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DEPENDENT AUTOMORPHISMS IN PRIME RINGS

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ABSTRACT. For each $n \geq 4$ we construct a class of examples of a minimal C-dependent set of n automorphisms of a prime ring R, where C is the extended centroid of R. For $n = 4$ and $n = 5$ it is shown that the preceding examples are completely general, whereas for $n = 6$ an example is given which fails to enjoy any of the nice properties of the above example.

1. **Introduction.** Let R be a prime ring with extended centroid C, central closure $A = RC$, and symmetric ring of quotients Q. We note that the ring $\mathrm{End}(Q)$ of endomorphisms of $(Q, +)$ is a right C-space. Let $G = \mathrm{Aut}(R)$, the group of automorphisms of R, and let G_i be the normal subgroup of X-inner automorphisms of R. We recall that $g \in G$ is said to be X-inner if there exists an invertible element s in Q such that $x^g = sxs^{-1}$ for all $x \in R$. Such an element s is called a normalizing element for R and the set of all such s will be denoted by N. We let G_0 be a set of representatives of G modulo G_i. It is well-known (see, *e.g.*, [B, Prop. 2.5.3]) that any $g \in G$ can be extended uniquely to an automorphism of Q and so we have $G \subseteq \mathrm{Aut}(Q) \subseteq \mathrm{End}(Q)$.

A set $S = \{g_1, g_2, \ldots, g_n\}$ of n distinct elements of G is a dependent set if there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$, not all zero, such that

$$\sum_{i=1}^{n} g_i \lambda_i = 0.$$

If each λ_i is nonzero then we shall say that S is a nontrivial dependent set. If no proper subset of S is a dependent set we shall say that S is a minimal dependent set. Clearly any minimal dependent set is nontrivial. It is also natural at this point to define $S = \{g_1, g_2, \ldots, g_n\}$ to be equivalent to $T = \{h_1, h_2, \ldots, h_n\}$ if there exists $g \in G$ such that $h_i = gg_i$, $i = 1, 2, \ldots, n$. Clearly the properties of dependence, nontrivial dependence, and minimal dependence are each invariant under this relation, with the coefficients λ_i remaining unchanged.

Before outlining our paper, we shall mention a few words of motivation. Certainly one should begin by citing the well-known theorem of Artin that any set of distinct automorphisms of a field C is independent over C (see [B, Theorem 7.6.6], for a more general result). Thus in our situation, if R is commutative, there are no dependent sets of automorphisms. Next, at least two special cases of linearly dependent automorphisms of prime rings have already been treated in the literature. The first one is the case of algebraic automorphisms (see, *e.g.*, [B, pp. 371–374]). The second one is the case when $g, h \in G$

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satisfy $g+g^{-1}=h+h^{-1}$. The problem of characterizing automorphisms g, h satisfying this identity has been considered extensively in von Neumann and C^* algebras (see, *e.g.*, [A] and references given there), and has recently been solved for prime rings [C].

In Section 2 we use the Kharchenko theory of generalized identities with automorphisms to reduce the study of dependent sets of automorphisms to the situation where the automorphisms are X-inner. We point out (Theorem 2.4) that dependent sets of 2 or 3 automorphisms cannot exist, and from this we conclude (Corollary 2.5) that for $4 \leq n \leq 7$ any nontrivial dependent set $g_1, g_2, \ldots, g_n$ is equivalent to $1, h_2, \ldots, h_n$, h_i X-inner.

In Section 3 we construct, for each $n \geq 2$, a class of examples of minimal dependent sets of $n+2$ automorphisms, namely, if $s \in N$ is algebraic over C of degree n then the set

$$1, \operatorname{inn}(s), \operatorname{inn}(s-\alpha_1), \ldots, \operatorname{inn}(s-\alpha_n)$$

is a minimal dependent set for appropriate $\alpha_1, \alpha_2, \ldots, \alpha_n \in C$. Theorem 3.2 gives a precise statement of this result.

In Section 4 we are able to characterize minimal dependent sets of automorphisms when $n = 4$ or $n = 5$ (Theorem 4.1 and Theorem 4.2), namely, any such minimal dependent set must be of the form given by Theorem 3.2.

In Section 5 we present an example of a minimal dependent set S of 6 inner automorphisms which fails to have the nice properties of the examples given by Theorem 3.2, *e.g.*, S is not a commuting set and not all of its elements are algebraic. This indicates that the problem of characterizing minimal sets of n automorphisms may be a challenging one for $n \geq 6$.

