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TOPOLOGICAL VITALI MEASURE SPACES

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The properties of Lebesgue outer measures embodied in the Vitali covering theorem, the Vitali-Carathéodory theorem, the Lusin theorem, the density theorem, outer regularity and inner regularity, and the relation between measurability and approximate continuity are studied in a general abstract space, called a topological Vitali measure space. The main theme is the mutual equivalence of these properties.

1. Introduction

The properties of Lebesgue outer measures embodied in the Vitali covering theorem (Saks [11]; p. 109), the Lusin theorem (Saks [11]; p. 72), the density theorem (Saks [11]; p. 129), outer regularity and inner regularity, and the relation between measurability and approximate continuity (Saks [11]; Theorem 10.6, p. 132) have been studied by many authors (see the references, except for Kelley [8]) in various abstract spaces, but in a somewhat isolated manner. The Vitali covering property is an almost indispensable tool for the study of differentiation. Some authors actually prove it under suitable conditions, and others assume it in some form or other.

In this paper, assuming a very weak Vitali property (Definition 2.6) we introduce the notion of a topological Vitali measure space (Definition 3.2), which seems to be the most general structure suitable for a systematic study of the above mentioned properties and of differentiation and integration, simultaneously. Our main results are (Theorem 3.2) the

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mutual equivalence of the above mentioned properties and the property embodied in the Vitali-Carathéodory theorem (Saks [11]; p. 75), (Theorem 4.2) the equivalence of measurability and approximate continuity a.e. (which extends a result of Sion [14]), and (Theorem 4.3) separability of the range of an approximately continuous function (which extends a result of Goffman and Waterman [4]). Various aspects of the present theory are illustrated by appropriate examples. The theory of differentiation will be considered at length in a subsequent paper.

2. Preliminaries

Let X be a topological space (Kelley [8]) endowed with an outer measure (Halmos [6]) μ defined on the power set of X. Given $E \subset X$, $\bar{E}$ will denote the closure, and E^0 the interior, of E. The set E is called measurable (μ) if

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E) \quad \text{for every } A \subset X .$$

If for every $A \subset X$ there is a measurable set $E \supset A$ with $\mu(E) = \mu(A)$, then μ is called regular.

A function f from a subset E of X to an arbitrary topological space Y is called measurable if, for every open set G in Y, the set $f^{-1}(G)$ is measurable. This is equivalent to saying that $f^{-1}(B)$ is measurable for every Borel set B in Y. For, the family of sets $C \subset Y$ with $f^{-1}(C)$ measurable is a σ-ring, and the Borel family in Y is the smallest σ-ring containing all the open sets in Y.

We now give some new definitions and lemmas.

DEFINITION 2.1. A family H, of arbitrary subsets of a topological space Y, is called a pseudo base for Y if every open set in Y is the union of members of H. The space Y is called pseudo countable if there is a countable pseudo base H for Y; if, further, the members of H are Borel sets of Y, then Y is called Borel countable.

A second countable space is clearly Borel countable, and, hence, pseudo countable, but not conversely (see the examples below).

Given a subset T of the set $\mathbb{R}$ of real numbers, we denote by T^* the topology where open sets are all subsets $G \subset \mathbb{R}$ with the following property: For every $x \in G$ there is an open interval I containing x, such that $T \cap I \subset G$ and $|I \setminus G| = 0$ if $x \in T$, and $I \subset G$ if $x \notin T$. ($|E|$ denotes the Lebesgue outer measure of a subset $E \subset \mathbb{R}$.)

We observe that the space $(\mathbb{R}, T^*)$ is always pseudo countable, since the open intervals with rational end points together with their intersections with T evidently form a countable pseudo base for the space. Further, every ordinary open set of the real line is open in this space, and, hence, every ordinary Borel set is also a Borel set in this space. Consequently, if T is an ordinary Borel set, then the space $(\mathbb{R}, T^*)$ is necessarily Borel countable.

EXAMPLE 2.1. Fix $T \subset \mathbb{R}$ with $|T| > 0$ and inner measure equal to 0 (compare Halmos [6]; Theorem D, p. 69). In the space $(\mathbb{R}, T^*)$, every open set, and, hence, every Borel set is clearly measurable. Consider now any sequence $\{B_n\}$ of Borel sets of this space covering $\mathbb{R}$. Let A_n denote the set of points $x \in B_n$ such that $|B_n \cap (x,y)| = 0$ for some $y > x$; then $|A_n| = 0$. Noting that $|T| > 0$, we fix any point $c \in T \setminus \cup\{A_n\}$ and set

$$I_n = (c+(n+1)^{-1}, c+n^{-1}),\ B_{kn} = B_k \cap I_n \setminus T,\quad (k \leqslant n) .$$

We form a countable set F by selecting just one point from each nonvoid set B_{kn}, $k \leqslant n$, $n = 1,2,3,\ldots$. Clearly $\mathbb{R} \setminus F$ is open, and $c \in \mathbb{R} \setminus F$. Now, consider any $B_k \ni c$. Since $c \notin A_k$, we must have $|B_k \cap I_n| > 0$ for some $n > k$. But, $B_k \cap I_n$ is measurable, and T has inner measure 0 . Therefore, the set B_{kn} is nonvoid, and, hence, by the construction of F, we do not have $B_k \subset \mathbb{R} \setminus F$. This shows that the space $(\mathbb{R}, T^*)$ is not Borel countable, although it is pseudo countable as noted above.

EXAMPLE 2.2. Fix an ordinary nonvoid Borel set T of the real line R, such that $|I \setminus T| > 0$ for every open interval I. In the space (R, T^*), there does not exist any countable base at any point $c \in T$, as can be easily seen by constructing a set like F of the preceding example. So the space is not second countable, nor even first countable, although it is Borel countable as noted above.

DEFINITION 2.2. A subset $E \subset X$ is called μ-open if $\mu(E \setminus E^0) = 0$, μ-closed if $\mu(\bar{E} \setminus E) = 0$, and μ-proper if $0 < \mu(E) < \infty$.

It is easy to verify that a set is μ-open if and only if its complement is μ-closed. Also, finite intersection and countable union of μ-open sets are μ-open, and finite union and countable intersection of μ-closed sets are μ-closed. We shall employ μ-open sets and μ-closed sets in several places, where it is usual to take open sets and closed sets, respectively. This is a nontrivial generalization, the necessity of which is shown in Example 3.1.

DEFINITION 2.3. A family V of nonvoid subsets of X is said to converge to a point $x \in X$, or, to be x-convergent, if every neighborhood of x contains some member of V.

DEFINITION 2.4. Let D be a function which assigns to each point $x \in X$ a nonvoid collection $\mathsf{D}(x)$ of x-convergent families of μ-proper μ-closed subsets of X, such that if B belongs to $\mathsf{D}(x)$, so does every x-convergent subfamily of B. Then D is called a Vitali covering function on X. A sequence of Vitali covering functions D_n on X is called increasing if $\mathsf{D}_n(x) \subset \mathsf{D}_{n+1}(x)$ for all n, x.

In what follows, D will denote an arbitrary Vitali covering function on X. For any $G \subset X$, we set

$$\mathsf{D}[G] = \{V \mid V \in \bigcup_{x \in X} \cup \mathsf{D}(x),\ V \subset G\}.$$

DEFINITION 2.5. A family $\mathsf{V} \subset \mathsf{D}[x]$ is called a Vitali D-covering of a subset $E \subset X$ if, for every $x \in E$, there is a $\mathsf{B} \in \mathsf{D}(x)$ such that $\mathsf{B} \subset \mathsf{V}$. The covering V is called measurable if every member of V is measurable.

