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Abelian varieties and the general Hodge conjecture

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Abstract. We investigate the relationship between the usual and general Hodge conjectures for abelian varieties. For certain abelian varieties A, we show that the usual Hodge conjecture for all powers of A implies the general Hodge conjecture for A.

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1. Introduction

The arithmetic filtration on the cohomology of a smooth complex projective variety X is defined by letting $F_a^r H^n(X, \mathbf{Q})$ be the linear span of cohomology classes supported on algebraic subvarieties of codimension at least r. When $n = 2r$, $F_a^r H^{2r}(X, \mathbf{Q})$ is the space spanned by the fundamental classes of algebraic subvarieties of codimension r. The other terms of the arithmetic filtration provide subtler information about subvarieties of X.

The general Hodge conjecture([Gr], [Ho]) asserts the equality of the arithmetic filtration with Grothendieck's corrected Hodge filtration, which we denote by $F_{\mathbf{Q}}^r H^n(X, \mathbf{Q})$. The special case $n = 2r$ of the general Hodge conjecture asserts that $F_a^r H^{2r}(X, \mathbf{Q})$, the space of algebraic cycles of codimension r, equals $H^{r,r}(X) \cap H^{2r}(X, \mathbf{Q})$, the space of Hodge cycles. This special case is (usually) called the usual Hodge conjecture.

The general Hodge conjecture is equivalent to the usual Hodge conjecture for all X, together with the assertion that for any irreducible Hodge structure $V \subset F_{\mathbf{Q}}^r H^n(X, \mathbf{Q})$, the Tate twist $V(-r)$ is isomorphic to a Hodge substructure of $H^{n-2r}(Y, \mathbf{Q})$ for some smooth projective variety Y (Proposition 2.1). In this paper we prove this second conjecture for a certain class of abelian varieties (Theorem 5.1).

The abelian varieties we consider are those whose Hodge rings are characterized by endomorphisms. To understand what this means we need to consider certain families of abelian varieties (see Section 4.6 for details). In a series of papers

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([Sm3] is a good survey) Shimura studied families of abelian varieties called PEL-families; these are solutions to moduli problems for abelian varieties with prescribed polarizations, endomorphisms, and, level structures. Mumford [Mm1] obtained a larger class of families of abelian varieties by considering arbitrary Hodge classes on powers of an abelian variety A instead of endomorphisms. Indeed, given an abelian variety A, Mumford constructed the family of all abelian varieties whose Hodge groups are 'contained' in the Hodge group of A (see [Mm1, p. 348] for the precise meaning). We shall say that A is of PEL-type if the family of abelian varieties constructed by Mumford from A is a PEL-family in Shimura's sense. (We note here that Shimura assumed the endomorphism algebras to be division algebras; this was only for convenience, and we do not make this assumption here.) Thus A is of PEL-type if and only if the Hodge group of A equals the Hodge group of the generic fiber of a PEL-family; i.e., each power of A has no Hodge cycles other than those it is required to have by virtue of the endomorphisms of A.

In Theorem 5.1 we show that for an abelian variety A of PEL-type whose Hodge group is semisimple, and whose simple factors of type III satisfy an additional condition, the usual Hodge conjecture for all powers of A implies the general Hodge conjecture for A (the definition of the type of a simple abelian variety is recalled in Sect. 4.1). This class of abelian varieties includes all abelian varieties A such that the Hodge ring of each power of A is generated by divisors, and such that each simple factor of A is of type I or type II. The general Hodge conjecture for such abelian varieties has also recently been proved by Tankeev [T] and Hazama [Ha] independently.

Our results also cover the abelian varieties which Weil [W] considered as possible counterexamples to the usual Hodge conjecture. For such an abelian variety A, the usual Hodge conjecture for A implies the usual Hodge conjecture for all powers of A. Schoen [Sc1] has proved the usual Hodge conjecture for a 4-dimensional abelian variety of Weil type with endomorphisms by $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ (see also [vG, p. 238]). We may now conclude the general Hodge conjecture for any power of such an abelian variety.

In the final section of this paper, we look at some examples of abelian varieties which do not satisfy the hypotheses of our main theorem, and briefly discuss the possibility of generalizing it.

2. The general Hodge conjecture

Let k be either $\mathbf{Q}$ or $\mathbf{R}$. A *k-Hodge structure* of *weight* n is a finite dimensional vector space V over k, together with a direct sum decomposition $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$ such that $V^{p,q} = \overline{V^{q,p}}$. The *type* of V is then the set $\{(p,q) \mid V^{p,q} \neq 0\}$. We define the *height* of V to be the smallest p such that $V^{p,n-p} \neq 0$.

