Horseshoes for autonomous Hamiltonian systems using the Melnikov integral

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Abstract. This paper applies the Melnikov method to autonomous perturbations of completely integrable Hamiltonian systems. The forcing of the perturbed system is caused by internal oscillations which are not necessarily decoupled. A unified treatment is presented which relates some results of Holmes and Marsden with a result of Lerman and Umanskii. It is also shown that two forms of the Melnikov function by integrals are in fact equal.

1. *Statement of results*

During the last 20 years, there has been much interest and progress in determining hyperbolic horseshoes for perturbations of Hamiltonian systems. In particular, the existence of such horseshoes implies that the system is not integrable and has chaotic motion. One basic method was introduced by Melnikov [13] using ideas which go back to Poincaré and is often called the Melnikov method or Poincaré–Melnikov–Arnold method. See [12] for a discussion of the various contributions of Arnold and others. Often in these references, the perturbation is an external time periodic perturbation or at least is the effect of an oscillation which decouples (as in the restricted three-body problem considered by Silnikov.) Recently, there have been several papers applying Melnikov's method to autonomous Hamiltonian systems where the perturbation forcing is internal (rather than an external time-dependent forcing) and the oscillation does not decouple. In particular, there are the three papers by Holmes and Marsden (see [8], [9] and [10]) which study orbits homoclinic to hyperbolic closed orbits and quasi-periodic motion on tori, and the paper by Lerman and Umanskii [11] which studies orbits homoclinic to a hyperbolic fixed point. The paper by Gruendler [6] has some related results on higher-dimensional systems but with external time periodic forcing.

This paper presents a unified treatment of these results and gives an alternative formula to that given by Holmes and Marsden for the the general case for systems with an S^1 symmetry. In some of these cases the integrals are only conditionally convergent. We indicate carefully how the sequence of limits of integration must be chosen so that the result has the correct dynamic meaning. Salem *et al.* also noticed that the limits must be carefully chosen (see [19] or [12]).

In either case we consider a Hamiltonian with two degrees of freedom, four total dimensions, with Hamiltonian function

$$H^\varepsilon(z) = H_0(z) + \varepsilon H_1(z) + \varepsilon^2 H_2(z) + \cdots . \tag{1.1}$$

To form the associated vector field or differential equation let

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be the two by two matrix and

$$J = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix}$$

be the standard four by four skew symmetric non-degenerate matrix. For any real-valued function $G(z)$ let $\dot{z} = J\nabla G = X_G$ be the associated Hamiltonian vector field. In particular, for H^ε, let

$$\dot{z} = J\nabla H^\varepsilon = X_{H^\varepsilon}(z) = X^\varepsilon(z)$$
$$= J\nabla H_0 + \varepsilon J\nabla H_1 + \cdots = X_0(z) + \varepsilon X_1(z) + \cdots. \qquad (1.2)$$

We let $z = \phi(t, z_0, \varepsilon)$ be the solutions of $\dot{z} = X^\varepsilon(z)$ with $\phi(0, z_0, \varepsilon) = z_0$. Also, $\phi_0(t, z_0) = \phi(t, z_0, 0)$. The basic assumption for this paper is that X_0 has a second real-valued independent integral F:

$$DF \cdot X_0 \equiv 0 \text{ so } F \text{ is constant on solution of } X_0;$$

$$\nabla F \text{ and } \nabla H_0 \text{ are independent.}$$

(In particular, they are independent on the homoclinic connection Γ discussed below.) With this assumption, F is a valid measure of one of the directions in the energy surface $H_0^{-1}(h)$ which is transverse to the solutions of X_0. A Hamiltonian system in dimension four with such a second integral is called completely integrable.

1A. In the first case, we assume H_0 has a hyperbolic closed orbit γ with a homoclinic connection. In other words, at least one branch of the stable manifold $W^s(\gamma, H_0)$ coincides with a branch of the unstable manifold $W^u(\gamma, H_0)$. Let Σ be a transversal to the flow and so it is three-dimensional. Assume $H_0(\gamma) = 0$. The energy surface $H_0^{-1}(0)$ is also three-dimensional, and their intersection $\Sigma \cap H_0^{-1}(0)$ is two-dimensional. Since X_0 has a second integral, if $W^u(\gamma, H_0)$ and $W^s(\gamma, H_0)$ intersect, they coincide on components of $W^s(\gamma, H_0) - \gamma$ and $W^u(\gamma, H_0) - \gamma$. Let Γ be one such component of $[W^s(\gamma, H_0) - \gamma] \cap W^u(\gamma, H_0)$. (Note that it can be shown that F and H_0 are necessarily dependent on γ since F is constant on $W^s(\gamma, H_0)$, $W^u(\gamma, H_0)$ and γ.)

Now, for z in Γ, let N_z be a plane transverse to Γ at z (or even normal). Note that Γ is two-dimensional so a plane is transverse in the total space. For $\varepsilon > 0$, let $z^u(\varepsilon)$ be a point in $W^u(\gamma_\varepsilon, H_\varepsilon) \cap N_z$ where the perturbed unstable manifold intersects N_z. Such a choice of points can be made smoothly in ε, because γ_ε varies smoothly as a function of ε since γ is hyperbolic and $W^u(\gamma_\varepsilon, H^\varepsilon)$ varies smoothly on compact subsets. Similarly define $z^s(\varepsilon) = W^s(\gamma_\varepsilon, H^\varepsilon) \cap N_z$. Now F can be used to measure the relative displacement of $W^s(\gamma_\varepsilon, H^\varepsilon)$ and $W^u(\gamma_\varepsilon, H^\varepsilon)$; define

$$M(z) = \frac{\partial}{\partial \varepsilon} \{F(z^u(\varepsilon)) - F(z^s(\varepsilon))\}\big|_{\varepsilon=0}. \qquad (1.3)$$

If $M(z)$ has a non-degenerate zero as z varies on Γ, then $W^s(\gamma_\varepsilon, H_\varepsilon)$ and $W^u(\gamma_\varepsilon, H_\varepsilon)$ have a transverse intersection nearby because M measures the lowest-order separ-

ation of these manifolds. Theorem 6 discusses the case when M merely changes sign at a point z_0.

The following theorem gives a formula to calculate M in terms of an integral along the orbits of the unperturbed system. It is the basic fact which makes the integral applicable.