DEFINITION 2.6. D is said to have the weak Vitali property on a subset $E \subset X$, if, for every $\varepsilon > 0$, there are a sequence of subsets $E_n \subset E$ with $\mu(E \setminus \cup\{E_n\}) < \varepsilon$ and a sequence of real numbers $p_n \in (0,1)$, such that, for every $A \subset E_n$ and for every measurable Vitali D-covering V of A, there is a finite family V_0 of pairwise disjoint members of V such that $\mu(A \setminus \cup\mathsf{V}_0) \leqslant p_n \cdot \mu(A)$.

DEFINITION 2.7. $\mathcal{D}$ is said to have the finite covering property on a subset $G \subset X$ if, for every $E \subset G$ and every measurable Vitali $\mathcal{D}$-covering $\mathcal{V}$ of E and every $\varepsilon > 0$, there is a finite family $\mathcal{V}_0$ of pairwise disjoint members of $\mathcal{V}$ such that $\mu(E \setminus \cup\mathcal{V}_0) < \varepsilon$.

LEMMA 2.1. *Let* $G \subset X$, $\mu(G) < \infty$, *and* $p \in (0,1)$. *Suppose, for every* $A \subset G$ *and for every measurable Vitali* $\mathcal{D}$*-covering* $\mathcal{V}$ *of* A, *there is a finite family* $\mathcal{V}_0$ *of pairwise disjoint members of* $\mathcal{V}$ *such that* $\mu(A \setminus \cup\mathcal{V}_0) \leqslant p\cdot\mu(A)$. *Then* $\mathcal{D}$ *has the finite covering property on* G.

Proof. Given a measurable Vitali $\mathcal{D}$-covering $\mathcal{V}$ of a subset $E \subset G$, let ℓ denote the infimum of the numbers $\mu(E \setminus \cup\mathcal{V}_0)$ for all finite families $\mathcal{V}_0$ of pairwise disjoint members of $\mathcal{V}$. Then $0 \leqslant \ell < \infty$, since $\mu \geqslant 0$ and $\mu(E) \leqslant \mu(G) < \infty$. So, for any $\varepsilon > 0$, there is a finite family $\mathcal{V}_1$ of pairwise disjoint members of $\mathcal{V}$ such that

(1) $$\mu(E \setminus F) < \ell + \varepsilon \quad \text{where} \quad F = \cup\mathcal{V}_1 .$$

Let $\mathcal{V}'$ denote the family of sets $V \in \mathcal{V}$ such that $V \cap \bar{F} = \emptyset$. Then $\mathcal{V}'$ is evidently a measurable Vitali $\mathcal{D}$-covering of $E \setminus \bar{F}$. So, by hypothesis, there is a finite family $\mathcal{V}_2$ of pairwise disjoint members of $\mathcal{V}'$ such that

(2) $$\mu((E \setminus \bar{F}) \setminus W) \leqslant p\cdot\mu(E \setminus \bar{F}) \quad \text{where} \quad W = \cup\mathcal{V}_2 .$$

Now, $\mathcal{V}_0 = \mathcal{V}_1 \cup \mathcal{V}_2$ is a finite family of pairwise disjoint members of $\mathcal{V}$, and, since the members of $\mathcal{V}_1$ are μ-closed, it follows readily from (1) and (2) that $\mu(E \setminus \cup\mathcal{V}_0) < p\cdot(\ell+\varepsilon)$. Therefore $0 \leqslant \ell < p\cdot(\ell+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary and $p < 1$, it follows that $\ell = 0$, which in fact proves the lemma.

LEMMA 2.2. *Let* $G \subset X$, $\mu(G) < \infty$, *and let* $\mathcal{D}$ *have the weak Vitali property on* G. *Then* $\mathcal{D}$ *has the finite covering property on* G.

Proof. Clearly $\mathcal{D}$ has the weak Vitali property on every $E \subset G$. Given $\varepsilon > 0$, then let $\{E_n\}$, $\{p_n\}$ be the sequences as furnished by Definition 2.6. Then

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$$\mu(E \setminus \cup\{E_n\}) < \varepsilon - \sum \varepsilon_n$$

for a suitable sequence of positive numbers ε_n, and, for each n, Lemma 2.1 applies to any $F_n \subset E_n$ with $p = p_n$.

Now, let $\mathcal{V}$ be a measurable Vitali $\mathcal{D}$-covering of E. We define, recursively, a sequence of families $\mathcal{U}_n \subset \mathcal{V}$ as follows. Let $W_0 = \emptyset$, and W_n denote the union of the members of $\mathcal{U}_n$, whenever defined. We set

$$\mathcal{V}_n = \{V \in \mathcal{V} \mid V \cap \bigcup_{i=1}^{n} \bar{W}_{i-1} = \emptyset\},$$

and note that $\mathcal{V}_n$ is a measurable Vitali $\mathcal{D}$-covering of $F_n = E_n \setminus \bigcup_{i=1}^{n} \bar{W}_{i-1}$, and, by Lemma 2.1, $\mathcal{D}$ has the finite covering property on F_n. Then we select a finite family $\mathcal{U}_n$ of pairwise disjoint members of $\mathcal{V}_n$ such that $\mu(F_n \setminus W_n) < \varepsilon_n$, that is,

$$\mu(E_n \setminus \bigcup_{i=1}^{n} \bar{W}_i) < \varepsilon_n .$$

Then, taking $\mathcal{U}_0 = \cup\{\mathcal{U}_n\}$ and noting that the sets W_i are μ-closed, we have

$$\begin{aligned}\mu(E \setminus \cup\mathcal{U}_0) &\leqslant \mu(\cup\{E_n\} \setminus \cup\mathcal{U}_0) + \mu(E \setminus \cup\{E_n\}) \\ &\leqslant \mu(E_n \setminus \bigcup_{i=1}^{n} \bar{W}_i) + \varepsilon - \sum \varepsilon_n \\ &< \sum \varepsilon_n + \varepsilon - \sum \varepsilon_n = \varepsilon .\end{aligned}$$

Since $\mathcal{U}_0$ is a countable family of pairwise disjoint measurable sets, and since $\mu(E) \leqslant \mu(G) < \infty$, it follows that there is a finite subfamily $\mathcal{V}_0 \subset \mathcal{U}_0$ satisfying $\mu(E \setminus \cup\mathcal{V}_0) < \varepsilon$. This proves the lemma.

LEMMA 2.3. *Let* $\{A_n\}$ *be an increasing sequence of subsets of* X, *and let* $\mathcal{D}(x) = \bigcup_{i=1}^{n} \mathcal{D}_n(x)$ *for all* $x \in X$, *where* $\{\mathcal{D}_n\}$ *is an increasing sequence of Vitali covering functions on* X *such that, for each* n, $\mathcal{D}_n$

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has the finite covering property on A_n . *Then, for every measurable Vitali* $\mathcal{D}$*-covering* $\mathcal{V}$ *of* $E = \cup\{A_n\}$, *there is a countable family* $\mathcal{V}_0$ *of pairwise disjoint members of* $\mathcal{V}$ *such that* $\mu(E \setminus \cup\mathcal{V}_0) = 0$.