Let $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$ and $W_{\mathbf{C}} = \bigoplus_{p+q=m} W^{p,q}$ be Hodge structures. A *morphism of Hodge structures* of *degree* r is a k-linear map $\varphi\colon V \to W$ such that $\varphi(V^{p,q}) \subset W^{p+r,q+r}$ for all pairs p, q. We say that the Hodge structures V and

W are *isomorphic* if they are isomorphic in the category whose objects are Hodge structures and whose morphisms are morphisms of Hodge structures of degree 0. We say that they are *equivalent* if they are isomorphic in the category whose objects are again Hodge structures, but whose morphisms are of arbitrary degree. Thus two Hodge structures are isomorphic if and only if they are equivalent and have the same weight.

Let S be the real algebraic group such that $S(\mathbf{R})$ is the unit circle in the complex plane. Then any $\mathbf{Q}$-Hodge structure V determines a morphism of real algebraic groups $\varphi\colon S \to \mathrm{GL}(V_{\mathbf{R}})$ such that $e^{i\theta}$ acts on $V^{p,q}$ as multiplication by $e^{(p-q)i\theta}$. The *Hodge group* of V is defined to be the smallest $\mathbf{Q}$-subgroup, G, of $\mathrm{GL}(V)$ such that $G_{\mathbf{R}}$ contains the image of φ. The Hodge substructures of V are the same as the G-submodules of V, and the $\mathbf{R}$-Hodge substructures of $V_{\mathbf{R}}$ are the same as the $G(\mathbf{R})$-submodules of $V_{\mathbf{R}}$.

Let X be a smooth projective variety over $\mathbf{C}$. The *arithmetic filtration* $F_a^{\bullet}$ on $H^n(X,\mathbf{Q})$ is defined by taking $F_a^r H^n(X,\mathbf{Q})$ to be the set of cohomology classes supported on algebraic subvarieties of codimension at least r.

The *Hodge filtration* is given by

$$F^r H^n(X,\mathbf{C}) := \bigoplus_{\substack{p+q=n \\ p\geqslant r}} H^{p,q}(X).$$

The *rational Hodge filtration* is obtained by defining $F_{\mathbf{Q}}^r H^n(X,\mathbf{Q})$ to be the largest $\mathbf{Q}$-Hodge substructure of $F^r H^n(X,\mathbf{C}) \cap H^n(X,\mathbf{Q})$.

We have $F_a^r H^n(X,\mathbf{Q}) \subset F_{\mathbf{Q}}^r H^n(X,\mathbf{Q})$. The *general Hodge conjecture* [Ho], as amended by Grothendieck [Gr], states that this inclusion is an equality. The case $n = 2r$ is the *usual Hodge conjecture*. The following proposition gives an equivalent formulation due to Grothendieck (see also [Sc2, Lem. 0.1, p. 139]).

PROPOSITION 2.1. (Grothendieck [Gr, p. 301]). *The general Hodge conjecture is equivalent to the usual Hodge conjecture together with*

(2.2) *For any smooth complex projective variety X, and any irreducible Hodge substructure V of $H^n(X,\mathbf{Q})$, there exists a smooth projective variety Y, a nonnegative integer s, and a Hodge substructure $W \subset H^s(Y,\mathbf{Q})$ such that W has height* 0 *and V is equivalent to W.*

Proof. Let d be the dimension of X, and h the height of V.

Assume the general Hodge conjecture. Then V is supported on an algebraic subvariety Z of codimension h. Let $Y \to Z$ be a desingularization of Z, and $f\colon Y \to X$ its composition with the inclusion. Then

$$f_*\colon H^s(Y,\mathbf{Q}) \to H^n(X,\mathbf{Q})$$

is a morphism of Hodge structures, where $s = n - 2h$. Since the image of f_* contains V, $H^s(Y,\mathbf{Q})$ has a Hodge substructure W which is equivalent to V.

(We have used here the fact that our Hodge structures are polarizable, and hence semisimple. This follows, for example, from the fact that their Hodge groups are reductive (see [D, Prop. 3.6, p. 44]).) Since the degree of f_* is h, the height of W must be 0.

Conversely, assume (2.2). Note that the height of V is $h = (n-s)/2$. Let k be the dimension of Y. Then there exists a Hodge cycle $\zeta \in H^{2k+2h}(Y \times X, \mathbf{Q})$ inducing an equivalence φ of Hodge structures from W to V. The usual Hodge conjecture for $Y \times X$ implies that ζ is the class of an algebraic cycle Z. Since $\varphi(\alpha) = p_{2*}(p_1^*\alpha \wedge \zeta)$, we see that any element of V is supported on $p_{2*}(Z)$. The codimension of $p_{2*}(Z)$ is h. This shows that V is contained in $F_a^h H^n(X, \mathbf{Q})$. □

We shall refer to (2.2) as the *unusual Hodge conjecture*. Then the general Hodge conjecture is equivalent to the usual Hodge conjecture together with the unusual Hodge conjecture.