THEOREM 1. *Assume the Hamiltonian vector field for H_0 has a closed orbit γ with a homoclinic connection Γ. Further, assume that H_0 is completely integrable with a second integral F which is independent from H_0 on Γ. Then*

$$M(z) = \lim_{j \to \infty} \int_{-T_j^*}^{T_j} (DF \cdot X_1)_{\phi_0(t,z)} \, dt \tag{1.4}$$

if T_j and T_j^* are chosen such that $\lim_{j \to 0} |\phi_0(T_j, z) - \phi_0(-T_j^*, z)| = 0$. Thus $\phi_0(T_j, z)$ and $\phi_0(-T_j^*, z)$ converge to the same point on the closed orbit γ. If $DF \cdot X_1 = 0$ for points on γ, then the improper integral converges absolutely. In particular, if γ remains a closed orbit when $\varepsilon > 0$ (and so X_1 is parallel to X_0 on γ), then the integral converges absolutely.

The proof of this theorem is similar to those in the literature, but is given in § 2 for completeness and because it is not very long. The limits of integration need to be chosen as specified in the theorem because the closed orbit can move as ε becomes positive. The proof shows that

$$M(z) = \int_{-T_j^*}^{T_j} (DF \cdot X_1)_{\phi_0(t,z)} \, dt + R(-T_j^*, T_j),$$

where

$$R(-T_j^*, T_j) = \frac{\partial}{\partial \varepsilon} F \circ \phi(-T_j^*, z^u(\varepsilon), \varepsilon) - \frac{\partial}{\partial \varepsilon} F \circ \phi(T_j, z^s(\varepsilon), \varepsilon).$$

Lemma 2.1 proves that as T_j and $T_j^* \to \infty$, $R(-T_j^*, T_j)$ converges to the difference of the infinitesimal displacement of γ_ε near $\phi_0(-T_j^*, z)$ and $\phi_0(T_j, z)$ as measured by F. If these converge to the same point, then the remainder goes to zero. The following example taken from [10, example 6.5] shows that this is indeed the case for a particular Hamiltonian.

Example 1.5. Let

$$H^\varepsilon(r, p, \theta, I) = p^2/2 + U(r) + I^2/(2r^2) + \varepsilon \sin \theta. \tag{1.5}$$

This rises from a slightly asymmetric central force problem. Since $\dot{I} = -\varepsilon \cos \theta$, $F = I$ can be taken as the second integral of H_0. Choose $U(r)$ with a simple maximum and I_0 such that the effective potential $U(r) + I_0^2/(2r^2)$ has a minimum at r_- and a maximum at r_+. Then H_0 has a saddle closed orbit with $r = r_+$, $v = 0$, $I = I_0$ and θ varying. It also has a homoclinic orbit $(r_0(t), p_0(t), \theta_0(t) + \theta^0, I_0)$ with $p_0(0) = 0$, $\theta_0(0) = 0$, $r_0(-t) = r_0(t)$ and $\theta_0(-t) = -\theta(t)$ for any choice of the initial condition θ^0. Then $DF \cdot X_{H_1} = -\cos(\theta_0(t) + \theta^0)$ and

$$M(\theta^0) = -\lim \int_{-T_j^*}^{T_j} \cos(\theta_0(t) + \theta^0) \, dt.$$

Note that any orbit on Γ can be realized by a suitable choice of θ^0, so it is the only variable used for the function M. Let $\theta_j = \theta_0(T_j)$ and $-\theta_j^* = \theta_0(-T_j^*)$. Since $\dot{\theta} = I_0/r^2$,

$$M(\theta^0) = -\cos\theta^0 \lim_j \frac{1}{I_0} \int_{-\theta_j^*}^{\theta_j} r_0(t)^2 \cos\theta \, d\theta$$

$$+ \sin\theta^0 \lim_j \frac{1}{I_0} \int_{-\theta_j^*}^{\theta_j} r_0(t)^2 \sin\theta \, d\theta,$$

where $r_0(t)$ is evaluated at the t corresponding to θ. The limits need to be chosen so that $|\theta_j + \theta^0 - (-\theta_j^* + \theta^0)| = |\theta_j + \theta_j^*|$ goes to zero modulo 2π. Here, this is most easily done by taking $\theta_j = 2\pi j$ and $-\theta_j^* = -2\pi j$. Since $r_0(t)^2 \sin\theta$ is an odd function, the limit of the second integral is zero. (This is easier to see if the limits are $\pm 2\pi j$, but is true in any case using the fact that $r_0(t) \to r_+$ as $t \to \pm\infty$.) The limit of the first integral is non-zero for generic choices of $U(r)$. Note that the limit of each integral can vary by $\pm r_+^2$ by choosing the limits θ_j and $-\theta_j^*$ incorrectly. Finally, integration by parts on the first integral yields

$$M(\theta^0) = \cos\theta^0 \left(\frac{2}{I_0}\right) \int_{-\infty}^{\infty} r_0(t) p_0(t) \sin\theta_0(t) \, dt,$$

since $r_0(T_j)^2 \sin\theta_j - r_0(-T_j^*)^2 \sin(-\theta_j^*) \to 0$. This integral converges absolutely because $p_0(t)$ goes to zero exponentially. This answer is just the negative of the answer in [10] as theorem 3 below shows it must be.

1B. The second case we discuss was considered by Lerman and Umanskii [11]. It assumes the system has hyperbolic fixed point P. They prove the comparable result which we state as theorem 2. We wish to thank P. Holmes for pointing out this result to us after he had seen a version of theorem 1 used in some joint work with J. Koiller.

THEOREM 2. *Assume H_0 is completely integrable with second integral F and has a hyperbolic fixed point P with non-empty homoclinic connection $\Gamma = [W^s(P, H_0) - \{P\}] \cap W^u(P, H_0)$. Assume the integral F is independent of H_0 along Γ. Then, for $z \in \Gamma$, the formula for $M(z)$ is as given in theorem 1 and the integral always converges absolutely.*

The proof of this theorem is exactly the same as theorem 1, so it follows from § 2. The reason that the integral converges absolutely is that there is only one point on P, so the limit

$$\lim_{t \to 0} |\phi_0(t, z) - \phi_0(-t, z)| = 0$$

without taking sequences of times.