Proof. Let E_n denote the set of points $x \in A_n$ for which there is a $B \in \mathcal{D}_n(x)$ such that $B \subset V$. Then, by the monotonicity of $\{A_n\}$ and $\{\mathcal{D}_n\}$, we have

$$E = \cup\{E_n\} \quad \text{and} \quad E_n \subset E_{n+1} \quad \text{for all } n .$$

Now we define, recursively, a sequence of families $\mathcal{U}_n \subset \mathcal{V}$ as follows. Let $W_0 = \emptyset$, and W_n denote the union of the members of $\mathcal{U}_n$, whenever defined. We set

$$\mathcal{V}_n = \{V \in \mathcal{V} \cap \mathcal{D}_n[X] \mid V \cap \bigcup_{i=1}^{n} \bar{W}_{i-1} = \emptyset\} ,$$

and note that $\mathcal{V}_n$ is a measurable Vitali $\mathcal{D}_n$-covering of $F_n = E_n \setminus \bigcup_{i=1}^{n} \bar{W}_{i-1} \subset A_n$, and that $\mathcal{D}_n$ has the finite covering property on F_n . Then we select a finite family $\mathcal{U}_n$ of pairwise disjoint members of $\mathcal{V}_n$ such that $\mu(F_n \setminus W_n) < 1/n$, that is,

$$\mu(E_n \setminus \bigcup_{i=1}^{n} \bar{W}_i) < 1/n .$$

Then the proof is completed by taking $\mathcal{V}_0 = \cup\{\mathcal{V}_n\}$, and by noting that $\mu(E \setminus \cup\mathcal{V}_0) \leq \sum \mu(E_n \setminus \cup\mathcal{V}_0) = 0$, since, the sets W_i being μ-closed, for every n and every $k > n$ we have

$$\mu(E_n \setminus \cup\mathcal{V}_0) \leq \mu(E_k \setminus \bigcup_{i=1}^{k} W_i) = \mu(E_k \setminus \bigcup_{i=1}^{k} \bar{W}_i) < 1/k .$$

3. Vitali measure space

Henceforth, we shall be dealing with a fixed space $(X, \mu, \{X_n\})$ where X is a topological space, μ is an outer measure defined on the

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power set of X, and $\{X_n\}$ is an increasing sequence of μ-proper μ-open sets with $\cup\{X_n\} = X$ (Thus, μ is σ-finite.). This will simply be called the measure space X.

DEFINITION 3.1. If every open set (and, hence, every Borel set) in X is measurable (μ), then μ is called topological; and then X is called a topological measure space.

DEFINITION 3.2. Let $\{\mathrm{C}_n\}$ be an increasing sequence of Vitali covering functions defined on X, such that, for each n, C_n has the weak Vitali property on X_n. Then the space $(X, \mu, \{X_n\}, \{\mathrm{C}_n\})$ is called a Vitali measure space; if, further, μ is topological, then it is called a topological Vitali measure space.

DEFINITION 3.3. Let $(X, \mu, \{X_n\}, \{\mathrm{C}_n\})$ be a Vitali measure space. We define the increasing sequence of Vitali covering functions C_n^* on X, by letting $\mathrm{C}_n^*(x)$ be the collection of the families V of μ-closed subsets of X for which there is a $\mathrm{B} \in \mathrm{C}_n(x)$ such that, for every $B \in \mathrm{B}$ there is a $V \in \mathrm{V}$, and conversely, with $V \subset B$ and $\mu(B) \leqslant n\cdot\mu(V)$. We also define the Vitali covering functions C, C^* on X, by setting

$$\mathrm{C}(x) = \bigcup_{n=1}^{\infty} \mathrm{C}_n(x) , \quad \mathrm{C}^*(x) = \bigcup_{n=1}^{\infty} \mathrm{C}_n^*(x) .$$

Note. $\mathrm{C}_n(x) \subset \mathrm{C}_n^*(x)$ and $\mathrm{C}(x) \subset \mathrm{C}^*(x)$ for all n, x.

In the sequel, whenever we speak of X as a Vitali measure space, we shall mean the space in Definition 3.2.

THEOREM 3.1 (Vitali covering theorem). *Let X be a Vitali measure space, and let V be a measurable Vitali C-covering of a subset $E \subset X$. Then there is a countable family V_0 of pairwise disjoint members of V such that $\mu(E \setminus \cup\mathrm{V}_0) = 0$.*

Proof. By Lemma 2.2, every C_n has the finite covering property on X_n. Then the theorem follows from Lemma 2.3, by noting that $E = \cup\{E \cap X_n\}$ and $E \cap X_n \subset E \cap X_{n+1}$ for all n.

DEFINITION 3.4. Let X be a Vitali measure space, and let $E \subset X$ and $x \in X$. The supremum [infimum] of the numbers ℓ, for which there is a $\mathsf{V} \in \mathsf{C}^*(x)$ such that $\mu(E \cap V) \geqslant \ell \cdot \mu(V)$ $[\mu(E \cap V) \leqslant \ell \cdot \mu(V)]$ for every $V \in \mathsf{V}$, is called the upper [lower] density, $\bar{d}(E,x)$ $[\underline{d}(E,x)]$, of E at x. If $\bar{d}(E,x) = \underline{d}(E,x)$, then this common equal value is called the density, $d(E,x)$, of E at x.

Note. $0 \leqslant \underline{d}(E,x) \leqslant \bar{d}(E,x) \leqslant 1$, since when every V belongs to $\mathsf{C}^*(x)$, so does every x-convergent subfamily of V.

Our primary interest lies in the following eight properties in a Vitali measure space X, with special attention to the last four in the general measure space X.

(μ_1) (Outer regularity property). For every $E \subset X$ and $\varepsilon > 0$, there is a μ-open set $G \supset E$ with $\mu(G) \leqslant \mu(E) + \varepsilon$.

(μ_2) (Vitali covering property). If V is a Vitali C^*-covering of a subset $E \subset X$, then for every $\varepsilon > 0$ there is a countable family V_0 of pairwise disjoint members of V such that

$$\Sigma\mu(\mathsf{V}_0) \leqslant \mu(E) + \varepsilon \quad \text{and} \quad \mu(E \setminus \cup\mathsf{V}_0) = 0 .$$

(By $\Sigma\mu(\mathsf{V}_0)$ we mean the sum of all $\mu(V)$, $V \in \mathsf{V}_0$.)

(μ_3) (Density property). Every $E \subset X$ has density 1 a.e. on E.

(μ_4) (Basic Vitali covering property). If V is a Vitali C_n-covering of a subset $E \subset X_n$, for some n, then for every $\varepsilon > 0$ there is a finite family V_0 of pairwise disjoint members of V such that

$$\Sigma\mu(\mathsf{V}_0) < \mu(E) + \varepsilon \quad \text{and} \quad \mu(E \setminus \cup\mathsf{V}_0) < \varepsilon .$$

(μ_5) (Inner regularity property). For every measurable $E \subset X$ and $\varepsilon > 0$, there is a μ-closed set $F \subset E$ with $\mu(E \setminus F) < \varepsilon$.

(μ_6) (Lusin property). If $f : E \to Y$ is measurable, where $E \subset X$ and Y is a Borel countable space, then for every $\varepsilon > 0$ there is a μ-closed set $F \subset E$ with $\mu(E \setminus F) < \varepsilon$, such that $f|F$ is continuous.

(μ_7) (Basic Vitali-Carathéodory property). If $f : X \to [0,\infty]$ is measurable and finite a.e., then for every $\varepsilon > 0$ there is a measurable lower semicontinuous function $h : X \to [0,\infty]$, such that $h \geqslant f$ a.e. and $\int_X (h-f)\,d\mu < \varepsilon$.

(μ_8) (Vitali-Carathéodory property). If $f : X \to [-\infty,\infty]$ is measurable and finite a.e., then there exist two sequences of measurable functions $\ell_n, u_n : X \to [-\infty,\infty]$ satisfying the following conditions:

(i) for every n, ℓ_n is lower semicontinuous and u_n is upper semicontinuous;

(ii) $\ell_{n+1} \leqslant \ell_n$ and $u_{n+1} \geqslant u_n$ for all n;

(iii) for every n, $\inf \ell_n > -\infty$ and $\sup u_n < \infty$;

(iv) for every n, $\ell_n \geqslant f \geqslant u_n$ a.e.;

(v) $\lim_n \ell_n(x) = f(x) = \lim_n u_n(x)$ a.e. on X;

(vi) if f is integrable (μ) on a measurable set $E \subset X$, so are the functions ℓ_n and u_n, and, further,

$$\lim_n \int_E \ell_n d\mu = \int_E f d\mu = \lim_n \int_E u_n d\mu .$$

We prove below a series of propositions, which by successive implications lead us to the following main results.