2. **Reduction to X-inner automorphisms.** In approaching the study of dependent set of automorphisms of a prime ring R, we will need to make use of Kharchenko's theory of generalized identities with automorphisms. A complete account of this theory is given, *e.g.*, in [B, Chapter 7], but we will only need the following special case. Let X be a single indeterminate, let the Cartesian product of $\{X\}$ and G be written suggestively as X^G, and let $C\langle X^G\rangle$ be the free C-algebra on the set X^G. One can then form the coproduct

$$Q_C\langle X^G\rangle = Q \coprod_C C\langle X^G\rangle;$$

this is the so-called "home" for generalized identities with automorphisms. We recall that $\phi \in Q_C\langle X^G\rangle$ is called a T-identity on R if ϕ is mapped to 0 under all substitutions of the form

$$X \longrightarrow x, \quad X^g \longrightarrow x^g, \quad x \in R, \quad g \in G.$$

We are in fact only interested in a piece of $Q_C\langle X^G\rangle$, namely, the (Q,Q)-bimodule $QX^{G_0}Q$ of so-called linear reduced elements. In this situation we have available a powerful theorem of Kharchenko [B, Theorem 7.5.6] a special case of which we state as

REMARK 2.1. If $\phi \in QX^{G_0}Q$ is a T-identity on R, then $\phi = 0$.

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We are now in a position to begin our analysis of dependent sets of automorphisms of R. We first make note of the well-known coset decomposition of G relative to G_i. This results from the following equivalence relation: for $g, g' \in G$, $g \equiv g' (\mathrm{mod}\, G_i)$ if $g' = gh$ for some $h \in G_i$. Now let $g_1, g_2, \ldots, g_n$ be any set of elements of G. We sort these into r nonoverlapping subsets, each subset lying in a distinct coset. We will refer to r as the coset number of $g_1, g_2, \ldots, g_n$. Thus, with suitable reordering, these elements may be listed as follows:

$$g_1h_{11}, \ldots, g_1h_{1k_1}; \quad g_2h_{21}, \ldots, g_2h_{2k_2}; \ldots ; g_rh_{r1}, \ldots, g_rh_{rk_r} \tag{2.1}$$

where $h_{ij} = \mathrm{inn}(s_{ij}) \in G_i$, $g_1, g_2, \ldots, g_r$ distinct representatives of G modulo G_i.

We assume now that we have given a dependent set of n distinct automorphisms of R written in the form (2.1). This means that

$$\sum_{j=1}^{k_1} \lambda_{1j} s_{1j} x^{g_1} s_{1j}^{-1} + \sum_{j=1}^{k_2} \lambda_{2j} s_{2j} x^{g_2} s_{2j}^{-1} + \cdots + \sum_{j=1}^{k_r} \lambda_{rj} s_{rj} x^{g_r} s_{rj}^{-1} = 0$$

for all $x \in R$, where the λ_{ij}'s are appropriate elements of C. In $QX^{G_0}Q$ we now set

$$\phi = \sum_{j=1}^{k_1} \lambda_{1j} s_{1j} X^{g_1} s_{1j}^{-1} + \cdots + \sum_{j=1}^{k_r} \lambda_{rj} s_{rj} X^{g_r} s_{rj}^{-1}.$$

By Remark 2.1 we conclude that ϕ is the zero element of $QX^{G_0}Q$ which means that

$$\sum_{j=1}^{k_i} \lambda_{ij} s_{ij} X^{g_i} s_{ij}^{-1} = 0, \quad i = 1, 2, \ldots, r$$

or, equivalently,

$$\sum_{j=1}^{k_i} \lambda_{ij} s_{ij} \underset{C}{\otimes} s_{ij}^{-1} = 0, \quad i = 1, 2, \ldots, r.$$

For $r = 1$ we restate the preceding observations.

REMARK 2.2. If $g_1, g_2, \ldots, g_n$ is a dependent set whose coset number is 1 then $g_1, g_2, \ldots, g_n$ is equivalent to $h_1, h_2, \ldots, h_n$ where $h_i = \mathrm{inn}(s_i)$ are X-inner and $h_1 = 1$. Furthermore the dependency is given by

$$\lambda_1 s_1 \otimes s_1^{-1} + \lambda_2 s_2 \otimes s_2^{-1} + \cdots + \lambda_n s_n \otimes s_n^{-1} = 0.$$

Since the coset number of any minimal dependent set is clearly 1 we have

REMARK 2.3. Any minimal dependent set is equivalent to $\{1, h_2, \ldots, h_n\}$, $h_i = \mathrm{inn}(s_i)$ X-inner.