1C. Next we want to relate this formula to the one derived by Holmes and Marsden in the most general case with S^1 symmetry [10, p. 299ff]. Such a symmetry naturally gives a second integral I. In fact, by using the related action angle coordinates, they consider $z = (y, \theta, I)$ and Hamiltonians

$$H^\varepsilon(z) = H_0(y, I) + \varepsilon H_1(z) + \cdots,$$

where H_0 is independent of θ. If the motion is restricted to a single energy surface, then θ can be considered a new time and the system as having a time-dependent

perturbation by X_1, i.e. a θ-dependent perturbation. Then the usual Melnikov integral applies. They show the resulting formulae are as follows. Let $\Omega = \partial H_0/\partial I$ and

$$Y_1 = \begin{pmatrix} \dfrac{\partial}{\partial y_2}(H_1/\Omega) \\ -\dfrac{\partial}{\partial y_1}(H_1/\Omega) \end{pmatrix}$$

If X_0 has a closed orbit with homoclinic connection, and if $z_0 = (y_0, \theta_0, I_0)$ is the initial condition, then they get a Melnikov integral

$$M^*(z_0) = \int_{-\infty}^{\infty} (DH_0 Y_1)_{\phi_0(t, z_0)} \, dt. \tag{1.6}$$

This integral is always absolutely convergent because DH_0 vanishes at points on the closed orbit γ so $DH_0 \cdot Y_1 = 0$ at such points. Our next theorem relates their function $M^*(z)$, which can be calculated with formula (1.6), with our function $M(z)$, which can be calculated by formula (1.4).

THEOREM 3. *Under the conditions above,* $M(z) = -M^*(z)$.

The proof of this theorem is given in § 3. The principal step in the proof is integration by parts. Note that because integration by parts involves evaluating a product at the limits of integration in addition to the two integrals, there is no contradiction to the fact that M^* converges absolutely and M only conditionally. See example 1.5 above.

Since these two integrals differ by only a sign and both measure the separation of stable and unstable manifolds, either formula can be applied to a given problem. Their integral has the advantage that it is absolutely convergent and may be simpler in some problems. The difficulty is that, to calculate their integral, the system must be converted explicitly into action angle coordinates, the θ and I, in order to calculate $\Omega = \partial H_0/\partial I$. In some problems this difficult to attain. In these cases $M(p)$ may be the better choice. Correspondence with J. Koiller about such problems is what motivated much of this work.

1D. Next, Holmes and Marsden considered higher degrees of freedom in the case with a $(S^1)^n$ symmetry [9]. In particular, they assumed $\theta = (\theta_1, \ldots, \theta_n)$ and $I = (I_1, \ldots, I_n)$, and that not only is H_0 independent of θ, but also that

$$H^\varepsilon(q, p, \theta, I) = F(q, p) + \sum_{j=1}^{n} G_j(I_j) + \varepsilon H_1(q, p, \theta, I). \tag{1.7}$$

They assume F has a homoclinic orbit in (q, p)-space to a hyperbolic fixed point $y^* = (q^*, p^*)$. As a result, $H_0 = F + \sum G_j$ has an invariant n-torus $\{(q^*, p^*, \theta, I_0)\} = T_0^n$, where $I_0 = (I_{10}, \ldots, I_{n0})$ is chosen so the motion is quasi-periodic. This torus has two normal directions in which it is hyperbolic and one component of the stable and unstable manifolds coincides,

$$[W^s(T_0^n, H_0) - T_0^n] \cap W^u(T_0^n, H_0) = \Gamma.$$

This system is completely integrable for $\varepsilon = 0$ with integrals $(F, I_1, \ldots, I_{n-1})$ or $(I_1, \ldots, I_n)$ or $(F, G_1, \ldots, G_{n-1})$ or some variation on these. Together one of these

sets of integrals measures the distances transverse to

$$\Gamma = W^s(T_0^n, H_0) \cap [W^u(T_0^n, H_0) - T_0^n] \qquad \text{in } H_0^{-1}(h).$$

They then form a Melnikov vector using the integrals $(I_1, \ldots, I_{n-1}, F)$. We proceed slightly differently than they did and define this vector-valued function in a manner consistent with the other cases considered above.

Consider

$$H^\varepsilon(z) = H_0(z) + \varepsilon H_1(z) + \varepsilon^2 H_2(z, \varepsilon), \tag{1.8}$$

where z has $2n+2$ dimensions. We need the following assumptions:

(H1) H_0 is completely integrable, with integrals $\{F_1, \ldots, F_n\}$ in addition to H_0.

(H2) The Hamiltonian vector field X_{H_0} for H_0 has an invariant n-torus T_0^n on which the motion is quasi-periodic, given by a frequency vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ satisfying the Diophantine conditions $|(\omega, j)| \geq c|j|^{-\tau}$, with two positive constants $c > 0$ and $\tau > n - 1$, for all integer vectors $j \in Z^n$ with $|j| = \sum_{\nu=1}^n |j_\nu| > 0$.

(H3) There are two dimensions normal to T_0^n in which the motion is hyperbolic (with one contracting direction and one expanding direction). Let $W^s(T_0^n, H_0)$ and $W^u(T_0^n, H_0)$ be the $(n+1)$-dimensional exponentially attracting and repelling manifolds for T_0^n. Assume T_0^n has a homoclinic orbit and let $\Gamma = W^s(T_0^n, H_0) \cap [W^u(T_0^n, H_0) - T_0^n]$. Assume the integrals F_j are independent of H_0 on Γ. Thus the flow of the vector fields X_{F_j} must leave Γ invariant, so Γ is $(n+1)$-dimensional, i.e. the manifolds intersect on components of $W^s(T_0^n, H_0) - T_0^n$ and $W^u(T_0^n, H_0) - T_0^n$.

(H4) Let $W^c(T_0^n, H_0)$ be the (local) centre manifold of dimension $2n$. Assume that the matrix of second partial derivatives $D^2 H_0$ is non-degenerate in the n-dimension normal to T_0^n in $W^c(T_0^n, H_0)$. (For system (1.7) this assumption is just that $G_j''(I_{j_0}) \neq 0$ for all j. Also, see conditions in [14, § 6].)

Because $W^c(T_0^n, H_0)$ is normally hyperbolic by (H3), for $\varepsilon > 0$ small enough there persists a (locally) invariant submanifold W_ε^c. Also, the canonical 2-form is necessarily non-degenerate on $W^c(T_0^n, H_0)$ and so on W_ε^c. Because of the non-degeneracy assumption (H4), the torus T_0^n persists for $\varepsilon > 0$ by KAM theory to give a torus $T_\varepsilon^n \subset W_\varepsilon^c$ on which the motion is quasi-periodic. In fact, T_ε^n depends differentiably on ε. In the case where H is analytic, Moser proved this in [14, § 6]; in the differentiable case it can be proved, for example, using the techniques in [20].