THEOREM 3.2. *(a) In a topological Vitali measure space X in which μ is regular, the properties (μ_1) through (μ_8) are mutually equivalent.*

(b) In a topological measure space X, the properties (μ_5) through (μ_8) are mutually equivalent.

(c) A Vitali measure space X has all the properties (μ_1) through (μ_8) if and only if μ is topological and (μ_1) holds.

(d) If a Vitali measure space X has the property (μ_4), then μ is topological and regular, and all the properties (μ_1) through (μ_8) hold.

Part (a) follows from Propositions 3.3 through 3.11, and (b) from Propositions 3.7 through 3.10; both (c) and (d) follow from (a) and Propositions 3.1, 3.2.

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The following examples illustrate the above results.

EXAMPLE 3.1. Let λ denote the Lebesgue outer measure on the real line $\mathbb{R}$. Fix $M \subset \mathbb{R}$ with inner measure equal to 0, such that $\lambda(A \cap M) = \lambda(A)$ for every λ-measurable set A (see Halmos [6]; Theorem E, p. 70). Define

$$\mu(E) = \lambda(E \cap M) \quad \text{for every} \quad E \subset R .$$

Then μ is an outer measure in $\mathbb{R}$, and every λ-measurable set is μ-measurable with equal value of the measure. In particular, μ is topological. Given any $E \subset \mathbb{R}$ and $\varepsilon > 0$, there is an open set $G_0 \supset E \cap M$ with $\lambda(G_0) \leqslant \lambda(E \cap M) + \varepsilon$. Since $\mu(E \setminus M) = 0$, the set $G = G_0 \cup (E \setminus M)$ is μ-open, $G \supset E$, and we have $\mu(G) = \mu(G_0) = \lambda(G_0) \leqslant \mu(G) + \varepsilon$, which verifies the property (μ_1). If, for each n, we define $I_n = (-n, n)$ and $C_n(x)$ to be the collection of all x-convergent families of closed intervals containing x, then the classical Vitali covering theorem trivially implies the weak Vitali property of C_n on I_n under μ. Thus $(\mathbb{R}, \underline{\mu}, \{I_n\}, \{C_n\})$ becomes a topological Vitali measure space satisfying (μ_1), and, hence, also every (μ_i). We note, however, that there is an abundance of subsets $E \subset \mathbb{R} \setminus M$ with $\lambda(E) > 0$, and for every such E we have $\mu(E) = 0$, while $\mu(H) = \lambda(H) \geqslant \lambda(E) > 0$ for every open set $H \supset E$. This shows that we cannot, in general, replace 'μ-open' by 'open' in (μ_1). Also, for any λ-measurable set A with $\lambda(A) > 0$, the set $A \cap M$ is μ-measurable and $\mu(A \cap M) = \lambda(A) > 0$, while $\mu(F) = \lambda(F) = 0$ for every closed set $F \subset A \cap M$, since M has inner λ-measure 0. This shows that we cannot, in general, replace 'μ-closed' by 'closed' in (μ_5) and (μ_6).

EXAMPLE 3.2. With $\mathbb{R}$, λ, M, $\{I_n\}$ and $\{C_n\}$ as in the preceding example, we now define

$$\mu(E) = \lambda(E) + \lambda(E \cap M) \quad \text{for every} \quad E \subset \mathbb{R} .$$

Then μ is an outer measure in $\mathbb{R}$, and a set is μ-measurable if and only if it is λ-measurable. In particular, μ is topological. Given any measurable set A and $\varepsilon > 0$, there is a closed set $F \subset A$ with

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$\lambda(A \setminus F) < \varepsilon/2$. Then $\mu(A \setminus F) = \lambda(A \setminus F) + \lambda((A \setminus F) \cap M) < \varepsilon$, which verifies (μ_5). Also, as before, C_n has the weak Vitali property on I_n under μ. Thus $(\mathbb{R}, \mu, \{I_n\}, \{\mathsf{C}_n\})$ becomes a topological Vitali measure space satisfying (μ_5), and, hence also (μ_6), (μ_7) and (μ_8). We note, however, that (μ_1) does not hold, since, for any $E \subset \mathbb{R} \setminus M$ and any μ-open set $G \supset E$, we have $\mu(G) = 2 \cdot \lambda(G) \geqslant 2 \cdot \lambda(E) = 2 \cdot \mu(E)$. So also, μ is not regular and (μ_4) does not hold.

PROPOSITION 3.1. *If μ is topological and (μ_1) holds, then μ is regular; specifically, for every $E \subset X$ there is a measurable set $A \supset E$ with $\mu(A) = \mu(E)$, such that A is the union of a G_δ-set and a set of measure zero.*

Proof. By (μ_1), there is a sequence of μ-open sets $G_n \supset E$ such that $\mu(G_n) \leqslant \mu(E) = 1/n$. Setting $G = \cap\{G_n^o\}$, we have $\mu(G) \leqslant \mu(E)$, and $\mu(E \setminus G) = 0$ since $E \setminus G \subset \cup\{G_n \setminus G_n^o\}$. Since μ is topological, the proof finishes by taking $A = G \cup (E \setminus G)$.

PROPOSITION 3.2. *If (μ_4) holds, then μ is topological and regular.*

Proof. We first show that, if A, B are any two subsets of some X_n, such that there are open sets $G_1 \supset A$ with $G_1 \cap B = \emptyset$ and $G_2 \supset B$ with $G_2 \cap A = \emptyset$, then $\mu(A) + \mu(B) = \mu(A \cup B)$.

Since $\mathsf{V} = \mathsf{C}_n[G_1] \cup \mathsf{C}_n[G_2]$ is a Vitali C_n-covering of $A \cup B \subset X_n$, given $\varepsilon > 0$ there is, by (μ_4), a subfamily $\mathsf{V}_0 \subset \mathsf{V}$ such that

$$\Sigma\mu(\mathsf{V}_0) < \mu(A \cup B) + \varepsilon \quad \text{and} \quad \mu((A \cup B) \setminus \cup\mathsf{V}_0) < \varepsilon$$

If $\mathsf{V}_1 = \{V \in \mathsf{V}_0 \mid V \cap A \neq \emptyset\}$ and $\mathsf{V}_2 = \{V \in \mathsf{V}_0 \mid V \cap B \neq \emptyset\}$, then

$$\mu(A \setminus \cup\mathsf{V}_1) = \mu(A \setminus \cup\mathsf{V}_0) \leqslant \mu((A \cup B) \setminus \cup\mathsf{V}_0) < \varepsilon ,$$

$$\mu(B \setminus \cup\mathsf{V}_2) = \mu(B \setminus \cup\mathsf{V}_0) \leqslant \mu((A \cup B) \setminus \cup\mathsf{V}_0) < \varepsilon .$$

Also, $\mathsf{V}_1 \cap \mathsf{V}_2 = \emptyset$ by the choices of G_1, G_2 and V. So,

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$$\Sigma\mu(V_1) + \Sigma\mu(V_2) \leqslant \Sigma\mu(V_0) < \mu(A \cup B) + \varepsilon .$$

It follows, therefore, that

$$\begin{aligned}\mu(A) + \mu(B) &\leqslant \mu(A \cap \cup V_1) + \mu(A \setminus \cup V_1) \\ &\qquad + \mu(B \cap \cup V_2) + \mu(B \setminus \cup V_2) \\ &\leqslant \Sigma\mu(V_1) + \varepsilon + \Sigma\mu(V_2) + \varepsilon \\ &< \mu(A \cup B) + 3\varepsilon .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, and since μ is subadditive, it follows that $\mu(A) + \mu(B) = \mu(A \cup B)$.

Next we show that, if G is any μ-open subset of some X_n , then $\mu(A \setminus G) + \mu(A \cap G) = \mu(A)$ for every $A \subset X_n$.