In view of [D, Corollary 5C] it follows from Remark 2.3 that distinct automorphisms of the free noncommutative algebra over a field are independent.

We proceed now to examine some low order cases.

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THEOREM 2.4. *No set of 2 or 3 distinct automorphisms of R can be dependent.*

PROOF. We first assume that g_1, g_2 are dependent, *i.e.*, $g_2 = g_1\lambda$, $\lambda \in C$. Pick any $a, b \in R$ such that ab (and hence $a^{g_1}b^{g_1}$) is nonzero. From

$$\lambda a^{g_1} b^{g_1} = \lambda (ab)^{g_1} = (ab)^{g_2} = a^{g_2} b^{g_2} = \lambda^2 a^{g_1} b^{g_1}$$

we conclude that $\lambda^2 = \lambda$, whence the contradiction $\lambda = 1$.

Next we assume that g_1, g_2, g_3 are dependent. Clearly the coset number is 1 and by Remark 2.2 we have the following tensor product relation

$$1 \otimes 1 + \lambda_2 s_2 \otimes s_2^{-1} + \lambda_3 s_3 \otimes s_3^{-1} = 0$$

where $s_2, s_3 \in N$, 1, s_2, s_3 are pairwise C-independent, and λ_2, λ_3 are nonzero elements of C. It follows that $s_3 = \alpha + \beta s_2$, α, β nonzero elements of C, and so (2.2) may be rewritten as

$$1 \otimes (1 + \lambda_3 \alpha s_3^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \lambda_3 \beta s_3^{-1}) = 0.$$

This implies that $1 + \lambda_3 \alpha s_3^{-1} = 0$, whence the contradiction $s_3^{-1} \in C$.

COROLLARY 2.5. *Any nontrivial dependent set $g_1, g_2, \ldots, g_n$, where $4 \leq n \leq 7$, is equivalent to $1, h_2, \ldots, h_n$, h_i X-inner.*

PROOF. If the coset number $r > 1$ then there is a dependent set of at most 3 elements. But this is ruled out by Theorem 2.4. The conclusion then follows from Remark 2.2.

3. **A class of minimal dependent sets.**

Our aim in this section is to produce for each $n \geq 2$ a class of examples of a minimal dependent set of $n + 2$ automorphisms. In order to make this construction we will need the following "inverse" formula.

LEMMA 3.1. *Let s be an invertible element of Q which is algebraic of degree n over C (thus s satisfies a polynomial $f(X) = 1 - \sum_{i=1}^{n} \beta_i X^i$, $\beta_i \in C$, $\beta_n \neq 0$). Let $\alpha \in C$ be such that $f(\alpha) \neq 0$. Then $s - \alpha$ is invertible with*

$$(s - \alpha)^{-1} = \frac{1}{f(\alpha)} \sum_{i=1}^{n} \beta_i (s^{i-1} + \alpha s^{i-2} + \cdots + \alpha^{i-1}).$$

PROOF. For $i = 1, 2, \ldots, n$ we may write

$$s^i = (s - \alpha)(s^{i-1} + \alpha s^{i-2} + \cdots + \alpha^{i-1}) + \alpha^i.$$

Then we have

$$1 = \sum_{i=1}^{n} \beta_i s^i = (s - \alpha) \sum_{i=1}^{n} \beta_i (s^{i-1} + \alpha s^{i-2} + \cdots + \alpha^{i-1}) + \sum_{i=1}^{n} \beta_i \alpha^i$$

from which the formula for $(s - \alpha)^{-1}$ easily follows.

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THEOREM 3.2. *Let R be a prime ring and let $s \in N$ such that s is algebraic over C of degree n (thus s satisfies $f(X) = 1 - \sum_{i=1}^{n} \beta_i X^i$, $\beta_i \in C$, $\beta_n \neq 0$). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct nonzero elements of C such that for each $i = 1, 2, \ldots, n$ $s - \alpha_i \in N$, and let $g_i = \mathrm{inn}(s - \alpha_i)$. Then 1, $g_0 = \mathrm{inn}(s)$, $g_1, g_2, \ldots, g_n$ is a minimal dependent set of automorphisms of R.*