Now, for $z \in \Gamma$, let N_z be $(n+1)$-subspace transverse to Γ at z. Then, for $\varepsilon > 0$, there are points $z^s(\varepsilon)$ and $z^u(\varepsilon)$ corresponding to z given by $z^\sigma(\varepsilon) = N_z \cap W^\sigma(T_\varepsilon^n, H^\varepsilon)$ for $\sigma = u, s$. These points vary smoothly in ε, because the stable and unstable manifolds vary smoothly with ε on compact sets (and T_ε^n varies smoothly with ε). The Melnikov vector $M(z) = (M_1(z), \ldots, M_n(z))$ is defined by

$$M_k(z) = \frac{\partial}{\partial \varepsilon} \{F_k(z^u(\varepsilon)) - F_k(z^s)(\varepsilon))\}\big|_{\varepsilon=0}. \tag{1.9}$$

The vector $M(z)$ clearly measures the infinitesimal separation of $W^u(T_\varepsilon^n, H^\varepsilon)$ and $W^s(T_\varepsilon^n, H^\varepsilon)$ in the n-directions transverse to Γ in $(H^\varepsilon)^{-1}(h)$. The following theorem gives an integral to calculate $M(z)$.

THEOREM 4. *Let $H^\varepsilon(z)$ be as in (1.8) and satisfy assumptions* (H1)–(H4) *above. Let*

$M(z)$ *be defined by* (1.9). *Then the components of M can be calculated by the following integrals*:

$$M_k(z) = \lim_{j \to 0} \int_{-T_j^*}^{T_j} (DF_k \cdot X_{H_1})_{\phi_0(t,z)} \, dt, \tag{1.10}$$

where T_j and $T_j^* \to \infty$ are chosen so that the distance

$$\lim_{j \to 0} d(\phi_0(-T_j^*, z), \phi_0(T_j, z)) = 0.$$

As in theorem 1, the convergence is absolute if $DF_j \cdot X_{H_1} = 0$ for points of T_0^n. Furthermore, if $M(z)$ has non-degenerate zero, then the system has a transverse homoclinic intersection for $\varepsilon > 0$ small enough.

The easiest way to satisfy the assumptions on the T_j and T_j^* is to take the T_j and T_j^* so that $\phi_0(T_j, z)$ and $\phi_0(-T_j^*, z)$ both converge to the same point on T_0^n. For example, for system (1.7) the only condition that is necessary is that $\lim |\theta(-T_j^*, z_0) - \theta(T_j, z_0)| = 0$ in $(S^1)^n$. Since $\theta_k(t) = \theta_{k_0} + \Omega_k t$, where $\Omega_k = G_k'(I_j)$, this means $\lim |\Omega_k(T_j - T_j^*)| = 0 \bmod 2\pi$ for all k. Taking T_j and T_j^* so that $\Omega_n(T_j - T_j^*) = 2\pi N_j$ for some integers N_j, this means we need $\lim |2\pi\Omega_k N_j/\Omega_n| = 0 \bmod 2\pi$ for $1 \le k \le n - 1$. This is clearly possible by the quasi-periodicity on T_0^n. Also note that in Arnold's orginal paper [1] the torus did not vary with ε, so $DF_k \cdot X_{H_1} = 0$ on T_0^n and the convergence was absolute.

In [9] Holmes and Marsden mention that the integrals are not necessarily absolutely convergent, but they do not state explicitly how the times T_j^* and T_j have to be chosen so the limit is equal to M, i.e. so the conditionally convergent integral has the desired dynamical meaning. (It is not enough to make any choice such that the limit exists, as example 1.5 shows.) Salem also noticed this and the situation was partially clarified in [19]. The proof of this theorem is similar to theorem 1 and is discussed in § 2.

1E. Gruendler [6] has a different way of constructing a vector-valued Melnikov integral in higher dimensions and does not assume the unperturbed problem is completely integrable. We sketch his ideas. He considers a case like theorem 2 with a fixed point and periodic time perturbation

$$\dot{x} = f(x) + h(x, t, \varepsilon), \tag{1.11}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ and $h(x, t, 0) \equiv 0$. He considers the most general case, but we explain only the case when f is Hamiltonian (or at least div $f \equiv 0$) and $W^u(P, f) = W^s(P, f) = \Gamma$. If x_0 is in R^{2n} (he uses n), he constructs $\{\psi^{(1)}(t, x_0), \ldots, \psi^{(2n)}(t, x_0): x_0$ is in $\Gamma\}$, where (i) each $\psi^{(j)}$ satisfies the variation equation $\dot{u} = Df(\phi_0(t, x_0))u$ and (ii) $\psi^{(n+1)}, \ldots, \psi^{(2n)}$ are tangent to Γ and the $\psi^{(j)}(t, x_0)$ grow at different rates as t goes to $\pm\infty$ due to the different eigenvalues at the fixed point P. For x_0 in Γ let $\Pi(x_0)$ be the plane transverse to Γ spanned by $\{\psi^{(1)}(0, x_0), \ldots, \psi^{(n)}(0, x_0)\}$. Let $x^u(t, x_0, \xi, \varepsilon)$ be the solution of (1.1) with initial condition at $t = \xi$ given by $x^u(\xi, x_0, \xi, \varepsilon) \in \Pi(x_0) \cap W^u(P, f + h(\cdot, \varepsilon))$. Then functions Δ_i^u are defined by

$$x^u(\xi, x_0, \xi, \varepsilon) - x_0 = \sum_{j=1}^{n} \Delta_i^u(x_0, \xi, \varepsilon)\psi^{(j)}(\xi, x_0).$$

If Ω is a volume, he notes that

$$\Delta_i^u(x_0, \xi, \varepsilon) = \Omega(\psi^{(1)}, \ldots, \psi^{(i-1)}, x^u(\xi, x_0, \xi, \varepsilon) - x_0, \psi^{(i+1)}, \ldots, \psi^{(2n)}),$$

where the $\psi^{(j)}$ are evaluated at (ξ, x_0). It is then shown that the derivative of Δ_i^u with respect to the vector ε at $\varepsilon = 0$ in the direction ε satisfies

$$M_i^u(x_0, \xi)\varepsilon = D_\varepsilon \Delta_i^u(x_0, \xi, 0)\varepsilon = \Omega(\psi^{(1)}, \ldots, \psi^{(i-1)}, (D_\varepsilon x^u) \cdot \varepsilon, \psi^{(i+1)}, \ldots, \psi^{(2n)}),$$

where the $\psi^{(j)}$ are evaluated at (ξ, x_0), and

$$M_i^u(x_0, \xi)\varepsilon = \int_{-\infty}^{\xi} \Omega(\psi^{(1)}, \ldots, \psi^{(i-1)}, (D_\varepsilon h)\varepsilon, \psi^{(i+1)}, \ldots, \psi^{(2n)}) \, dt,$$

where the $\psi^{(j)}$ are evaluated as

$$\psi^{(j)}(t, \phi_0(t - \xi, x_0)) \qquad \text{and} \qquad (D_\varepsilon h)\varepsilon = (D_\varepsilon h(\phi_0(t - \xi, x_0), t, 0))\varepsilon.$$