Since $\mu(G \setminus G^o) = 0$, we may and do suppose here that G is open. Then $C_n[G]$ is a Vitali C_n-covering of $A \cap G \subset X_n$. So, given $\varepsilon > 0$, by (μ_4) there is a finite subfamily $V \subset C_n[G]$ such that

$$\mu(A \cap \dot{G} \setminus F) < \varepsilon \quad \text{where} \quad F = \cup V \subset G .$$

Now $X \setminus \bar{F} \supset (A \setminus G) \setminus \bar{F}$ and $(X \setminus \bar{F}) \cap (A \cap F) = \emptyset$; also $G \supset A \cap F$ and $G \cap ((A \setminus G) \setminus \bar{F}) = \emptyset$. So, by the preceding result,

$$\mu((A \setminus G) \setminus \bar{F}) + \mu(A \cap F) = \mu(((A \setminus G) \setminus \bar{F}) \cup (A \cap F)) \leqslant \mu(A) .$$

But, F is μ-closed, $G \supset F$ and $\mu(A \cap G) \leqslant \mu(A \cap F) + \mu(A \cap B \setminus F) < \mu(A \cap F) + \varepsilon$. Hence it follows that $\mu(A \setminus G) + \mu(A \cap G) - \varepsilon < \mu(A)$. This implies, as before, the desired result.

Now we show that, for every $A \subset X$, $\mu(A_n) \to \mu(A)$ where $A_n = A \cap X_n$. Since $X = \cup\{X_n\}$, we have $A = \cup\{B_n\}$ where $B_n = A_n \setminus X_{n-1}$, $X_0 = \emptyset$. Also, since X_{i-1} is a μ-open subset of X_i , by the above we have

$$\begin{aligned}\mu(A_i) &= \mu(A_i \cap X_{i-1}) + \mu(A_i \setminus X_{i-1}) \\ &= \mu(A_{i-1}) + \mu(B_i) , \quad A_o = \emptyset .\end{aligned}$$

Therefore $\sum_{i=1}^{n} \mu(B_i) = \mu(A_n)$, whence $\lim_n \mu(A_n) = \Sigma\mu(\{B_i\}) \geqslant \mu(\cup\{B_i\}) = \mu(A)$. Since $A_n \subset A$ for all n , the desired result follows at once.

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Finally, consider any open set $G \subset X$. Since $G \cap X_n$ is μ-open, by the above results we have, for every $A \subset X$,

$$\mu(A) = \lim_n \mu(A \cap X_n)$$

$$= \lim_n [\mu((A \cap X_n) \cap (G \cap X_n)) + \mu((A \cap X_n) \setminus (G \cap X_n))]$$

$$= \lim_n [\mu(A \cap G \cap X_n) + \mu((A \setminus G) \cap X_n)]$$

$$= \mu(A \cap G) + \mu(A \setminus G) .$$

Hence G is measurable, showing that μ is topological.

Next, consider any $E \subset X$ and any $\varepsilon > 0$. We define, recursively, a sequence of finite families U_n of μ-closed sets as follows. Let $W_o = \emptyset$, and W_n denote the union of the members of U_n, whenever defined. We set

$$\mathsf{V}_n = \{v \in \mathsf{C}_n[X] \mid v \cap \bigcup_{i=1}^{n} \bar{W}_{i-1} = \emptyset\} ,$$

and note that V_n is a Vitali C_n-covering of $F_n = E_n \setminus \bigcup_{i=1}^{n} \bar{W}_{i-1} \subset X_n$ where $E_n = E \cap X_n$. Then, applying (μ_4), we select a finite family $\mathsf{U}_n \subset \mathsf{V}_n$ such that

$$\mu(W_n) < \mu(F_n) + 2^{-n}\cdot\varepsilon, \ \mu(F_n \setminus W_n) < 2^{-n}\cdot\varepsilon .$$

Now, since μ is topological, the μ-closed sets W_n are clearly measurable; also, they are pairwise disjoint. Therefore, setting $W = \cup\{W_n\}$, we have

$$\mu(W) = \Sigma\mu(W_n) \leqslant \Sigma[\mu(F_n) + 2^{-n}\cdot\varepsilon]$$

$$= \Sigma[\mu(F_n \cap W_n) + \mu(F_n \setminus W_n)] + \varepsilon$$

$$\leqslant \Sigma\mu(E \cap W_n) + \Sigma\, 2^{-n}\cdot\varepsilon + \varepsilon$$

$$\leqslant \mu(E) + 2\varepsilon .$$

Further, as in the last part of the proof of Lemma 2.3, we have $\mu(E \setminus W) = 0$. Thus, we have a measurable set $A = W \cup (E \setminus W)$ such that $A \supset E$ and

$\mu(A) = \mu(W) \leqslant \mu(E) + 2\varepsilon$. This clearly implies that μ is regular; which completes the proof.

PROPOSITION 3.3. *For topological* μ , (μ_1) *implies* (μ_2) .

Proof. We first observe that, since μ is topological, every Vitali covering is measurable. Now, for any n and any $A \subset X_n$, consider any Vitali C_n^*-covering V of A . Recalling Definition 3.3, we can find a Vitali C_n-covering W of A such that, for every $W \in W$ there is a $V \in V$ with $V \subset W$ and $\mu(W) \leqslant n\cdot\mu(V)$. By (μ_1) , there is a μ-open set $G \supset A$ such that

$$\mu(G) \leqslant \mu(A) + \frac{1}{2n}\cdot\mu(A) \ , \tag{1}$$

where we take $G = A$ if $\mu(A) = 0$. Let W_1 denote the family of sets $W \in W$ such that $W \subset G^0$. Then W_1 is a Vitali C_n-covering of $A \cap G^0$. Since, by Lemma 2.2, C_n has the finite covering property on X_n , there is a finite family $\{W_i\}$ of pairwise disjoint members of W_1 such that $\mu(A \cap G^0 \setminus \cup\{W_i\}) \leqslant \mu(A)/(2n)$, that is,

$$\mu(A \cup \{W_i\}) \leqslant (1/2n)\cdot\mu(A) \tag{2}$$

since $A \subset G$ and G is μ-open. For each i , we select a $V_i \in V$ such that $V_i \subset W_i$ and $\mu(W_i) \leqslant n\cdot\mu(V_i)$. Then, noting that the sets W_i , V_i are measurable, we have

$$\begin{aligned}
\mu(A \setminus \cup\{V_i\}) &= \mu(A \cap \cup\{W_i\} \setminus \cup\{V_i\}) + \mu((A \setminus \cup\{W_i\}) \setminus \cup\{V_i\}) \\
&\leqslant \Sigma\mu(W_i \setminus V_i) + (1/2n)\cdot\mu(A) \qquad \text{by } (2) \\
&= \Sigma[\mu(W_i) - \mu(V_i)] + (1/2n)\cdot\mu(A) \\
&\leqslant (1-1/n)\cdot\Sigma\mu(W_i) + (1/2n)\cdot\mu(A)) \\
&\leqslant (1-1/n)\cdot\mu(G) + (1/2n)\cdot\mu(A) \\
&\leqslant (1-1/2n^2)\cdot\mu(A) \qquad \text{by } (1) .
\end{aligned}$$

Hence, by Lemma 2.1, C_n^* has the finite covering property on X_n .

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Let now $\mathcal{V}$ be a Vitali $\mathcal{C}^*$-covering of a subset $E \subset X$, and $\varepsilon > 0$. By (μ_1), there is a μ-open set $G \supset E$ such that $\mu(G) \leqslant \mu(E) + \varepsilon$. Let $\mathcal{V}_1$ denote the family of sets $V \in \mathcal{V}$ such that $V \subset G^0$. Then $\mathcal{V}_1$ is a Vitali $\mathcal{C}^*$-covering of $E \cap G^0$. Since $E \cap G^0$ is the union of the increasing sequence of sets $E \cap X_n \cap G^0$, and since every $\mathcal{C}_n^*$ has the finite covering property on X_n, so, by Lemma 2.3, there is a countable subfamily $\mathcal{V}_0 \subset \mathcal{V}_1$ of pairwise disjoint measurable sets such that $\mu(E \cap G^0 \setminus \cup\mathcal{V}_0) = 0$. Since $\cup\mathcal{V}_0 \subset G^0$ and G is μ-open, we have $\Sigma\mu(\mathcal{V}_0) \leqslant \mu(G) \leqslant \mu(E) + \varepsilon$ and $\mu(E \setminus \cup\mathcal{V}_0) = 0$. This verifies (μ_2) and completes the proof.