PROOF. We must show that there is a dependency

$$1\mu + g_0\lambda_0 + g_1\lambda_1 + \cdots + g_n\lambda_n = 0, \quad \mu, \lambda_i \in C \tag{3.1}$$

with the required properties but no dependency of any proper subset of $1, g_0, g_1, \ldots, g_n$. Using the formula for $(s - \alpha_j)^{-1}$ given by Lemma 3.1 we see that the tensor product formulation of (3.1) is

$$\begin{aligned} 0 = \mu 1 \otimes 1 + \lambda_0 s \otimes \sum_{i=1}^{n} \beta_i s^{i-1} + \sum_{j=1}^{n} \lambda_j (s - \alpha_j) \\ \otimes \frac{1}{f(\alpha_j)} \sum_{i=1}^{n} \beta_i (s^{i-1} + \alpha_j s^{i-2} + \cdots + \alpha_j^{i-1}) \end{aligned} \tag{3.2}$$

The right hand side of (3.2) can be expanded and rewritten as a C-linear combination of the $2n$ terms

$$1 \otimes 1, 1 \otimes s, \ldots, 1 \otimes s^{n-1}, \quad s \otimes 1, s \otimes s, \ldots, s \otimes s^{n-1}. \tag{3.3}$$

Since $1, s, \ldots, s^{n-1}$ are C-independent the terms in (3.3) are C-independent, and so the coefficients of these terms must each be 0. We write these resulting $2n$ homogeneous equations in $n+2$ unknowns $\mu, \lambda_0, \lambda_1, \ldots, \lambda_n$ in the form of a $2n \times (n+2)$ matrix (the rows being in the same order as the terms in (3.3)):

(3.4)
$$\begin{bmatrix} 1 & 0 & \frac{-\alpha_1}{f(\alpha_1)}(\beta_1 + \beta_2\alpha_1 + \cdots + \beta_n\alpha_1^{n-1}) & \cdots & \frac{-\alpha_n}{f(\alpha_n)}(\beta_1 + \beta_2\alpha_n + \cdots + \beta_n\alpha_n^{n-1}) \\ 0 & 0 & \frac{-\alpha_1}{f(\alpha_1)}(\beta_2 + \beta_3\alpha_1 + \cdots + \beta_n\alpha_1^{n-2}) & \cdots & \frac{-\alpha_n}{f(\alpha_n)}(\beta_2 + \beta_3\alpha_n + \cdots + \beta_n\alpha_n^{n-2}) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \frac{-\alpha_1}{f(\alpha_1)}\beta_n & \cdots & \frac{-\alpha_n}{f(\alpha_n)}\beta_n \\ 0 & \beta_1 & \frac{1}{f(\alpha_1)}(\beta_1 + \beta_2\alpha_1 + \cdots + \beta_n\alpha_1^{n-1}) & \cdots & \frac{1}{f(\alpha_n)}(\beta_1 + \beta_2\alpha_n + \cdots + \beta_n\alpha_n^{n-1}) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \beta_n & \frac{1}{f(\alpha_1)}\beta_n & \cdots & \frac{1}{f(\alpha_n)}\beta_n \end{bmatrix}$$

We shall show that (3.4) has rank $\leq n+1$, and thus there is a nontrivial solution, *i.e.*, $1, g_0, g_1, \ldots, g_n$ is a dependent set.

By applying a series of elementary row operations in a methodical way to (3.4) one arrives at the matrix

$$(3.5)\qquad \begin{bmatrix} 1 & 0 & \frac{-\alpha_1^n\beta_n}{f(\alpha_1)} & \cdots & \frac{-\alpha_n^n\beta_n}{f(\alpha_n)} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{f(\alpha_1)}\alpha_1^{n-1}\beta_n & \cdots & \frac{1}{f(\alpha_n)}\alpha_n^{n-1}\beta_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \frac{1}{f(\alpha_1)}\alpha_1\beta_n & \cdots & \frac{1}{f(\alpha_n)}\alpha_n\beta_n \\ 0 & \beta_n & \frac{1}{f(\alpha_1)}\beta_n & \cdots & \frac{1}{f(\alpha_n)}\beta_n \end{bmatrix}.$$

Clearly the above matrix has rank $\leq n+1$.

Our next task is to show that any $n+1$ of the automorphisms $1, g_0, g_1, \ldots, g_n$ form an independent set. This is equivalent to showing that if any single column of the above matrix (3.5) is deleted the resulting $2n \times (n+1)$ matrix has rank $n+1$. No matter which column is omitted we claim that the $(n+1) \times (n+1)$ submatrix consisting of row 1 plus rows $n+1$ through $2n$ has determinant $\neq 0$. Indeed, if either column 1 or 2 is omitted then there is an obvious $n \times n$ Vandermonde determinant. If some other column is omitted then the $(n-1) \times (n-1)$ submatrix consisting of rows $n+1$ through $2n-1$ is a Vandermonde matrix.