Similarly Δ_i^s is defined and $M_i^s(x_0, \xi)\varepsilon = D_\varepsilon \Delta_i^s(x_0, \xi, 0)\varepsilon$. Thus

$$M_i(x_0, \xi)\varepsilon = \int_{-\infty}^{\infty} \Omega(\psi^{(1)}, \ldots, \psi^{(i-1)}, (D_\varepsilon h)\varepsilon, \psi^{(i+1)}, \ldots, \psi^{(2n)}) \, dt.$$

Considering $M_i(x_0, \xi)$ as a row vector, then

$$M(x_0, \xi) = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix}$$

is an n by N matrix. If it has rank n, then n of the parameters ε_j can be solved in terms of the others and still give a zero of $\Delta_i(x_0, \xi, \varepsilon) = \Delta_i^u(x_0, \xi, \varepsilon) - \Delta_i^s(x_0, \xi, \varepsilon) = 0$ and so a homoclinic orbit [6, theorem 3.2]. If this point is a non-degenerate zero as (x_0, ξ) varies in Γ, then the homoclinic intersection is transverse [6, theorem 4.1].

1F. The final results of this paper turn away from different contexts in which Melnikov integrals can be defined and consider conditions which imply hyperbolic horseshoes without transverse intersections. In particular, the assumption that M has a non-degenerate zero is replaced with the assumption that (a scalar-valued Melnikov integral) M changes sign. The following theorem considers the case of a Hamiltonian system in four dimensions, where X_0 has a hyperbolic closed orbit with a homoclinic connection.

THEOREM 5. A. *Make the assumptions of theorem* 1. *If* $M(y)$ *changes sign at* y_0 *as* y *varies on* $\Gamma = [W^s(\gamma, H_0) - \gamma] \cap W^u(\gamma, H_0)$, *then, for* $\varepsilon > 0$ *small enough, the manifolds* $W^s(\gamma_\varepsilon, H^\varepsilon)$ *and* $W^u(\gamma_\varepsilon, H^\varepsilon)$ *have a topologically transverse intersection near* y_0 (*they cross each other*).

B. *If* $H^\varepsilon(z)$ *is analytic in* z, *then, for* $\varepsilon > 0$ *small enough, the system has hyperbolic horseshoes arbitrarily near these topologically transverse intersections and is non-integrable.*

The only reason $H^\varepsilon(z)$ needs to be analytic in part B is to be able to conclude that the original topologically transverse intersection, corresponding to a zero of M, has finite order of contact.

Several people have noted that theorem 5B is true, including de Carvalho and Roussarie [2], Conley as given in [3], and probably others. Most of these papers prove the result by showing that arbitrarily near the intersection of finite order these are necessarily (differentiably) transverse intersections. It also follows by a direct proof following [18, proposition 5.1]. A sketch of this later proof is given in § 4 for the case of a hyperbolic fixed point considered in theorem 6 below. This proof is essentially (implicitly) contained in [18] for the case of theorem 5B. That paper, in turn, is a direct evolution of the theorems and methods of Newhouse [15]. Earlier, Gavrilov and Silnikov [5] had used some of these ideas in their work. Some related ideas appear in the work on bifurcations from Morse/Smale systems by Newhouse and Palis [16]. They consider what bounds on the angles of intersections of stable and unstable manifolds are sufficient to give a hyperbolic horseshoe. Also, Patterson considered some questions related to theorem 5 in his thesis [17]. I had several useful conversations with him about ideas related to these before this paper was written.

Similar ideas for non-transverse intersections can be applied to the case of hyperbolic fixed point with non-zero imaginary part of the eigenvalues. When the manifolds are transverse, Devaney proved there is a hyperbolic horseshoe [4]. In fact, if the system has a topologically transverse homoclinic intersection with finite-order contact, then the following theorem states the same result holds.

THEOREM 6. A. *Let H be an analytic Hamiltonian function in dimension four with a hyperbolic fixed P point whose eigenvalues have non-zero imaginary parts. If $W^u(P, H)$ and $W^s(P, H)$ have a topologically transverse intersection at z_0 in $H^{-1}(h_0)$, with $h_0 = H(P)$, then there is a horseshoe arbitrarily near z_0.*

B. *Let $H^\varepsilon = H_0 + \varepsilon H_1$ be as in theorem 2. If $M(z)$ changes sign at $z_0 = \Gamma = W^s(P, H_0) \cap W^u(P, H_0)$, then, for $0 < \varepsilon \le \varepsilon_0$ small enough, the vector field X^ε for H^ε has a horseshoe arbitrarily near z_0.*

The proof of theorem 6A combines the methods of [4] and [18] and is sketched in § 4. Theorem 6B follows because M gives the lowest-order terms in ε.

2. *Proof of theorem* 1
In the definition of $M(z)$, we need to calculate

$$\frac{\partial}{\partial \varepsilon}(F \circ z^\sigma(\varepsilon))\big|_{\varepsilon=0} \qquad \text{for } \sigma = \text{u, s.}$$

Instead of merely calculating this, we calculate

$$\frac{\partial}{\partial \varepsilon}(F \circ \phi(t, z^\sigma(\varepsilon), \varepsilon))\big|_{\varepsilon=0}$$

and indeed

$$\frac{d}{dt}\frac{\partial}{\partial \varepsilon}(F \circ \phi(t, z^\sigma(\varepsilon), \varepsilon))\big|_{\varepsilon=0}.$$

Proceeding for $\sigma = u, s$, taking all derivatives at $\varepsilon = 0$,

$$\frac{d}{dt}\frac{\partial}{\partial\varepsilon} F \circ \phi(t, z^{\sigma}(\varepsilon), \varepsilon) = \frac{\partial}{\partial\varepsilon}\frac{d}{dt} F \circ \phi(t, z^{\sigma}(\varepsilon), \varepsilon)$$

$$= \frac{\partial}{\partial\varepsilon}(DF \cdot X^{\varepsilon})_{\phi(t, z^{\sigma}(\varepsilon), \varepsilon)}$$

$$= (DF \cdot X_1)_{\phi_0(t, z)} + \frac{\partial}{\partial\varepsilon}(DF \cdot X_0)_{\phi(t, z^{\sigma}(\varepsilon), \varepsilon)}$$

$$= (DF \cdot X_1)_{\phi_0(t, z)}.$$

The first term on the third line comes from differentiating the ε in X^{ε} and putting $\varepsilon = 0$ elsewhere, while the second term is the reverse. The second term is zero because $DF \cdot X_0 \equiv 0$ at all points. Integrating this from T_1 to T_2 gives

$$\frac{\partial}{\partial\varepsilon} F \circ \phi(T_2, z^{\sigma}(\varepsilon), \varepsilon) - \frac{\partial}{\partial\varepsilon} F \circ \phi(T_1, z^{\sigma}(\varepsilon), \varepsilon) = \int_{T_1}^{T_2} (DF \cdot X_1)_{\phi_0(t, z)} \, dt.$$