PROPOSITION 3.4. *For any* μ, (μ_2) *implies* (μ_3).

Proof. Given $E \subset X$, let E_n denote the set of points $x \in E \cap X_n$ at which $\underline{d}(E,x) < n/(n+1)$. If E_0 is the set of points of E at which E has density 1, then we have $E \setminus E_0 = \cup\{E_n\}$. It is therefore enough to show that $\mu(E_n) = 0$ for each n.

Now, for every $x \in E_n$ there is a $V \in \mathcal{C}^*(x)$ such that $\mu(E \cap V) < (n/(n+1))\cdot\mu(V)$ for every $V \in \mathcal{V}$. The family of all such V constitutes a Vitali $\mathcal{C}^*$-covering of E_n. So, by (μ_2), for any $\varepsilon > 0$ there is a countable family of sets V_i such that

$$\Sigma_i\mu(V_i) \leqslant \mu(E_n) + \varepsilon \quad \text{and} \quad \mu(E_n \setminus \cup\{V_i\}) = 0 ,$$

where $\mu(E \cap V_i) < (n/(n+1)).\mu(V_i)$ for every i. Then

$$\begin{aligned} \mu(E_n) &\leqslant \mu(E_n \cap \cup\{V_i\}) + \mu(E_n \setminus \cup\{V_i\}) \\ &\leqslant \Sigma_i\mu(E_n \cap V_i) \leqslant \Sigma_i\mu(E \cap V_i) \\ &\leqslant (n/(n+1))\ \Sigma_i\mu(V_i) \leqslant (n/(n+1))\cdot[\mu(E_n) + \varepsilon] \end{aligned}$$

Since $\mu(E_n) \leqslant \mu(X_n) < \infty$ and $\varepsilon > 0$ is arbitrary, it follows that $\mu(E_n) = 0$, which completes the proof.

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PROPOSITION 3.5. *For topological* μ , (μ_3) *implies* (μ_4) .

Proof. For any n , let V ve a Vitali C_n-covering of a subset $E \subset X_n$, and let $\varepsilon > 0$. Let E_1 denote the set of points of E at which E has density 1 . For every $x \in E_1$, there is a $\mathsf{B} \in \mathsf{C}_n(x)$ such that $\mathsf{B} \subset \mathsf{V}$, and, since $\underline{d}(E,x) = 1$, there must exist an x-convergent subfamily B_1 of B such that, for every $V \in \mathsf{B}_1$,

$$\mu(E \cap V) \geqslant (1/(1/p))\cdot\mu(V) \quad \text{where} \quad p = \varepsilon/(1+\mu(E)) .$$

So, the subfamily $V_1 \subset \mathsf{V}$ of all such V constitutes a Vitali C_n-covering of E_1 . Also, the members of V_1 are measurable, since μ is topological. Therefore, by Lemma 2.2, there is a finite subfamily $\{V_i\} \subset \mathsf{V}_1$ of pairwise disjoint measurable sets such that $\mu(E_1 \setminus \cup\{V_i\}) < \varepsilon$. Since by (μ_3) , $\mu(E \setminus E_1) = 0$, it follows that $\mu(E \setminus \cup\{V_i\}) < \varepsilon$; further, we have $\Sigma\mu(V_i) \leqslant (1+p)\cdot\Sigma\mu(E \cap V_i) \leqslant (1+p)\cdot\mu(E) < \mu(E) + \varepsilon$. This verifies (μ_4) and completes the proof.

PROPOSITION 3.6. *For any* μ , (μ_4) *implies* (μ_5) .

Proof. Let E be any measurable subset of X , and $\varepsilon > 0$. First suppose E is contained in some X_m . Consider any $p \in (0,\varepsilon)$ and any $q \in (0,p)$. Since $\mathsf{C}_m[X]$ is a Vitali C_m-covering of E , by (μ_4) there is a finite family V of μ-closed sets such that $\mu(E \setminus \cup\mathsf{V}) < q$ and $\Sigma\mu(\mathsf{V}) < \mu(E) + q$. Then $V = \cup\mathsf{V}$ is μ-closed, $\mu(E \setminus V) < q$ and, since E is measurable and $\mu(E) \leqslant \mu(X_m) < \infty$, we have

$$\begin{aligned} \mu(V \setminus E) &= \mu(V) - \mu(V \cap E) \\ &\leqslant \mu(\mathsf{V}) - [\mu(E) - \mu(E \setminus V)] < 2q . \end{aligned}$$

Hence, there is a sequence of μ-closed sets V_n with

$$\mu(E \setminus V_n) < p\cdot 2^{-n} \quad \text{and} \quad (V_n \setminus E) < p\cdot 2^{-n} , \; n - 1,2,\ldots .$$

If $W = \cap\{V_n\}$, then W is μ-closed, and obviously $\mu(W \setminus E) = 0$. Therefore $F = W \cap E$ is μ-closed, $F \subset E$ and, further,

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$\mu(E \setminus F) = \mu(E \setminus W) \leqslant \Sigma\mu(E \setminus V_n) < \Sigma p \cdot 2^{-n} \quad p < \varepsilon$.

In the general case, we set

$$E_n = E \cap X_n \setminus X_{n-1},\ X_0 = \emptyset\ ,\quad n = 1,2,\ldots\ .$$

Then $E_n \subset X_n$ and, since by Proposition 3.2 μ is topological, E_n is measurable. Hence, by above, there is a μ-closed set $F_n \subset E_n$ with $\mu(E_n \setminus F_n) < \varepsilon \cdot 2^{-n}$. Now the set $F = \cup\{F_n\}$ is μ-closed. To see this, it is enough to note that $\mu(X \setminus \cup\{X_n^0\}) = 0$ and that, for each n , $H_n = \bigcup_{i=1}^{n} F_i$ is μ-closed and $(\bar{F} \setminus F) \cap X_n^0 \subset \bar{H}_n \setminus H_n$, since $F \setminus H_n$ is contained in the closed set $X \setminus X_n^0$. Also, we have $F \subset \cup\{E_n\} = E$ and $\mu(E \setminus F) \leqslant \Sigma\mu(E_n \setminus F_n) < \varepsilon \cdot 2^{-n} = \varepsilon$. This verifies (μ_5) and completes the proof.

PROPOSITION 3.7. *For topological* μ , (μ_5) *implies* (μ_6) .

Proof. The proof will be done by a simple modification of the ingenious method of Sion ([14], Theorem 3.5, page 470). We first observe that, if E is any measurable subset of X and $\varepsilon > 0$, then there is an open set G such that $\mu(G \Delta E) < \varepsilon$. In fact, by (μ_5) , there is a μ-closed set $F \subset X \setminus E$ with $\mu((X \setminus E) \setminus F) < \varepsilon$. Then, it suffices to take $G = X \setminus \bar{F}$.

Now, let $f : E \to Y$ be measurable, where $E \subset X$ and Y is a Borel countable space. We fix a countable pseudo base $\{B_n\}$ for Y , where each B_n is a Borel set in Y . Since $f^{-1}(B_n)$ is measurable, by the above there is an open set G_n in X such that $\mu(G_n \Delta f^{-1}(B_n)) < \varepsilon \cdot 2^{-n-1}$.