We remark that in Theorem 3.2 the condition $s - \alpha \in N$ (provided $f(\alpha) \neq 0$) is always satisfied if $R = Q$. However, if $R \neq Q$ it does not always hold, as shown by the following concrete example (due to Bergman, see [D, Section 4]). Let $R = F\langle x, y\rangle$ subject to $x^2 = 1 = y^2$. Then the "exchange" automorphism $\sigma\colon x \leftrightarrow y$ is X-inner, given by $\sigma = \mathrm{inn}(s)$, $s = x + y$, but for any $\alpha \neq 0$, $s - \alpha \notin N$.

4. **Results for $n = 4$ and $n = 5$.** In this section we first show that any nontrivial dependent set of 4 automorphisms of a prime ring R is not only a minimal dependent set but must in fact be equivalent to the example given by Theorem 3.2. Without loss of generality, therefore, we are considering the equation

$$(4.1)\qquad 1 \otimes 1 + \lambda_2 s_2 \otimes s_2^{-1} + \lambda_3 s_3 \otimes s_3^{-1} + \lambda_4 s_4 \otimes s_4^{-1} = 0$$

where 1, s_2, s_3, s_4 are pairwise independent elements of N and λ_2, λ_3, λ_4 are nonzero elements of C. Suppose $1, s_2, s_3$ are C-independent. Thus we may write $s_4 = \alpha+\beta s_2+\gamma s_3$, $\alpha, \beta, \gamma \in C$, and rewrite (4.1) as

$$(4.2)\qquad 1 \otimes (1 + \alpha\lambda_4 s_4^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \beta\lambda_4 s_4^{-1}) + s_3 \otimes (\lambda_3 s_3^{-1} + \gamma\lambda_4 s_4^{-1}) = 0.$$

An immediate contradiction results since at least one of α, β, γ must be nonzero. Therefore, without loss of generality, we may assume that

$$(4.3)\qquad s_3 = \alpha + \beta s_2, \quad s_4 = \gamma + \delta s_2$$

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where each of $\alpha, \beta, \gamma, \delta$ is a nonzero element of C. We note that s_2, s_3, s_4 commute with each other. Using (4.3) we now rewrite (4.1) as

$$1 \otimes (1 + \alpha\lambda_3 s_3^{-1} + \gamma\lambda_4 s_4^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \beta\lambda_3 s_3^{-1} + \delta\lambda_4 s_4^{-1}) = 0. \tag{4.4}$$

From (4.4) we see that

$$1 + \alpha\lambda_3 s_3^{-1} + \gamma\lambda_4 s_4^{-1} = 0 \tag{4.5}$$

whereupon multiplication of (4.5) by $s_3 s_4$ yields

$$s_3 s_4 + \alpha\lambda_3 s_4 + \gamma\lambda_4 s_3 = 0. \tag{4.6}$$

Substituting (4.3) in (4.6) we have

$$(\alpha + \beta s_2)(\gamma + \delta s_2) + \alpha\lambda_3(\gamma + \delta s_2) + \gamma\lambda_4(\alpha + \beta s_2) = 0$$

and rearranging terms we have

$$\beta\gamma s_2^2 + (\alpha\delta + \beta\gamma + \alpha\lambda_3\delta + \gamma\lambda_4\beta)s_2 + (\alpha\gamma + \alpha\lambda_3\gamma + \alpha\lambda_4\gamma) = 0.$$

Thus s_2 is algebraic of degree 2 over C. The conditions of Theorem 3.2 having now been met, we have now proved

THEOREM 4.1. *Any nontrivial dependent set of* 4 *automorphisms of a prime ring* R *is a minimal dependent set and is equivalent to one of the form* $1, g_1, g_3, g_4$, *where* $g_1 = \mathrm{inn}(s)$, $g_3 = \mathrm{inn}(s - \alpha)$, $g_4 = \mathrm{inn}(s - \beta)$ *are elements of* G_i, α, β *distinct nonzero elements of* C, s *algebraic of degree* 2 *over* C.

A similar though somewhat more complicated result holds for $n = 5$.

THEOREM 4.2. *Any nontrivial dependent set* S *of* 5 *automorphisms of a prime ring* R *is equivalent to* 1, $\mathrm{inn}(s)$, $\mathrm{inn}(s_3)$, $\mathrm{inn}(s_4)$, $\mathrm{inn}(s_5)$ *where*

(a) *if* S *is not minimal then* $s_3 = s - \alpha$, $s_4 = s - \beta$, $s_5 = s - \gamma$, α, β, γ *distinct nonzero elements of* C, s, $s - \alpha$, $s - \beta$, $s - \gamma \in N$, s *algebraic of degree* 2 *over* C.