For $\sigma = u$ take $T_1 = -T_j^*$ and $T_2 = 0$, while for $\sigma = s$ take $T_1 = 0$ and $T_2 = T_j$. Adding the two terms gives

$$M(z) = \int_{-T_j^*}^0 (DF \cdot X_1)_{\phi_0(t, z)} \, dt + \int_0^{T_j} (DF \cdot X_1)_{\phi_0(t, z)} \, dt$$

$$+ \frac{\partial}{\partial\varepsilon} F \circ \phi(-T_j^*, z^{u}(\varepsilon), \varepsilon) - \frac{\partial}{\partial\varepsilon} F \circ \phi(T_j, z^{s}(\varepsilon), \varepsilon).$$

Taking the limit as $j \to \infty$, it is only necessary to prove the limit of the sum of the last two terms goes to zero with the choice of time $-T_j^*$ and T_j. The intuitive reason is that $\phi(-T_j^*, z_0, 0)$ and $\phi(T_j, z_0, 0)$, for $z_0 = z^{u}(0) = z^{s}(0)$, go to the same point on the closed orbit γ, and this term measures the displacement of γ in terms of the change of values of F. Proceeding more carefully, the following lemma completes the proof of theorem 1.

LEMMA 2.1

$$\lim_{j \to \infty} \left(\frac{\partial}{\partial\varepsilon} F \circ \phi(-T_j^*, z^{u}(\varepsilon), \varepsilon) - \frac{\partial}{\partial\varepsilon} F \circ \phi(T_j, z^{s}(\varepsilon), \varepsilon) \right) = 0 \tag{2.1}$$

with T_j and T_j^* chosen as in theorem 1.

Before proving lemma 2.1, we prove the special case of theorem 4 assuming H^{ε} is given by (1.7), where the explicit variables make the exposition simpler. The first part of the proof of theorem 4 is exactly like the proof above, where K is now either the integral F or $I_1, \ldots, I_{n-1}$. Given y_0 a homoclinic point for F and any $\theta_0 = (\theta_{10}, \ldots, \theta_{n0})$, there is a point on the stable or unstable manifold $z^{s}(\theta_0, \varepsilon)$ and $z^{u}(\theta_0, \varepsilon)$ for $\varepsilon > 0$. Also, the invariant torus for $\varepsilon > 0$ can be written as a function (or graph) $z = \psi(\theta, \varepsilon) = \psi^{\varepsilon}(\theta)$. The lemma which corresponds to 2.1 is the following:

LEMMA 2.2

$$\lim_{j \to \infty} \left| \frac{\partial}{\partial\varepsilon} K \circ \phi(-T_j^*, z^{u}(\theta_0, \varepsilon), \varepsilon) - \frac{\partial}{\partial\varepsilon} K \circ \phi(T_j, z^{s}(\theta_0, \varepsilon), \varepsilon) \right| = 0 \tag{2.2}$$

if $\lim_{j\to\infty} |\theta(T_j^*, z(\theta_0, 0)) - \theta(T_j, z(\theta_0, 0)| = 0$. *Here* $\theta(t, z_0)$ *is the* θ-*component of the flow for Hamiltonian* (1.7) *with* $\varepsilon = 0$ *and initial conditions* z_0.

Proof. We want to determine the nature of the limit of the vector $(\partial/\partial\varepsilon)\phi(T_j, z^s(\theta_0, \varepsilon), \varepsilon)$. In fact we want to decompose it into the component along the invariant torus and the normal component. To clarify the convergence, it is necessary to extend the flow into (z, ε)-space with $\dot{\varepsilon} = 0$. Label this flow as $\phi^*(t, z_0, \varepsilon)$. Also let $\theta(t, z_0, \varepsilon)$ be the θ-coordinate of the flow $\phi^*(t, z_0, \varepsilon)$ and $\theta(t, z_0) = \theta(t, z_0, 0)$. Let $\gamma = \{\psi(\theta, 0)\}$ be the original unperturbed torus and $\gamma^* = \{(\psi(\theta, \varepsilon), \varepsilon)\}$ be the extended perturbed torus with stable manifold $W^s(\gamma^*, H^\varepsilon) \subset \{(z, \varepsilon)\}$.

The vector $(\partial/\partial\varepsilon)(z^s(\theta_0, \varepsilon), \varepsilon)$ is tangent to $W^s(\gamma^*, H^\varepsilon)$ and so it follows that

$$\frac{\partial}{\partial\varepsilon}\phi^*(T_j, z^s(\theta_0, \varepsilon), \varepsilon) = \left(D_z\phi_{(T_j, z_0, 0)}\frac{\partial}{\partial\varepsilon}z^s(\theta_0, \varepsilon), 1\right)$$

is also. Because γ^* is normally attracting in $W^s(\gamma^*, H^\varepsilon)$, as T_j goes to infinity the component of the vector normal to γ^* goes to zero. Proceeding with explicit calculations,

$$\lim_{j\to\infty}\left(\frac{\partial}{\partial\varepsilon}K\circ\phi(-T_j^*, z^u(\theta_0, \varepsilon), \varepsilon) - \frac{\partial}{\partial\varepsilon}K\circ\phi(T_j, z^s(\theta_0, \varepsilon), \varepsilon)\right)$$

$$= \lim\left(DK\frac{\partial}{\partial\varepsilon}[\phi(-T_j^*, z^u(\theta_0, \varepsilon), \varepsilon)] - DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(-T_j^*, z^u(\theta_0, \varepsilon), \varepsilon), \varepsilon)]\right)$$

$$+ \lim\left(DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(-T_j^*, z^u(\theta_0, \varepsilon), \varepsilon), \varepsilon)] - DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(-T_j^*, z_0, 0), \varepsilon)]\right)$$

$$+ \lim\left(-DK\frac{\partial}{\partial\varepsilon}[\phi(T_j, z^s(\theta_0, \varepsilon), \varepsilon)] + DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon), \varepsilon)]\right)$$

$$+ \lim\left(-DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon), \varepsilon)] + DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(T_j, z_0, 0), \varepsilon)]\right)$$

$$+ \lim\left(DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(-T_j^*, z_0, 0), \varepsilon)] - DK\frac{\partial}{\partial\varepsilon}[\psi(\theta(T_j, z_0, 0), \varepsilon)]\right). \tag{2.3}$$

The fifth limit is zero because of the choice of T_j^* and T_j.