Setting $A = E \setminus \cup\{G_n \Delta f^{-1}(B_n)\}$, we have

$$\mu(E \setminus A) \leqslant \Sigma\mu(G_n \Delta f^{-1}(B_n)) < \Sigma\varepsilon \cdot 2^{-n-2} = \varepsilon/2\ .$$

Also, consider any $a \in A$. If H is any neighborhood of $f(a)$ in Y , then $f(a) \in B_n \subset H$ for some n . Since $a \in A \cap f^{-1}(B_n)$, so $a \in A \cap G_n$; further, $f(x) \in B_n \subset H$ for every $x \in A \cap G_n$. Hence $f \mid A$ is continuous. But, since μ is topological and $E = f^{-1}(Y)$ is measurable, clearly A is measurable. So, by (μ_5), there is a μ-closed set $F \subset A$ with $\mu(A \setminus F) < \varepsilon/2$. Then $\mu(E \setminus F) = \mu(E \setminus A) + \mu(A \setminus F) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, and $f \mid F$ is obviously continuous. This verifies (μ_6) and completes the proof.

PROPOSITION 3.8. *For topological* μ, (μ_6) *implies* (μ_7).

Proof. Let $f : X \to [0,\infty]$ be measurable and finite a.e., and $\varepsilon > 0$. Here the space $[0,\infty]$ is assumed to have the usual order topology, so that it is second countable, and, hence, Borel countable. Now, first suppose $M = \sup f < \infty$. By (μ_6), there is a μ-closed set F with $\mu(X \setminus F) < \varepsilon/M$ such that $f \mid F$ is continuous. We define $h(x) = f(x)$ if $x \in F$, $h(x) = M$ if $x \in X \setminus \bar{F}$, and $h(x) = \liminf_{y \to x} (f \mid F)(y)$ if $x \in \bar{F} \setminus F$. Then it can be easily seen that h fulfills all the conditions in (μ_7) (semicontinuity implies measurability, since μ is topological).

In the general case, we note that $f = \Sigma f_n$ a.e. on X, where $f_n(x) = f(x)$ if $n - 1 \leqslant f(x) < n$ and $f_n(x) = 0$ otherwise. By above, there is a sequence of nonnegative measurable lower semicontinuous functions h_n, such that $h_n \geqslant f_n$ a.e. on X and $\int_X (h_n - f_n) d\mu < \varepsilon \cdot 2^{-n}$. Then the function $h = \Sigma h_n$ evidently fulfills all the conditions in (μ_7), and this completes the proof.

PROPOSITION 3.9. *For any* μ, (μ_7) *implies* (μ_8).

The proof is similar to that of the Vitali Carathéodory theorem (Saks [*11*]; p. 75), (μ_7) serving for the Lemma used therein.

PROPOSITION 3.10. *For topological* μ, (μ_8) *implies* (μ_5).

Proof. Let E be a measurable subset of some X_m, and $\varepsilon > 0$. Let $f(x) = 1$ if $x \in E$, and $f(x) = 3$ if $x \in X \setminus E$. Then f is clearly measurable. Let $\{\ell_n\}$ denote the sequence of lower semicontinuous measurable functions furnished by (μ_8), corresponding to this f. Set

$$F_n = \{x \in X \mid \ell_n(x) \leqslant 2\} \ , \ n = 1,2,\ldots \ .$$

Then F_n is measurable and closed. Also, the condition $\ell_{n+1} \leqslant \ell_n$ gives $F_n \subset F_{n+1}$, the condition $\lim \ell_n = f$ a.e. gives $\mu(E \setminus \cup\{F_n\}) = 0$, and the condition $\ell_n \geqslant f$ a.e. gives $\mu(F_n \setminus E) = 0$.

Now, $\{E \cap F_n\}$ is an increasing sequence of measurable sets, and, hence, $\lim \mu(E \cap F_n) = \mu(E \cap \cup\{F_n\}) = \mu(E)$, since $\mu(E \setminus \cup\{F_n\}) = 0$. Since $\mu(E) \leqslant \mu(X_m) < \infty$, it follows that $\mu(E \cap F_k) > \mu(E) - \varepsilon$ for sufficiently large k. Since F_k is closed and $\mu(F_k \setminus E) = 0$, so $F = E \cap F_k$ is μ-closed; also, $F \subset E$ and we have $\mu(E \setminus F) = \mu(E \setminus F_k) = \mu(E) - \mu(E \cap F_k) < \varepsilon$. This verifies (μ_5) for the set $E \subset X_m$.

The general case now follows by the last part of the proof of Proposition 3.6.

PROPOSITION 3.11. *For regular* μ, (μ_5) *implies* (μ_1).

Proof. Since μ is regular, for any $E \subset X$ there is a measurable set $A \supset E$ with $\mu(A) = \mu(E)$. Since $X \setminus A$ is measurable, by (μ_5) there is a μ-closed set $F \subset X \setminus A$ such that $\mu((X \setminus A) \setminus F) < \varepsilon$. Then $G = X \setminus F$ is μ-open, $G \supset A \supset E$ and we have $\mu(G) = \mu(A) + \mu(G \setminus A) = \mu(E) + \mu((X \setminus A) \setminus F) \leqslant \mu(E) + \varepsilon$. This verifies (μ_1) and completes the proof.

4. Approximate continuity

In this section, we assume that X is a Vitali measure space satisfying (μ_4), so that, by Theorem 3.2, μ is topological and regular and all the properties (μ_1) through (μ_8) hold. The symbol Y will denote an arbitrary topological space. We shall study measurability of subsets of X and of functions from X to Y in terms of density and approximate continuity.

We first observe that, since μ is σ-finite and regular, for every $E \subset X$ there is (Halmos [6]; Theorem C, p. 50) a mesaurable set A, called a measurable cover of E, such that $A \supset E$, $\mu(A) = \mu(E)$ and

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$\mu(F) = 0$ for every measurable set $F \subset A \setminus E$.

LEMMA 4.1. *If* A *is any measurable cover of a subset* $E \subset X$ *, then* $\underline{d}(A,x) = \underline{d}(E,x)$ *and* $\bar{d}(A,x) = \bar{d}(E,x)$ *for all* $x \in X$.

Proof. Given any measurable set V , clearly we can find a measurable cover B of $E \cap V$ such that $B \subset A \cap V$. Then, since $A \cap V \setminus B$ is a measurable subset of $A \setminus E$, we have $\mu(A \cap V \setminus B) = 0$, and, hence, $\mu(A \cap V) = \mu(B) = \mu(E \cap V)$.

Now, since μ is topological, every μ-closed set is measurable. Hence, by above, $\mu(A \cap V) = \mu(E \cap V)$ for every $V \in C^*[X]$, whence the lemma follows at once.

THEOREM 4.1. *A subset* $E \subset X$ *is measurable if and only if* E *has density* 0 *a.e. on* $X \setminus E$.

Proof. By density property (μ_3) , $X \setminus E$ has density 1 a.e. on itself. Also, when E is measurable, then $\mu(V) = \mu(V \cap E) + \mu(V \cap (X \setminus E))$ for every $V \in C^*[X]$. From this the 'only if' part follows quite readily.

Next, suppose E has density 0 a.e. on $X \setminus E$. Then, choosing a measurable cover A of E , by Lemma 4.1 A has density 0 a.e. on $A \setminus E$, which by density property implies that $\mu(A \setminus E) = 0$. Since A is a measurable superset of E , it follows that E is measurable, and the proof ends.

DEFINITION 4.1. A function $f : X \to Y$ is said to be approximately continuous at a point $x \in X$ if, for every neighborhood H of $f(x)$ in Y , $X \setminus f^{-1}(H)$ has density 0 at x.

The following theorem extends a result of Sion [14] (Corollary 3.10, p. 473), mainly by relaxing his hypothesis that Y has a countable base. Sion's proof of the only if part involves, as usual, the Lusin property, which is not available in our general case. However, a more direct proof applies.