3. The group theoretic filtration

Define the *group theoretic filtration* on $H^n(X, \mathbf{Q})$ by letting $F_g^r H^n(X, \mathbf{Q})$ be the sum of those Hodge substructures of $H^n(X, \mathbf{Q})$ which are equivalent to Hodge substructures of $H^s(Y, \mathbf{Q})$ for some smooth projective variety Y, and some $s \leqslant n - 2r$. We say that a class $\mathfrak{A}$ of smooth projective varieties *dominates* X if each irreducible Hodge substructure of $F_g^r H^n(X, \mathbf{Q})$ is equivalent to a Hodge substructure of $H^s(Y, \mathbf{Q})$ for some $Y \in \mathfrak{A}$, and some $s \leqslant n - 2r$. As we shall see, this filtration is intimately related to representations of the Hodge group, hence its name.

The unusual Hodge conjecture may now be restated as asserting the equality of the rational Hodge filtration with the group theoretic filtration. The proof of Proposition 2.1 shows that to prove the general Hodge conjecture for X it is sufficient to prove (i) the unusual Hodge conjecture for X, and, (ii) the usual Hodge conjecture for $Y \times X$ for all Y in a class $\mathfrak{A}$ of algebraic varieties which dominates X.

The following two propositions are easy consequences of the definitions, Proposition 2.1, and its proof.

PROPOSITION 3.1. $F_g^r H^n(X, \mathbf{Q}) \subset F_{\mathbf{Q}}^r H^n(X, \mathbf{Q})$ *for any smooth projective variety* X. □

PROPOSITION 3.2. *If the usual Hodge conjecture is true for* $Y \times X$ *for all* Y *in a class* $\mathfrak{A}$ *of algebraic varieties which dominates* X, *then* $F_g^r H^n(X, \mathbf{Q}) \subset F_a^r H^n(X, \mathbf{Q})$. □

PROPOSITION 3.3. (cf. [Gr, p. 301]). *For any smooth, projective variety* X, *and any* r, *we have*

$$F_g^r H^{2r+1}(X, \mathbf{Q}) = F_{\mathbf{Q}}^r H^{2r+1}(X, \mathbf{Q})$$

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and the usual Hodge conjecture implies that

$$F_g^r H^{2r+1}(X, \mathbf{Q}) = F_a^r H^{2r+1}(X, \mathbf{Q}) = F_{\mathbf{Q}}^r H^{2r+1}(X, \mathbf{Q}).$$

Proof. $F_{\mathbf{Q}}^r H^{2r+1}(X, \mathbf{Q})$ is a Hodge structure of type $(r, r+1), (r+1, r)$. Hence the Tate twist $V := F_{\mathbf{Q}}^r H^{2r+1}(X, \mathbf{Q})(-r)$ is a Hodge structure of type (0, 1), (1, 0); since it is polarizable there exists an abelian variety A such that V is isomorphic to $H^1(A, \mathbf{Q})$. Therefore $F_g^r H^{2r+1}(X, \mathbf{Q}) = F_{\mathbf{Q}}^r H^{2r+1}(X, \mathbf{Q})$, and the usual Hodge conjecture for $A \times X$ implies that either of these equals $F_a^r H^{2r+1}(X, \mathbf{Q})$. □

4. Abelian varieties

4.1. Endomorphisms

Let A be a simple abelian variety over **C**. By Albert's classification, the endomorphism algebra $D(A) := \mathrm{End}(A) \otimes \mathbf{Q}$ of A is one of the following (see [Sm1, Sect. 1.3, p. 154])

(I) a totally real number field,
(II) a totally indefinite quaternion algebra over a totally real number field,
(III) a totally definite quaternion algebra over a totally real number field, or,
(IV) a division algebra whose center is a CM-field.

The type of $D(A)$ in the above classification is also called the *type* of A.

4.2. Known cases

Recall that any abelian variety A is isogenous to $A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_l^{n_l}$ where the n_i are positive integers, and the A_i are pairwise nonisogenous simple abelian varieties called the *simple factors* of A. The general Hodge conjecture is known for the following abelian varieties

(a) Any abelian variety A such that the Hodge ring of A^n is generated by divisors for all n, and such that each simple factor of A is of type I or type II (Hazama [Ha] and Tankeev [T]). Special cases of this were proved earlier by Mattuck [Ma] and Gordon [Go1, Go2].
(b) A power of an elliptic curve with complex multiplication (Shioda [So, p. 63]).
(c) A simple abelian variety A of CM-type such that $[\bar{K} : \bar{K}_0] = 2^d$, where $d := \dim A, K$ is the endomorphism algebra of A, K_0 is the maximal real subfield of K, and bars denote Galois closure (Tankeev [T]).
(d) A general fiber of a PEL family of abelian 4-folds with endomorphisms by $\mathbf{Q}(i)$, and polarization given by a hermitian form of signature (3, 1) (Schoen [Sc2]).