We consider only the third and fourth limits because the first two are similar. Considering the third limit, the vector

$$\frac{\partial}{\partial\varepsilon}\phi^*(T_j, z^s(\theta_0, \varepsilon), \varepsilon) = D\phi^*_{(T_j, z_0, 0)}\left(\frac{\partial}{\partial\varepsilon}z^s(\theta_0, \varepsilon), 1\right)$$

is tangent to $W^s(\gamma^*, H^\varepsilon)$. As T_j goes to infinity, the component of this vector normal to γ^* goes to zero (by the contraction on this manifold), and further the two vectors

$$\frac{\partial}{\partial\varepsilon}\phi^*(T_j, z^s(\theta_0, \varepsilon), \varepsilon) \quad \text{and} \quad \frac{\partial}{\partial\varepsilon}\psi(\theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon), \varepsilon)$$

have the same components along γ^*, so the third limit is zero.

Finally, for the fourth limit,

$$\frac{\partial}{\partial \varepsilon} \psi(\theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon), \varepsilon) - \frac{\partial}{\partial \varepsilon} \psi(\theta(T_j, z_0, 0), \varepsilon) = \frac{\partial \psi^0}{\partial \theta} \frac{\partial}{\partial \varepsilon} \theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon).$$

This last vector is tangent to γ (not just γ^*), and so acting on it by DK gives zero. Thus the fourth limit is zero. This completes the proof of lemma 2.2. □

Proof of lemma 2.1. The main difference in this proof is that there is no explicit coordinate corresponding to θ in the proof of lemma 2.2 which can be used to associate a point on the closed orbit γ or γ^*. However, the stable manifold theorem says that points on $W^s(\gamma^*, H^\varepsilon)$ approach γ^* in phase with some point on γ^*, i.e. if $(z_\varepsilon, \varepsilon)$ is on $W^s(\gamma^*, H^\varepsilon)$, then there is a point (z_1, ε) on γ^* such that $|\phi^*(t, z_\varepsilon, \varepsilon) - \phi^*(t, z_1, \varepsilon)|$ goes to zero as t goes to infinity. We use this to define a continuous function

$$\pi_1 : W^s(\gamma^*, H^\varepsilon) \to \gamma^*$$

by

$$\pi_1(z_\varepsilon, \varepsilon) = (z_1, \varepsilon),$$

and

$$\pi : W^s(\gamma^*, H^\varepsilon) \to \gamma$$

by

$$\pi(z_\varepsilon, \varepsilon) = \psi^{-1} \pi_1(z_\varepsilon, \varepsilon),$$

where $\{\psi(z, \varepsilon): z \text{ in } \gamma)\} = \gamma_\varepsilon \times \{\varepsilon\}$. Then in equation (2.3) $\theta(t, z, \varepsilon)$ is replaced by $\pi \circ \phi^*(t, z, \varepsilon)$ and the initial condition is specified by z_0 and not just θ_0, e.g. $\psi(\theta(T_j, z^s(\theta_0, \varepsilon), \varepsilon), \varepsilon)$ is replaced by $\psi(\pi \circ \phi^*(T_j, z^s(z_0, \varepsilon), \varepsilon), \varepsilon)$. The rest of the proof is the same. □

General case of theorem 4. Again the first part of the proof is exactly as before but F_k replaces F. The calculation of the limit is like the proof of lemma 2.1 except that the closed orbit γ is replaced by the torus T_0^n and π is a projection onto T_0^n. The rest of the proof is the same. □

3. *Equivalence of two integrals: theorem* 3

We start with the formula for $M(z)$. In this case the coordinates are $z = (y, \theta, I)$ and the second integral is $F(z) = I$. If X_1 is the vector field for H_1, then $DF \cdot X_1 = -\partial H_1/\partial \theta$. We want to compare this with $(d/dt)H_1 \cdot \phi_0(t, z_0)$:

$$\frac{d}{dt} H_1 \circ \phi_0(t, z_0) = D_y H_1 \cdot X_0 + \frac{\partial H_1}{\partial \theta} \cdot \dot{\theta} + \frac{\partial H_1}{\partial I} \cdot \dot{I}, \tag{3.1}$$

where D_y is the partial derivative with respect to y variables. But $\dot{\theta} = \partial H_0/\partial I = \Omega$ and $\dot{I} = 0$. Also $D_y H_1 \cdot X_0 = -D_y H_0 \cdot X_1$ either by writing out or using the result for Poisson brackets. Combining and solving for $-\partial H_1/\partial \theta$ in (3.1) gives

$$DF \cdot X_1 = -\frac{\partial H_1}{\partial \theta} = -\frac{1}{\Omega} D_y H_0 \cdot X_1 - \frac{1}{\Omega} \frac{d}{dt} H_1 \circ \phi_0(t, z_0).$$

Thus

$$M(z_0) = \lim_{j \to \infty} - \int_{-T_j^*}^{T_j} \frac{1}{\Omega} D_y H_0 \cdot X_1 \, dt + \int_{-T_j^*}^{T_j} -\frac{1}{\Omega} \frac{d}{dt} (H_1 \circ \phi_0(t, z_0)) \, dt.$$

Using integration by parts in the second integral with $u = -1/\Omega$ and $dv = (d/dt)(H_1 \circ \phi_0(t, z)) \, dt$ gives

$$du = \frac{1}{\Omega^2} \left(D_y \Omega \cdot X_0 + \frac{\partial \Omega}{\partial I} \dot{I} + \frac{\partial \Omega}{\partial \theta} \dot{\theta} \right) dt = \frac{1}{\Omega^2} D_y \Omega \cdot X_0 \, dt = -\frac{1}{\Omega^2} D_y H_0 \cdot X_\Omega \, dt$$

because $\dot{I} = 0$, $\partial \Omega / \partial \theta = 0$ and $D_y \Omega \cdot X_0 = -D_y H_0 \cdot X_\Omega$. Thus

$$M(z_0) = \lim_{j \to \infty} - \int_{-T_j^*}^{T_j} D_y H_0 \left(\frac{X_1}{\Omega} - \frac{H_1}{\Omega^2} X_\Omega \right) dt - \lim_{j \to \infty} \frac{1}{\Omega} H_1 \circ \phi_0(t, z_0) \Big|_{-T_j^*}^{T_j}$$

$$= \int_{-\infty}^{\infty} D_y H_0 \cdot Y_1 \, dt.$$

The boundary terms go to zero by the choice of T_j and T_j^*.