THEOREM 4.2. *Let* Y *be a pseudo countable space, then a function* $f : X \to Y$ *is measurable if and only if it is approximately continuous a.e. on* X . *(The restriction on* Y *is unnecessary for the 'if' part.)*

Proof. Suppose f is approximately continuous a.e. on X . Then, for any open set H in Y, $X \setminus f^{-1}(H)$ has density 0 a.e. on $f^{-1}(H)$.

Therefore, by Theorem 4.1, $X \setminus f^{-1}(H)$ is measurable, and, hence, $f^{-1}(H)$ is measurable. Thus f is measurable, proving the 'if' part.

To prove the 'only if' part, let $\{H_n\}$ denote a countable pseudo base for Y. By the density property (μ_3), $f^{-1}(H_n)$ has density 1 at each point of $f^{-1}(H_n) \setminus A_n$ for some $A_n \subset f^{-1}(H_n)$ with $\mu(A_n) = 0$. Let $E = X \setminus \cup\{A_n\}$, then clearly $\mu(X \setminus E) = 0$. Also, consider any $x \in E$. If H is any open neighborhood of $f(x)$ in Y, then $f(x) \in H_n \subset H$ for some n. But $x \notin A_n$ since $x \in E$. So $x \in f^{-1}(H_n) \setminus A_n$, and, hence, $f^{-1}(H_n)$ has density 1 at x. Since $f^{-1}(H_n) \subset f^{-1}(H)$, and since measurability of f implies that $f^{-1}(H)$ is measurable, it readily follows that $x \setminus f^{-1}(H)$ has density 0 at x. Thus f is approximately continuous at each point of E, which completes the proof.

Goffman and Waterman [4] (Theorem 1, p. 117) showed that the range of an approximately continuous function from an euclidean space to a metric space is separable. In our final theorem, following the definitions below, we prove a much wider and stronger result.

DEFINITION 4.2. A family of nonvoid subsets of a subset $Y_o \subset Y$ is said to be dense in Y_o if every neighborhood of every point of Y_o contains a member of the family.

DEFINITION 4.3. A function $f : X \to Y$ is called μ-positive at a point $x \in X$ if $\mu(f^{-1}(H)) > 0$ for every neighborhood H of $f(x)$ in Y.

DEFINITION 4.4. A sequence $\{\mathrm{H}_n\}$ of open coverings of the space Y is called a contraction for Y if, for every $y_o \in Y$ and every neighborhood H_o of y_o, there is a k such that $y_o \in H \in \mathrm{H}_k$ implies $H \subset H_o$. The space Y is called contractive if there is a contraction for Y.

Every pseudo metrizable space is contractive. For, if p is a pseudo metric compatible with the topology of Y, let H_n denote the

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family of open spheres of p-radius $1/n$. Then each H_n is an open covering of Y. Also, given any neighborhood H_o of any point $y_o \in Y$, there is a k such that H_o contains the open sphere of p-radius $2/k$ about y_o. Using triangle inequality, we readily obtain that $y_o \in H \in \mathsf{H}_k$ implies $H \subset H_o$. Hence $\{\mathsf{H}_n\}$ is a contraction for Y, and Y is contractive.

On the other hand, consider the set $I = [0,1]$ endowed with the topology consisting of all subsets $E \subset I$ such that $E \cap \{0,1\} \neq \emptyset$ implies $(0,x) \subset E$ for some $x \in (0,1)$. Since $\{0\}$ and $\{1\}$ are disjoint closed sets having no disjoint neighborhoods, the space is not normal, and, hence, not pseudo metrizable (Kelley [8]; Theorem 10, p. 120). In connection with our proposed theorem, it may also be noted that the space I is not separable ($\{x\}$ is open for every $x \in (0,1)$). But, let A_n denote the family of the sets $[0,1/n)$, $\{1\} \cup (0,1/n)$ and all singletons $\{x\}$ with $x \in (0,1)$. Then it can be readily verified that $\{\mathsf{A}_n\}$ is a contraction for the space I, so that I is contractive.

THEOREM 4.3. *Let Y be contractive, and let $f : X \to Y$ be μ-positive everywhere and approximately continuous a.e. on X. Then, for every $\varepsilon \in (0,1/2)$, there is a countable family U of μ-closed subsets of X such that for every $F \in \mathsf{U}$ there is a $B \in \mathsf{C}_1[X]$ with $F \subset B$ and $\mu(F) > (1 - \varepsilon)\mu(B)$, and such that the countable family $\{f(F) \mid F \in \mathsf{U}\}$ is dense in $f(X)$.*

Proof. Let $\{\mathsf{H}_n\}$ denote a contraction for Y, and let A_o denote the set of points of X at which f is not approximately continuous. We note that $\mu(A_o) = 0$, and by Theorem 4.2 f is measurable.

Consider any H_n, any $H \in \mathsf{H}_n$, and any $x \in E \setminus A_o$ where $E = f^{-1}(H)$. Since H is open, clearly E has density 1 at the point x. So, there is a $\mathsf{B} \in \mathsf{C}_1(x)$ such that $\mu(E \cap B) > (1 - \varepsilon)\cdot\mu(B)$ for every $B \in \mathsf{B}$. Since $E \cap B$ is measurable, by inner regularity property (μ_5) there is a μ-closed set $B^* \subset E \cap B$ with $\mu(E \cap B \setminus B^*) < \mu(E \cap B) - (1 - \varepsilon)\mu(B)$. Then, we have $\mu(B^*) > (1 - \varepsilon)\cdot\mu(B) > (1/2)\cdot\mu(B)$. Since $\mathsf{C}_1(x) \subset \mathsf{C}_2(x)$, it follows that $\{B^* \mid B \in \mathsf{B}\} \in \mathsf{C}_2^*(x)$.

Now, let V_n denote the family of μ-closed sets $F \in \mathsf{C}_2^*[X]$ for which there are a set $H \in \mathsf{H}_n$ and a set $B \in \mathsf{C}_1[X]$ such that $F \subset f^{-1}(H) \cap B$ and $\mu(F) > (1 - \varepsilon)\cdot\mu(B)$. Then, since H_n covers Y , the above demonstration shows that V_n is a Vitali C_2^*-covering of $X \setminus A_o$. Therefore, by Vitali covering property (μ_2) , there is a countable subfamily $\mathsf{U}_n \subset \mathsf{V}_n$ such that

$$\mu(A_n) = 0 \quad \text{where} \quad A_n = (X \setminus A_o) \setminus \cup\mathsf{U}_n .$$

Let U denote the countable family $\cup\{\mathsf{U}_n\}$ of μ-closed sets thus obtained. Then, by construction, for every $F \in \mathsf{U}$ there is a $B \in \mathsf{C}_1[X]$ such that $F \subset B$ and $\mu(F) > (1 - \varepsilon)\cdot\mu(B)$. Also, consider any $x_o \in X$ and any open neighborhood H_o of $f(x_o)$ in Y . Since f is μ-positive at x_o , we have $\mu(f^{-1}(H_o)) > 0$. But, $\mu(A) = 0$ where $A = \cup\{A_{n-1}\}$. Therefore, there is at least one point $u_o \in f^{-1}(H_o) \setminus A$. Then, since H_o is a neighborhood of $f(u_o)$, and since $\{\mathsf{H}_n\}$ is a contraction for Y, there is a k such that $H_o \supset H$ whenever $f(u_o) \in H \in \mathsf{H}_k$. Now, $u_o \notin A_o \cup A_k$ since $u_o \notin A$. So, $u_o \in F$ for some $F \in \mathsf{U}_k$, and, by construction, there is an $H \in \mathsf{H}_k$ such that $F \subset f^{-1}(H)$. Then $f(u_o) \in H \in \mathsf{H}_k$, and, hence, $H \subset H_o$. Therefore $f(F) \subset H \subset H_o$. This shows that $\{f(F) \mid F \in \mathsf{U}\}$ is dense in $f(X)$ and the proof ends.

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