4.3. HERMITIAN GROUPS

We shall now review some facts about algebraic groups of hermitian type. For further details see Satake [Sa3]. Let G be a connected, semisimple, linear algebraic group over **R**. Assume that $G(\mathbf{R})$ is hermitian, so that $X := G(\mathbf{R})^0/K$ is a bounded symmetric domain, where K is a maximal compact subgroup of $G(\mathbf{R})^0$. Let $\mathfrak{g} := \mathrm{Lie}\, G(\mathbf{R})$, $\mathfrak{k} := \mathrm{Lie}\, K$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Let x be the unique fixed point of K in X. Differentiating the natural map $G(\mathbf{R})^0 \to X$ gives an isomorphism of $\mathfrak{p}$ with $T_x(X)$, the tangent space of X at x, and there exists a unique $H_0 \in Z(\mathfrak{k})$, called the *H-element* at x, such that $\mathrm{ad}\, H_0 \,|\, \mathfrak{p}$ is the complex structure on $T_x(X)$.

Let β be a nondegenerate alternating form on a finite dimensional real vector space V. The symplectic group $\mathrm{Sp}(V, \beta)$ is of hermitian type; the corresponding symmetric domain is the Siegel space

$$\mathfrak{S}(V,\beta) := \{J \in \mathrm{GL}(V) \,|\, J^2 = -1 \quad \text{and} \quad \beta(x, Jy) \quad \text{is symmetric, positive definite}\}$$

$\mathrm{Sp}(V,\beta)$ acts on $\mathfrak{S}(V,\beta)$ by conjugation. The H-element at a point $J \in \mathfrak{S}(V,\beta)$ is $H_0 = \frac{1}{2}J$. To obtain matrix representations, consider the real Hodge structure $V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$, where $V^{1,0}$ (resp. $V^{0,1}$) is the eigenspace of the complexification of J for the eigenvalue i (resp. $-i$). Let $e_1, \ldots, e_m$ be a basis of $V^{1,0}$, and $e_{j+m} = \bar{e}_j$. Then $e_1, \ldots, e_{2m}$ is a symplectic basis for $(V_{\mathbf{C}}, i\beta)$. Note that $\mathrm{Sp}(V_{\mathbf{C}}, \beta) = \mathrm{Sp}(V_{\mathbf{C}}, i\beta)$. With respect to this basis, we have

$$i\beta = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}, \qquad H_0 = \begin{pmatrix} (i/2)I_m & 0 \\ 0 & (-i/2)I_m \end{pmatrix}. \tag{4.3.1}$$

Let G and G' be hermitian groups with symmetric domains X and X', and, H-elements H_0 and H_0', at the points $x \in X$ and $x' \in X'$, respectively. A representation $\rho\colon G \to G'$, defined over **R**, is said to satisfy the *H_1-condition* if

$$[\mathrm{d}\rho(H_0) - H_0', \mathrm{d}\rho(g)] = 0 \quad \text{for all} \;\; g \in \mathfrak{g},$$

and to satisfy the *H_2-condition* if

$$\mathrm{d}\rho(H_0) = H_0'.$$

Let $\tau : X \to X'$ be holomorphic. We say that τ is *strongly equivariant* with $\rho\colon G \to G'$, if ρ satisfies the H_1-condition, and $\tau(gx) = \rho(g)\tau(x)$ for all $g \in G(\mathbf{R})^0$.

4.4. HODGE GROUPS

The *Hodge group* $G(A)$ of a complex abelian variety A is defined to be the Hodge group of $H^1(A, \mathbf{Q})$. The derived group G^{der} of the Hodge group is a connected, semisimple, linear algebraic group over $\mathbf{Q}$ such that $G^{\mathrm{der}}(\mathbf{R})$ is of Hermitian type and G^{der} has no nontrivial, connected normal $\mathbf{Q}$-subgroup H such that $H(\mathbf{R})$ is compact [Mm1]. Let $V := H^1(A, \mathbf{Q})$, and let β be a Riemann form for A. Then $G(A) \subset \mathrm{Sp}(V, \beta)$, and the inclusion of G^{der} into $\mathrm{Sp}(V, \beta)$ satisfies the H_1-condition. It satisfies the H_2-condition if and only if $G(A)$ is semisimple (see [A2, Prop. 2.2, p. 1124]). If two abelian varieties have the same simple factors then their Hodge groups are the same.

