_t \in E - e^{-tQ}x) \\ &\geq P^0(X_t^c \in E_1 - e^{-tQ}x, Y_t \in E_2) \\ &\geq P^0(X_t^c \in E_1 - e^{-tQ}x)\frac{\mu(E_2)}{2} \\ &\geq P^x(X_t^c \in E_1)\frac{\mu(E_2)}{2}. \end{aligned}$$

Hence we obtain $\int_0^\infty tP^x(X_t \in E)dt = \infty$, and this completes the proof.

Using these lemmas, we shall prove Theorem 2. Let $P_t(x, E)$ and $P_t^c(x, E)$ be transition probabilities of X and X^c, respectively. And let P_t and P_t^c be their transition operators.

Proof of Theorem 2. For any $a > 0$, the Fourier transform of $h_a(x) = \Pi_{j=1}^d ((a - |x_j|) \vee 0)$ is expressed as

$$\hat{h}_a(z) = \int e^{i\langle x,z\rangle} h_a(x)dx = \prod_{j=1}^d 4z_j^{-2}\sin^2(2^{-1}az_j).$$

We have the inversion formula

$$h_a(x) = \frac{1}{(2\pi)^d}\int e^{-i\langle x,z\rangle}\hat{h}_a(z)dz.$$

At first we suppose that X is strongly transient. By virtue of Lemma 3.2, we obtain that X^c is strongly transient for any c. Let $\hat{P}_t(x, z)$ and $\hat{P}_t^c(x, z)$ be the characteristic functions of $P_t(x, dy)$ and $P_t^c(x, dy)$, respectively. There is a positive constant N satisfying that $|e^{-sQ}z| \leq N|z|$ for all $s > 0$ and $z \in \mathbf{R}^d$. Choose a large enough. We have $h_a(e^{-sQ}x + z) \geq h_{a-N|x|}(z)$. Then, from Theorem 1,

$$\begin{aligned} \infty &> \int_0^\infty dt\ t\int P_t^c(x, dy)h_a(y) \\ &= \int_0^\infty dt\ t\int P_t^c(0, dy)h_a(e^{-tQ}x + y) \\ &\geq \int_0^\infty dt\ t\int P_t^c(0, dy)h_{a-N|x|}(y) \end{aligned}$$

$$= \frac{1}{(2\pi)^d} \int_0^\infty dt\ t \int dz \hat{h}_{a-N|x|}(z) \hat{P}_t^c(0,-z)$$
$$= \frac{1}{(2\pi)^d} \int_0^\infty dt\ t \int dz \hat{h}_{a-N|x|}(z) \mathrm{Re}\hat{P}_t^c(0,-z).$$

We have

$$\hat{P}_t^c(0,z) = \exp\left[\int_0^t ds \int \left(\exp(i\langle z, e^{-sQ}x\rangle) - 1\right) \rho_c(dx)\right].$$

Hence we get

$$\mathrm{Re}\hat{P}_t^c(0,z) = \cos F_c(t,z) \exp G_c(t,z),$$

where

$$F_c(t,z) = \int_0^t ds \int \sin\langle z, e^{-sQ}x\rangle \rho_c(dx),$$

$$G_c(t,z) = \int_0^t ds \int \left(\cos\langle z, e^{-sQ}x\rangle - 1\right) \rho_c(dx).$$

Use Lemma 3.1 (ii) for $I(x) = \sin x$. Then

$$|F_c(t,z)| \le K_1 \rho(|x| \ge c),$$

where K_1 is a constant depending only on Q. Choose c so large that $K_1\rho(|x| \ge c) < \pi/4$. Then $\cos F_c(t,z) \ge 1/\sqrt{2}$. Therefore,

$$\infty > \int_0^\infty dt\ t \int dz \hat{h}_{a-N|x|}(z) \exp G_c(t,z).$$

We have $\hat{h}_{a-N|x|}(z) > 0$ for $|z| < \frac{2\pi}{a-N|x|}$. Hence, for some z with $0 < |z| < 1$,

$$\infty > \int_0^\infty dt\ t \exp G_c(t,z)$$
$$\int_0^\infty dt\ t \exp\left[\int_0^t ds \int \left(e^{-|\langle z, e^{-sQ}x\rangle|} - 1\right) \rho_c(dx) + H_c(t,z)\right],$$

where

$$H_c(t,z) = \int_0^t ds \int \left(\cos\langle z, e^{-sQ}x\rangle - e^{-|\langle z, e^{-sQ}x\rangle|}\right) \rho_c(dx).$$

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Letting $I(x) = 2^{-1}\left(\cos x - e^{-|x|}\right)$ in (ii) of Lemma 3.1, we have

$$\begin{aligned}\infty > &\int_0^\infty dt\; t \exp\left[\int_0^t ds \int \left(e^{-|\langle z, e^{-sQ}x\rangle|} - 1\right)\rho_c(dx)\right]\\ \geq &\int_0^\infty dt\; t \exp\left[\int_0^t ds \int \left(e^{-|e^{-sQ}x|} - 1\right)\rho_c(dx)\right].\end{aligned}$$