(b) *if* S *is minimal, then either* $s_3 = s - \alpha$, $s_4 = s - \beta$, $s_5 = s - \gamma$, s *algebraic of degree* 3 *over* C *or* $s_3^{-1} = s^{-1} - \alpha$, $s_4^{-1} = s^{-1} - \beta$, $s_5^{-1} = s^{-1} - \gamma$, s^{-1} *algebraic of degree* 3 *over* C.

PROOF. As usual, without loss of generality, we may begin by considering the equation

$$1 \otimes 1 + \lambda_2 s_2 \otimes s_2^{-1} + \lambda_3 s_3 \otimes s_3^{-1} + \lambda_4 s_4 \otimes s_4^{-1} + \lambda_5 s_5 \otimes s_5^{-1} = 0 \tag{4.7}$$

where 1, s_2, s_3, s_4, s_5 are pairwise independent elements of N and λ_2, λ_3, λ_4, λ_5 are nonzero elements of C.

We first make the assumption that any 3-element subset $\{1, s_i, s_j\}$ of $\{1, s_2, s_3, s_4, s_5\}$ is a dependent set. In particular we may write

$$s_3 = \alpha_1 + \alpha_2 s_2, \quad s_4 = \beta_1 + \beta_2 s_2, \quad s_5 = \gamma_1 + \gamma_2 s_2 \tag{4.8}$$

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where each of α_i, β_i, γ_i, $i = 1, 2$ is a nonzero element of C. We remark that the s_i's commute with each other. Substitution of (4.8) in (4.7) now yields

$$1 \otimes (1 + \alpha_1 \lambda_3 s_3^{-1} + \beta_1 \lambda_4 s_4^{-1} + \gamma_1 \lambda_5 s_5^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \alpha_2 \lambda_3 s_3^{-1} + \beta_2 \lambda_4 s_4^{-1} + \gamma_2 \lambda_5 s_5^{-1}) = 0$$

whence in particular

$$1 + \alpha_1 \lambda_3 s_3^{-1} + \beta_1 \lambda_4 s_4^{-1} + \gamma_1 \lambda_5 s_5^{-1} = 0. \tag{4.9}$$

Multiplication of (4.9) by $s_3 s_4 s_5$ results in

$$s_3 s_4 s_5 + \alpha_1 \lambda_3 s_4 s_5 + \beta_1 \lambda_4 s_3 s_5 + \gamma_1 \lambda_5 s_3 s_4 = 0. \tag{4.10}$$

Substituting (4.8) in (4.10) and expanding in powers of s_2, we readily see that s_2 is algebraic over C of degree ≤ 3.

If S is a minimal dependent set we claim that s_2 must be algebraic of degree 3 over C. Indeed, suppose s_2 is algebraic of degree 2. Without changing the original dependency (4.7) we may replace s_3 by $s_2 - \alpha$ and s_4 by $s_2 - \beta$ for appropriate $\alpha, \beta \in C$. But now we know by Theorem 3.2 that 1, $\mathrm{inn}(s_2)$, $\mathrm{inn}(s_2 - \alpha)$, $\mathrm{inn}(s_2 - \beta)$ is a dependent set, in contradiction to the minimality of S. Therefore we must assume that s_2 is algebraic of degree 3. Again, replacing s_3, s_4, s_5 by $s_2 - \alpha$, $s_2 - \beta$, $s_2 - \gamma$ for appropriate α, β, γ, we have thus established (b) in the present situation. Still in the present situation, if S is not a minimal dependent set we claim that s_2 must be algebraic of degree 2. Indeed, suppose s_2 is algebraic of degree 3. Replacing s_3, s_4, s_5 by $s_2 - \alpha$, $s_2 - \beta$, $s_3 - \gamma$, we obtain from Theorem 3.2 the contradiction that

$$\bigl\{1, \mathrm{inn}(s_2), \mathrm{inn}(s_2 - \alpha), \mathrm{inn}(s_2 - \beta), \mathrm{inn}(s_2 - \gamma)\bigr\}$$

is a minimal dependent set. Thus in the present situation we have established (a).