PROPOSITION 4.4.1. *Let A and B be abelian varieties dominated by $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Assume that A and B satisfy the unusual Hodge conjecture and the Hodge group of $A \times B$ is the product of the Hodge groups of A and B. Then $A \times B$ also satisfies the unusual Hodge conjecture; it is dominated by $\{X \times Y \mid X \in \mathfrak{A},\ Y \in \mathfrak{B}\}$.*

Proof. Any irreducible Hodge structure in $H^n(A \times B, \mathbf{Q})$ is of the form $V \otimes W$, where $V \subset H^i(A, \mathbf{Q})$ and $W \subset H^j(B, \mathbf{Q})$ are irreducible Hodge structures and $i + j = n$. By assumption V and W are equivalent to Hodge structures $V' \subset H^a(X, \mathbf{Q})$ and $W' \subset H^b(Y, \mathbf{Q})$, where V' and W' have height 0, $X \in \mathfrak{A}$, and, $Y \in \mathfrak{B}$. Then $V' \otimes W' \subset H^{a+b}(X \times Y, \mathbf{Q})$ has height 0 and is equivalent to $V \otimes W$. □

4.5. THE LEFSCHETZ GROUP

Let A be a complex abelian variety, $D := \mathrm{End}\,(A) \otimes \mathbf{Q}$ its endomorphism algebra, $V := H^1(A, \mathbf{Q})$, β a Riemann form on A, and ρ the induced involution on D. The Lefschetz group $L(A)$ is defined by Murty [Mt1, p. 198] as the connected component of the centralizer of D in the symplectic group $\mathrm{Sp}(V, \beta)$. It is a reductive algebraic group over $\mathbf{Q}$ which contains the Hodge group of A.

Suppose now that A is simple. Let E be the center of D, and k the field of invariants of ρ. Then k is the maximal real subfield of E. Recall [Sm2, Lemma 1.2, p. 162] that there exists a unique E-bilinear form $T\colon V \times V \to D$ such that for all $x, y \in V$, and all $a, b \in D$,

$$\beta(x, y) = \mathrm{Tr}_{D/\mathbf{Q}}\, T(x, y), \qquad T(ax, by) = aT(x, y)b^{\rho}, \quad \text{and}$$
$$T(y, x) = -T(x, y)^{\rho}.$$

Then $L(A)$ is the connected component of the restriction of scalars of a unitary group

$$L(A) = R_{k/\mathbf{Q}}\, \mathrm{Aut}_D(V, T)^0. \tag{4.5.1}$$

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4.6. KUGA FIBER VARIETIES

Let G be a semisimple linear algebraic group over $\mathbf{Q}$ such that $G(\mathbf{R})$ is of hermitian type and has no compact factors defined over $\mathbf{Q}$. Denote by X the bounded symmetric domain associated to G. Let V be a finite dimensional vector space over $\mathbf{Q}$, β a nondegenerate alternating form on $V, \rho\colon G \to \mathrm{Sp}(V, \beta)$ a representation defined over $\mathbf{Q}$, which satisfies the H_1-condition, and $\tau\colon X \to \mathfrak{S}(V, \beta)$ a strongly equivariant map. Let Γ be a torsion-free arithmetic subgroup of G, and L a Γ-lattice in V on which β takes integer values. From this data Kuga ([K], cf. [Sa3, Sect. IV.7, pp. 195–202]) constructed a family of polarized abelian varieties, called a *Kuga fiber variety* over the arithmetic variety $\Gamma\backslash X$, such that the fiber over Γx is the torus $V_{\mathbf{R}}/L$ with the complex structure $\tau(x)$, and Riemann form β.

Let A be an abelian variety with Hodge group G and Lefschetz group L. The derived groups G^{der} and L^{der} are both hermitian, and their inclusions into the symplectic group of a Riemann form satisfy the H_1-condition. The corresponding Kuga fiber varieties are the 'Hodge families' of Mumford ([Mm1], [Mm2]), and the PEL-families of Shimura ([Sm3], see also [Sa3, p. 200, Example 2]), respectively. We shall say that A is of PEL-*type* if Mumford's Hodge family is a PEL-family. Thus A is of PEL-type if and only if $L^{\mathrm{der}} = G^{\mathrm{der}}$. If the Hodge ring of every power of A is generated by divisors then A is of PEL-type [Mt1, Theorem 3.1, p. 202]. From this we see that all the abelian varieties listed in Section 4.2 for which the general Hodge conjecture is currently known are of PEL-type.