Next we shall prove the converse. In order to prove that X is strongly transient, it suffices to prove that, for all x and for all small a,

$$\int_0^\infty dt\; t \int P_t(x, dz)\hat{h}_a(z) < \infty.$$

We have

$$\begin{aligned}\int P_t(x, dz)\hat{h}_a(z) &= \int dz h_a(z)\hat{P}_t(0, z)e^{i\langle e^{-tQ}x, z\rangle}\\ &\leq \int dz h_a(z)|\hat{P}_t(0, z)|\\ &\leq a^d \int_{|z|<a\sqrt{d}} dz|\hat{P}_t^c(0, z)|.\end{aligned}$$

Hence it suffices to prove, for some $c > 0$,

$$\int_0^\infty dt\; t \int_{|z|<1} dz|\hat{P}_t^c(0, z)| < \infty.$$

Furthermore,

$$\begin{aligned}&\int_{|z|<1} \left|\hat{P}_t^c(0, z)\right| dz\\ &= \int_{|z|<1} \exp G_c(t, z) dz\\ &= \exp\left[\int_0^t ds \int \left(e^{-|e^{-sQ}x|} - 1\right)\rho_c(dx)\right] \int_{|z|<1} \exp\left[H_c(t, z) + I_c(t, z)\right] dz,\end{aligned}$$

where $I_c(t, z) = \int_0^t ds \int \left(e^{-|\langle z, e^{-sQ}x\rangle|} - e^{-|e^{-sQ}x|}\right)\rho_c(dx)$. Denote by $\mathbf{S}^{d-1}$ the $d-1$ dimensional unit sphere. Define a set S_Q as

$$S_Q = \{\xi \in \mathbf{S}^{d-1} : |u^Q\xi| > 1 \quad \text{for } u > 1\}.$$

We can disintegrate ρ_c as

$$\rho_c(E) = \int_{S_Q} \sigma(d\xi) \int_0^\infty 1_E(r^Q\xi)\tau_\xi(dr)$$

for each Borel set E in $\mathbf{R}^d$, where σ is a probability measure on S_Q and τ_ξ is a measure on $(0,\infty)$ such that $\tau_\xi(B)$ is measurable in ξ for any Borel set B in $(0,\infty)$ and $\tau_\xi((0,\infty)) = \rho_c(\mathbf{R}^d)$. Hence setting $I(x) = e^{-|x|}$ in (i) of Lemma 3.1 and letting $e^{-s}r = \omega$, we obtain, for $z \in \mathbf{R}^d$ with $|z| < 1$,

$$\begin{aligned}
|I_c(t,z)| &= \int_{S_Q} \sigma(d\xi) \int_0^\infty \tau_\xi(dr) \int_0^t ds \left(e^{-|\langle z, e^{-sQ}r^Q\xi\rangle|} - e^{-|e^{-sQ}r^Q\xi|}\right) \\
&\leq \int_{S_Q} \sigma(d\xi) \int_0^\infty \tau_\xi(dr) \int_0^\infty \frac{d\omega}{\omega} \left(e^{-|\langle z, \omega^Q\xi\rangle|} - e^{-|\omega^Q\xi|}\right) \\
&\leq \rho_c(\mathbf{R}^d) \left(K_1 + K_2 \int_{S_Q} \log \frac{1}{|\langle z, \xi\rangle|} \sigma(d\xi)\right),
\end{aligned}$$

where K_1 and K_2 depend only on Q. From Lemma 2.2 of [7], choosing c large enough, we have

$$\begin{aligned}
&\int_{|z|<1} \exp\left[H_c(t,z) + I_c(t,z)\right] dz \\
&\leq \text{const} \int_{|z|<1} \exp\left[K_2\rho_c(\mathbf{R}^d) \int_{S_Q} \log\frac{1}{|\langle z,\xi\rangle|}\sigma(d\xi)\right] dz < \infty.
\end{aligned}$$

Here we used Lemma 3.1 (ii) for $I(x) = 2^{-1}\left(\cos x - e^{-|x|}\right)$. By virtue of Theorem 1 this theorem has been proved.

§4. Example

Now we present an example of Theorem 2. This example has been already treated in [7]. Let X be an OU type process with $Q = \alpha I$, $\alpha > 0$, and a Lévy measure ρ such that, for every Borel set E in $[0,\infty)$,

$$\int 1_E(|x|)\rho(dx) = a \int_E \frac{dr}{r(\log(r \vee b))^{\gamma+1}},$$

where $\gamma > 0, a > 0, b > 1$. Then we have the following.

(i) If $\gamma > 1$, X is recurrent.