4. *Proof of theorem* 6

The proof combines the ideas of the proof of Devaney [4] with the ideas when the intersection is only topologically transverse at a homoclinic intersection for a diffeomorphism in two dimensions [18, propositions 5.1 and 7.8].

Following Devaney, there are coordinates $(x, y) = (x_1, x_2, y_1, y_2)$ near the fixed point $P = (0, 0)$ where the flow is given by

$$F_t(x, y) = \begin{pmatrix} e^{At}(x_1 \cos \Omega t - x_2 \sin \Omega t) \\ e^{At}(x_1 \sin \Omega t + x_2 \cos \Omega t) \\ e^{-At}(y_1 \cos \Omega t + y_2 \sin \Omega t) \\ e^{-At}(-y_1 \sin \Omega t + y_2 \cos \Omega t) \end{pmatrix}$$

where $A = A(x, y)$ and $\Omega = \Omega(x, y)$ are analytic functions of the initial conditions (x, y). For small enough r, δ, there are transversals to the flow near the local unstable and stable manifolds:

$$\Sigma^u = \{(x, y): |x| = r, |y| \le \delta\},$$

$$\Sigma^s = \{(x, y): |x| \le \delta, |y| = r\}.$$

Assuming $H(P) = 0$, let $\Sigma_0^s = \Sigma^s \cap H^{-1}(0)$ and $\Sigma_0^u = \Sigma^u \cap H^{-1}(0)$. Also let $\sigma^s = \Sigma_0^s \cap \{x = 0\}$ (resp. $\sigma^u = \Sigma_0^u \cap \{y = 0\}$) be the intersection of the local stable (resp. unstable) manifold with Σ_0^s (resp. Σ_0^u). If γ is the orbit of the homoclinic intersection given in the theorem, let $q^s = \gamma \cap \sigma^s$ and $q^u = \gamma \cap \sigma^u$. Let D^s and D^u be two-dimensional discs in Σ_0^s and Σ_0^u centred at q^s and q^u.

There are Poincaré maps defined from points near q^s to itself (at least for some points). This map is the composition of two maps, $\Phi_0: (D^s - \sigma^s) \to \Sigma_0^u$ and $\Phi_1: D^u \to D^s$. Here Φ_1 is formed by following trajectories from points in D^u near q^u until they return to Σ_0^s near q^s and so in D^s (if D^u is small enough). Also any point on $D^s - \sigma^s$ will flow past P until it exists at some point in Σ_0^u; this defines Φ_0. If $\mathcal{D} = \Phi_0^{-1}(D^u)$, then $\Phi = \Phi_1 \circ \Phi_0: \mathcal{D} \subset (D^s - \sigma^s) \to D^s$.

Let $(\theta, z) = (\theta^s, z^s)$ be a coordinate system on D^s where $\sigma^s = \{z^s = 0\}$ and θ^s is an angular coordinate along σ^s with q^s corresponding to $(z^s, \theta^s) = (0, 0)$. If the intersection is topologically transverse, then $W^u(P, H) \cap D^s$ is given by the graph of a function $z = a\theta^j + h(\theta)$, where $h(\theta) = O(|\theta|^{j+1})$. The slope of $W^u(P, H)$ at a point (θ, z) on the graph is about $ja\theta^{j-1} \approx ja(z/a)^{(j-1)/j} = ja^{1/j}z^{1-1/j} \equiv \xi(z)$. For $\eta > 1$ near enough to 1, we can consider the sectors of vectors whose slopes are about $\xi(z)$,

$$S(\theta, z) = \{v = (v_s, v_u): \eta^{-1} < |v_u|/(|v_s|\xi(z)) < \eta\}.$$

Thus for v in $S(\theta, z)$ the slope of v is between $\eta^{-1}\xi(z)$ and $\eta\xi(z)$. For v in $S(\theta, z)$ let $v' = (D\Phi_0)v$ and $v'' = (D\Phi)v = (D\Phi_1)v'$. We show that for v in $S(\theta, z)$ the vector of v'' is also in $S(\Phi(\theta, z))$ and $\|v''\| > \|v\|$. Thus there is a hyperbolic structure.

If v is a vector in $S(\theta, z)$ with $v_s = 1$, then $|v_u| \approx jaz^{1-1/j}$. The calculation in [1, p. 435] shows that this component is stretched by a factor of $1/|z|$ by the time it reaches Σ_0^s, $|v_u'| \sim |z|^{1-1/j} \cdot |z|^{-1} = |z|^{-1/j}$. This component is arbitrarily large as $|z| \to 0$. Also the stable component is contracted as it goes past P, so $|v_s'| < |v_s| = 1$. Thus v' is a vector pointing nearly parallel to the unstable direction. Then $v'' = D\Phi v'$ is arbitrarily close to the direction of $W^u(P, H) \cap D^s$ and so it lies in the sector $S(\Phi(\theta, z))$. Also $\|v''\| \geq C_1\|v'\| \geq C_1|v_u'| \geq C_2|z|^{-1/j}\|v\|$. As $|z| \to 0$, $C_2|z|^{-1/j} \to \infty$, so the vector is stretched by an arbitrarily large factor when it returns to D^s. (On the other hand the length of time needed to return is proportional to $|z|^{-1}$, so the characteristic exponent is proportional to $|z|^{1-1/j}$ and goes to zero as $|z| \to 0$.) Dually the vectors in the stable direction are contracted by a similar factor (or expanded by this factor as the solutions are followed backward).

As discussed in [4], $D^s \cap \Phi(D^s)$ is made up of infinitely many vertical strips which accumulate on $W^u(P, H) \cap D^s$. Similarly, $D^s \cap \Phi^{-1}(D^s)$ is made up of infinitely many horizontal strips which accumulate on σ^s. If only finitely many of these are used of each type, then there is a hyperbolic invariant set, a horseshoe. As the number of symbols increases, the hyperbolicity of Φ actually increases on these new orbits, but the hyperbolicity relative to the period of the flow (the characteristic exponents) goes to zero.

Remark. In connection with the situation in theorem 6A, if H is completely integrable and $W^s(P, H) = W^u(P, H)$, then the above proof produces strips near the homoclinic connection which are mapped across themselves by the first return map. Thus it at first appears that H has a hyperbolic horseshoe, which is impossible because it is completely integrable. However, the map of these strips across themselves in this case is a sheer map, so there is no hyperbolic horseshoe and there is no contradiction.

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