5. The main theorem

THEOREM 5.1. *Let A be an abelian variety of* PEL-*type with semisimple Hodge group. Suppose that for every simple factor B of A, if B is of type* III, *then $H^1(B, \mathbf{Q})$ has odd dimension as a vector space over the endomorphism algebra of B. Then the group theoretic and rational Hodge filtrations on the cohomology of A coincide, A is dominated by any sufficiently large power of itself, and the usual Hodge conjecture for all powers of A implies the general Hodge conjecture for A.*

Proof. A is isogenous to a product $A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_l^{n_l}$ where the A_i are pairwise nonisogenous abelian varieties. By the multiplicativity of the L-group [Mt1, Lem. 2.1, p. 198] we have

$$L(A) = L(A_1) \times L(A_2) \times \cdots \times L(A_l).$$

Since

$$G(A) \subset G(A_1) \times G(A_2) \times \cdots \times G(A_l)$$

and $G(A)$ equals the derived group of $L(A)$, we conclude that each A_i is of PEL-type, and,

$$G(A) = G(A_1) \times G(A_2) \times \cdots \times G(A_l).$$

Proposition 4.4.1 now implies that it is enough to prove the theorem when A is a power of a simple abelian variety.

Let A be a power of a simple abelian variety A_0, and let $G := G(A)$. Let U be an irreducible Hodge structure contained in the cohomology of A. Let W be an irreducible G-submodule of $U_{\mathbf{R}}$. Suppose there exists a G-submodule W' of $H^a(A_0^b, \mathbf{R})$ for some a, b, such that W' contains $(a, 0)$-forms and is G-equivalent to W. Then the smallest rational Hodge structure U' containing W' is G-equivalent to U and has height 0. (We have used here the fact that two representations of G defined over $\mathbf{Q}$ are equivalent over $\mathbf{Q}$ if and only if they are equivalent over $\mathbf{R}$. This follows from the density of $G(\mathbf{Q})$ in $G(\mathbf{R})$ [BS, Theorem A, p. 26]. See [Sa2, Lemma 1, p. 220] or [A1, Lemma 2.1, p. 228] for more details.) Therefore, to complete the proof of the theorem, it suffices to show that such a W' always exists.

Let D be the endomorphism algebra of A_0, E the center of D, and k the maximal real subfield of E. From 4.5.1 we see that $G(\mathbf{R})$ is a product ΠG_α indexed by the set S of embeddings of k into $\mathbf{R}$. Let $\alpha \in S$. Then

$$D_\alpha := D \otimes_{k,\alpha} \mathbf{R} \cong \begin{cases} \mathbf{R} & \text{if } D \text{ is of type I or II,} \\ \mathbf{C} & \text{if } D \text{ is of type IV,} \\ \mathbf{H} & \text{if } D \text{ is of type III.} \end{cases}$$

From 4.1 we now see that, in the notation of Helgason [He, pp. 444–445],

$$G_\alpha \cong \begin{cases} \mathrm{Sp}(n, \mathbf{R}) & \text{if } D \text{ is of type I or II,} \\ \mathrm{SU}(p, q) & \text{if } D \text{ is of type IV,} \\ \mathrm{SO}^*(2n) & \text{if } D \text{ is of type III.} \end{cases}$$

If D is of type IV, our assumption that G is semisimple implies that $p = q$ (see [Sa3, Chap. IV, (4.14), p. 183]). In all cases, we find that G_α is not compact. It then follows from [Sa3, Sect. IV.5, pp. 185–186] that we have a decomposition $H^1(A_0, \mathbf{R}) = \bigoplus_{\alpha \in S} V_\alpha$, where each V_α is a real Hodge substructure of $H^1(A_0, \mathbf{R})$ on which G_γ acts trivially for $\gamma \neq \alpha$, and

$$G_\alpha \to \mathrm{Sp}(V_\alpha, \beta \,|\, V_\alpha \times V_\alpha)$$

is an H_2-morphism for each α. Thus to complete the proof, it suffices to show that for each irreducible G_α-submodule W_α in the exterior algebra of V_α, there exists an equivalent G_α-submodule W'_α of height 0 in the exterior algebra of V_α^b for some b.

To simplify the notation we drop the subscript α, writing D for D_α, G for G_α, and V for V_α. We divide the proof into three cases, according to whether $D = \mathbf{R}$, $\mathbf{C}$, or, $\mathbf{H}$.

Case 1: $D = \mathbf{R}$.