We now examine the remaining case in which some subset $\{1, s_i, s_j\}$, say $\{1, s_2, s_3\}$, is C-independent. If 1, s_2, s_3, s_4 are independent as well then from (4.7) we have $s_5 = \alpha_1 + \alpha_2 s_2 + \alpha_3 s_3 + \alpha_4 s_4$ and we may rewrite (4.7) as

$$\begin{aligned} &1 \otimes (1 + \alpha_1 \lambda_5 s_5^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \alpha_2 \lambda_5 s_5^{-1}) \\ &\quad + s_3 \otimes (\lambda_3 s_3^{-1} + \alpha_3 \lambda_5 s_5^{-1}) + s_4 \otimes (\lambda_4 s_4^{-1} + \alpha_4 \lambda_5 s_5^{-1}) = 0 \end{aligned} \tag{4.11}$$

an obvious contradiction since, *e.g.*, 1, s_5^{-1} are C-independent. Therefore we may write

$$s_4 = \alpha_1 + \alpha_2 s_2 + \alpha_3 s_3, \quad s_5 = \beta_1 + \beta_2 s_2 + \beta_3 s_3$$

and replace (4.7) by

$$\begin{aligned} &1 \otimes (1 + \alpha_1 \lambda_4 s_4^{-1} + \beta_1 \lambda_5 s_5^{-1}) + s_2 \otimes (\lambda_2 s_2^{-1} + \alpha_2 \lambda_4 s_4^{-1} + \beta_2 \lambda_5 s_5^{-1}) \\ &\quad + s_3 \otimes (\lambda_3 s_3^{-1} + \alpha_3 \lambda_4 s_4^{-1} + \beta_3 \lambda_5 s_5^{-1}) = 0. \end{aligned} \tag{4.12}$$

From (4.12) we conclude that

$$\{1, s_4^{-1}, s_5^{-1}\}, \quad \{s_2^{-1}, s_4^{-1}, s_5^{-1}\}, \quad \{s_3^{-1}, s_4^{-1}, s_5^{-1}\} \tag{4.13}$$

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are each dependent sets. Using strongly the fact that any dependency among 3 elements of $\{1, s_2^{-1}, s_3^{-1}, s_4^{-1}, s_5^{-1}\}$ requires each coefficient to be nonzero, we remark that it follows from the dependencies in (4.13) that any 3-element subset $\{1, s_i^{-1}, s_j^{-1}\}$ of $\{1, s_2^{-1}, s_3^{-1}, s_4^{-1}, s_5^{-1}\}$ is a dependent set. This brings us to the situation analogous to our first assumption, with the s_i^{-1}'s now playing the role of the s_i's. Thus we may write

$$s_3^{-1} = \alpha_1 + \alpha_2 s_2^{-1}, \quad s_4^{-1} = \beta_1 + \beta_2 s_2^{-1}, \quad s_5^{-1} = \gamma_1 + \gamma_2 s_2^{-1}.$$

If S is a minimal dependent set then s_2^{-1} is algebraic of degree 3 and, replacing s_3^{-1}, s_4^{-1}, s_5^{-1} by $s_2^{-1} - \alpha$, $s_3^{-1} - \beta$, $s_4^{-1} - \gamma$, we have established (b). If S is not a minimal dependent set then s_2^{-1} is algebraic of degree 2. It follows that s_2^{-1} (and hence s_3^{-1}, s_4^{-1}, s_5^{-1}) lies in the C-span of $\{1, s_2\}$, and we may then further conclude that s_3, s_4, s_5 also lie in the C-span of $\{1, s_2\}$. Without loss of generality we may then write

$$s_3 = s_2 - \alpha, \quad s_4 = s_2 - \beta, \quad s_5 = s_2 - \gamma$$

and so (a) has been established.

5. **An example for $n = 6$.** We know by Remark 2.3 that any minimal dependent set of automorphisms of a prime ring R is equivalent to a set of X-inner automorphisms. Can one say more in general? One might conjecture that for every $n \geq 4$ the above minimal dependent set of n X-inner automorphisms is in fact one of the examples given by Theorem 3.2. Theorem 4.1 and Theorem 4.2 show that this conjecture is true in case $n = 4$ or $n = 5$. A less demanding conjecture would be that the X-inner automorphisms at least share some properties in common with the examples given by Theorem 3.1. For instance, one could ask if each automorphism was algebraic and/or commuted with each other. However, we now proceed to give an example of a minimal dependent set of 6 inner automorphisms not all of which are algebraic and which do not commute among themselves.

Let R be any closed prime ring with 1 over C containing elements p, q such that

(a) $p^2 = q^2 = -1$
(b) $1, p, pq$ are C-independent
(c) $1, p, qp$ are C-independent
(d) $[pq, p] \neq 0$
(e) pq is transcendental.