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(ii) If $\gamma < 1$, X is strongly transient.

(iii) If $\gamma = 1$ and $a \leq \alpha$, X is recurrent.

(iv) If $\gamma = 1$ and $1 < a/\alpha \leq 2$, X is weakly transient.

(v) If $\gamma = 1$ and $2 < a/\alpha$, X is strongly transient.

The conditions do not depend on the dimension d. K. Sato [5] proved that d-dimensional strongly non-lattice Lévy processes are strongly transient if $d \geq 5$. Hence this example shows the difference between Lévy processes and OU type processes.

Proof. Here we shall prove (ii), (iv), and (v). All the rest has been already proved in [7]. From now on let $c = b$. By the change of variables (1.3) becomes

$$(4.1)\qquad \int_0^1 \frac{dv}{v} \log\frac{1}{v} \exp\left[\int_v^1 \frac{du}{u} \int \left(e^{-u^\alpha |x|} - 1\right) \rho_b(dx)\right]$$
$$= \frac{1}{\alpha^2} \int_0^1 \frac{dv}{v} \log\frac{1}{v} \exp\left[-\frac{a}{\alpha} \int_v^1 \frac{du}{u} \int_b^\infty \frac{1 - e^{-ur}}{r(\log r)^{\gamma+1}} dr\right].$$

Here we obtain

$$\begin{aligned}\int_v^1 \frac{du}{u} \int_b^\infty \frac{1-e^{-ur}}{r(\log r)^{\gamma+1}} dr &= \int_v^1 du \int_b^\infty \frac{dr}{r(\log r)^{\gamma+1}} \int_0^r e^{-us} ds \\ &= \frac{1}{\gamma} \int_v^1 du \int_b^\infty \frac{e^{-us} ds}{(\log s)^\gamma} + O(1) \\ &= \frac{1}{\gamma} \int_b^{2/v} \frac{e^{-vs}}{s(\log s)^\gamma} ds + O(1)\end{aligned}$$

as $v \downarrow 0$.

(ii) If $\gamma < 1$, then we get

$$\int_b^{2/v} \frac{e^{-vs}}{s(\log s)^\gamma} ds \geq \frac{e^{-2}}{1-\gamma} \left(\log\frac{2}{v}\right)^{1-\gamma} + O(1)$$

as $v \downarrow 0$. Hence we see the right-hand side of (4.1) is finite.

(iv) If $\gamma = 1$ and $1 < a/\alpha \leq 2$, then we get

$$\int_b^{2/v} \frac{e^{-vs}}{s \log s} ds \leq \log\log\frac{2}{v} + O(1)$$

as $v \downarrow 0$. Hence we see the right-hand side of (4.1) is infinite.

(v) If $\gamma = 1$ and $2 < a/\alpha$, then we get

$$\int_b^{2/v} \frac{e^{-vs}}{s \log s} ds \geq \int_b^{1/v^\beta} \frac{e^{-vs}}{s \log s} ds$$

for $v < 1$, where $\beta = \beta(v)$ is a function of v such that $0 < \beta < 1$. Then we have

$$\begin{aligned}\int_b^{1/v^\beta} \frac{e^{-vs}}{s \log s} ds &\geq e^{-v^{1-\beta}} \int_b^{1/v^\beta} \frac{ds}{s \log s} \\ &= e^{-v^{1-\beta}} \log\log \frac{1}{v^\beta} + O(1)\end{aligned}$$

as $v \downarrow 0$. We will calculate (4.1). Now choose δ small enough. Then

$$\begin{aligned}&\int_0^\delta \frac{dv}{v} \log \frac{1}{v} \exp\left[-\frac{a}{\alpha} \int_v^1 \frac{du}{u} \int_b^\infty \frac{1 - e^{-ur}}{r(\log r)^{\gamma+1}} dr\right] \\ &\leq \int_0^\delta \frac{dv}{v} \log \frac{1}{v} \exp\left[-\frac{a}{\alpha} e^{-v^{1-\beta}} \log\log \frac{1}{v^\beta} + O(1)\right] \\ &\leq \text{const} \int_0^\delta \frac{dv}{v} \exp\left[-\frac{a}{\alpha} e^{-v^{1-\beta}} \log \beta\right] \left(\log \frac{1}{v}\right)^{1 - \frac{a}{\alpha} e^{-v^{1-\beta}}} \\ &\leq \text{const} \int_0^\delta \frac{dv}{v} \left(\log \frac{1}{v}\right)^{1 - \frac{a}{\alpha} e^{-v^{1-\beta}}}\end{aligned}$$

Here we choose $\beta = \beta(v)$ such that $\beta(v) \to 1$ and $(1 - \beta(v)) \log \frac{1}{v} \to \infty$ as $v \downarrow 0$. As a result, it follows from $a/\alpha > 2$ that the last expression is finite.

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