In this case G is a symplectic group $\mathrm{Sp}(V_0, \beta)$, and V is equivalent as a G-module to either V_0 or $V_0 \oplus V_0$, according to whether A_0 is of type I or type II (see [Sa1, Sect. 3.4, p. 451] and [Mt1, pp. 201–202]). We may assume without loss of generality that $V_0 = V$. With the matrix representation of Section 4.3, the symplectic Lie algebra is given by

$$\mathfrak{sp}(V_{\mathbf{C}}, \beta) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| B, C \text{ symmetric} \right\},$$

and a Cartan subalgebra is given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \right\}.$$

Each λ_j is evidently a weight of $\mathfrak{sp}(V_{\mathbf{C}}, \beta)$ and e_j is a vector of weight λ_j. The fundamental weights of $\mathfrak{sp}(V_{\mathbf{C}}, \beta)$ are $\mu_j := \lambda_1 + \cdots + \lambda_j$. The representation of $\mathfrak{sp}(V_{\mathbf{C}}, \beta)$ on $\Lambda^j V_j$ has highest weight μ_j; it is equivalent to the direct sum of $\Lambda^{j-2} V$ and the irreducible representation π_j of highest weight μ_j [Vr, p. 394, Exercise 24]. A vector of highest weight in $\Lambda^j V$ is $e_1 \wedge e_2 \wedge \cdots \wedge e_j$. Since each $e_i \in V^{1,0}$, we see that the representation of highest weight equal to μ_j in $\Lambda^j V$ contains a vector in $V^{j,0}$.

Any irreducible representation π of $\mathfrak{sp}(V_{\mathbf{C}}, \beta)$ has highest weight $\mu = a_1\mu_1 + \cdots + a_n\mu_n$, where the a_j are nonnegative integers. μ is the highest weight of

$$\pi_1 \otimes \pi_1 \otimes \cdots \otimes \pi_2 \otimes \cdots \otimes \pi_n,$$

where we take the tensor product of a_j copies of π_j for each j. Let $a := a_1 + 2a_2 + \cdots + na_n$. From the Künneth decomposition

$$\bigwedge^a V^m \cong \bigoplus_{c_1 + c_2 + \cdots + c_m = a} \left(\bigwedge^{c_1} V \bigotimes \bigwedge^{c_2} V \bigotimes \cdots \bigotimes \bigwedge^{c_m} V \right) \tag{5.1}$$

we see that $\wedge^a V_{\mathbf{C}}^b$ contains a submodule W equivalent to π provided that

$$b \geqslant a_1 + a_2 + \cdots + a_n.$$

Furthermore, this submodule contains an $(a, 0)$-form.

Case 2: $D = \mathbf{C}$.

In this case G is the real Lie group $\mathrm{SU}(m, m)$ for some $m \geqslant 2$, and $G(\mathbf{C})$ is isomorphic to $\mathrm{SL}_{2m}(\mathbf{C})$. With respect to a suitable basis the Lie algebra of $G(\mathbf{R})$ is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_{12} \\ {}^t\bar{X}_{12} & X_2 \end{pmatrix} \in \mathfrak{sl}_{2m}(\mathbf{C}) \middle| \begin{matrix} X_i \in M_m(\mathbf{C}) \\ {}^t\bar{X}_i = -X_i \end{matrix} \right\},$$

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a Cartan subalgebra is given by

$$\mathfrak{t} = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{2m}) \in \mathfrak{sl}_{2m}(\mathbf{C}) \mid \lambda_j \in i\mathbf{R}\}, \tag{5.2}$$

and an H-element is given by (4.3.1) (see [Sa1, p. 430]). The fundamental weights of $\mathfrak{g}_{\mathbf{C}}$ relative to $\mathfrak{t}_{\mathbf{C}}$ are

$$\mu_k := \lambda_1 + \cdots + \lambda_k, \quad 1 \leqslant k \leqslant 2m-1.$$

The action of $\mathfrak{g}$ on V is given by the symplectic representation

$$\begin{pmatrix} X_1 & X_{12} \\ {}^t\bar{X}_{12} & X_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{X}_2 & 0 & 0 & {}^tX_{12} \\ 0 & X_1 & X_{12} & 0 \\ 0 & {}^t\bar{X}_{12} & X_2 & 0 \\ \bar{X}_{12} & 0 & 0 & \bar{X}_1 \end{pmatrix}, \tag{5.3}$$

(see [Sa1, Section 1.5, pp. 432–433, and, Section 3.2, p. 447] and [Mt1, pp. 201–202]). Then $V_{\mathbf{C}} = W \oplus \bar{W}$, where W and $\bar{W}$ are irreducible $\mathfrak{g}_{\mathbf{C}}$-modules with highest weights μ_1 and μ_{2m-1} respectively. Each of W and $\bar{W}$ is the sum of an m-dimensional space of $(1,0)$-forms and an m-dimensional space of $(0,1)$-forms. For $1 \leqslant k \leqslant m$, $\Lambda^k W$ is an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule of $\Lambda^k V_{\mathbf{C}}$ which contains $(k,0)$-forms and has highest weight μ_k. For $m < k \leqslant 2m-1$, $\Lambda^{2m-1-k}\bar{W}$ is an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule of $\Lambda^{2m-1-k}V_{\mathbf{C}}$ which contains $(2m-1-k, 0)$-forms and has highest weight μ_k. Thus any fundamental representation of $\mathfrak{g}_{\mathbf{C}}$ occurs in $\Lambda^a V_{\mathbf{C}}$ and contains $(a,0)$-forms for some a.