For example let $R = \mathbb{H}_1 \coprod_{\mathbb{R}} \mathbb{H}_2$, the coproduct of $\mathbb{H}_1$ and $\mathbb{H}_2$ over the reals $\mathbb{R}$, where $\mathbb{H}_1 = \mathbb{H}_2$ is the quaternions. It is well-known (see, *e.g.*, [D, Theorem 5]) that R is a closed prime ring over the reals $\mathbb{R}$. We choose $p \in \mathbb{H}_1$ with $p^2 = -1$, and $q \in \mathbb{H}_2$, with $q^2 = -1$. Then (b)–(e) follow easily from the fundamental properties of a coproduct. One next verifies that the 6 elements

$$1, \quad pq, \quad p + pq, \quad p - 1, \quad 2p - 1, \quad 2p + pq \tag{5.1}$$

have, respectively, the inverses

$$1, \quad qp, \quad -\tfrac{1}{2}(p - qp), \quad -\tfrac{1}{2}(p + 1), \quad -\tfrac{1}{5}(2p + 1), \quad -\tfrac{1}{5}(2p - qp).$$

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We now consider the 6 inner automorphisms of R determined by the elements in (5.1). To show these are dependent we proceed to solve the following tensor product equation:

$$\begin{aligned}\lambda_1(1\otimes 1)+\lambda_2 pq\otimes qp+\lambda_3(p+pq)\otimes\left[-\tfrac{1}{2}(p-qp)\right]\\ +\lambda_4(p-1)\otimes\left[-\tfrac{1}{2}(p+1)\right]+\lambda_5(2p-1)\otimes\left[-\tfrac{1}{5}(2p+1)\right]\\ +\lambda_6(2p+pq)\otimes\left[-\tfrac{1}{5}(2p-qp)\right]=0.\end{aligned}\tag{5.2}$$

We rewrite (5.2) as

$$\begin{aligned}1\otimes\left[\lambda_1+\frac{\lambda_4}{2}(p+1)+\frac{\lambda_5}{5}(2p+1)\right]\\ +p\otimes\left[-\frac{\lambda_3}{2}(p-qp)-\frac{\lambda_4}{2}(p+1)-\frac{2\lambda_5}{5}(2p+1)-\frac{2\lambda_6}{5}(2p-qp)\right]\\ +pq\otimes\left[\lambda_2 qp-\frac{\lambda_3}{2}(p-qp)-\frac{\lambda_6}{5}(2p-qp)\right]=0.\end{aligned}\tag{5.3}$$

By further expansion we may finally write (5.3) as a C-linear combination of the terms

$$1\otimes 1,\quad 1\otimes p,\quad p\otimes 1,\quad p\otimes p,\quad p\otimes qp,\quad pq\otimes p,\quad pq\otimes qp.$$

The coefficients of these seven terms must each equal 0, and so the solution of (5.2) is equivalent to solving a homogeneous system of seven equations in six unknowns. We note, however, that the coefficient of $1\otimes p$ is $\frac{\lambda_4}{2}+\frac{2\lambda_5}{5}$ and the coefficient of $p\otimes 1$ is $\frac{-\lambda_4}{2}-\frac{2\lambda_5}{5}$. We also note that the coefficient of $pq\otimes p$ is $\frac{-\lambda_3}{2}-\frac{2\lambda_6}{5}$ while the coefficient of $p\otimes qp$ is $\frac{\lambda_3}{2}+\frac{2\lambda_6}{5}$. Therefore the equations corresponding to $p\otimes 1$ and $pq\otimes p$ are redundant, and so the solution of (5.2) in fact reduces to solving five equations in six unknowns, which we write down as the following 5×6 matrix whose rows correspond respectively to the terms $1\otimes 1$, $1\otimes p$, $p\otimes p$, $p\otimes qp$, $pq\otimes qp$:

$$\begin{bmatrix}1 & 0 & 0 & \frac{1}{2} & \frac{1}{5} & 0\\ 0 & 0 & 0 & \frac{1}{2} & \frac{2}{5} & 0\\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{4}{5} & -\frac{4}{5}\\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{2}{5}\\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{5}\end{bmatrix}.$$

First, it is clear that there is a nontrivial solution (five equations in six unknowns) and so the six automorphisms are dependent. Secondly, one may easily check that removal of any one column (*i.e.*, setting any particular $\lambda_j=0$) gives a 5×5 matrix of rank 5, and hence no proper subset of the original six automorphisms is a dependent set.

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