Any irreducible representation π of $\mathfrak{g}_{\mathbf{C}}$ has highest weight $\mu = a_1\mu_1 + \cdots + a_{2m-1}\mu_{2m-1}$, where the a_j are nonnegative integers. μ is the highest weight of

$$\pi_1 \otimes \pi_1 \otimes \cdots \otimes \pi_2 \otimes \cdots \otimes \pi_{2m-1},$$

where we take the tensor product of a_j copies of π_j for each j. Let

$$a := a_1 + 2a_2 + \cdots + ma_m + (m-1)a_{m+1} + \cdots + a_{2m-1}.$$

From the Künneth formula (5.1) we see that $\Lambda^a V_{\mathbf{C}}^b$ contains a submodule equivalent to π provided that $b \geqslant a_1 + a_2 + \cdots + a_{2m-1}$. Furthermore this submodule contains an $(a,0)$-form.

Case 3. $D = \mathbf{H}$.

Let m be the dimension of $H_1(A_0, \mathbf{Q})$ over $D(A_0)$, which we are assuming to be odd. It follows from [Sm1, Prop. 15, p. 177] that $m \neq 1$, and thus $m \geqslant 3$. The real Lie group G is simple and of type D_m. Its Lie algebra is given by

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_{12} \\ -\bar{X}_{12} & \bar{X}_1 \end{pmatrix} \in \mathfrak{sl}_{2m}(\mathbf{C}) \;\middle|\; \begin{matrix} X_1, X_{12} \in M_m(\mathbf{C}) \\ {}^t\bar{X}_1 = -X_1, \quad {}^tX_{12} = -X_{12} \end{matrix} \right\},$$

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a Cartan subalgebra is given by (5.2), and an H-element is given by (4.3.1). The fundamental weights of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{t}_{\mathbf{C}}$ are $\mu_1, \ldots, \mu_m$, where

$$\mu_k = \lambda_1 + \cdots + \lambda_k \quad \text{for } 1 \leqslant k \leqslant m-2,$$
$$\mu_{m-1} = (\lambda_1 + \cdots + \lambda_{m-1} - \lambda_m)/2, \quad \text{and}$$
$$\mu_m = (\lambda_1 + \cdots + \lambda_{m-1} + \lambda_m)/2.$$

The inclusion of $\mathfrak{g}$ into $\mathfrak{su}(m,m)$ satisfies the H_2-condition. The action of $\mathfrak{g}$ on V is the composition of this inclusion with the symplectic representation defined in (5.3) (see [Sa1, Sect. 3.3, pp. 449–451] and [Mt1, pp. 201–202]).

Recall from *Case* 2 of this proof that $V_{\mathbf{C}} = W \oplus \bar{W}$. V is an irreducible $\mathfrak{g}$-module, but W is equivalent to $\bar{W}$, so $\mathfrak{g}_{\mathbf{C}}$ acts on $V_{\mathbf{C}}$ as two copies of the standard representation. W is a $2m$-dimensional vector space containing an m-dimensional subspace of (1, 0)-forms and an m-dimensional subspace of (0, 1)-forms. For $1 \leqslant k \leqslant m-2$, $\Lambda^k W$ is an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule of $\Lambda^k V_{\mathbf{C}}$ containing $(k,0)$-forms, and having highest weight μ_k. $\Lambda^{m-1} W$ is an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule of $\Lambda^{m-1} V_{\mathbf{C}}$ containing $(m-1,0)$-forms, and having highest weight $\mu_{m-1}+\mu_m$. However $\Lambda^m W$ splits as the direct sum of two irreducible $\mathfrak{g}_{\mathbf{C}}$-submodules, U_1 and U_2, having highest weights $2\mu_{m-1}$ and $2\mu_m$, respectively. U_2 contains $(m,0)$-forms, but U_1 does not.

The complex conjugate of μ_{m-1} is $(-\lambda_1 - \cdots - \lambda_{m-1} + \lambda_m)/2$ which is conjugate to μ_m under the action of the Weyl group when m is odd. It follows that for odd m, the complex conjugate of U_1 is equivalent to U_2, and hence any real $\mathfrak{h}$-submodule of $\Lambda^m V$ which contains U_1 must also contain a $\mathfrak{g}_{\mathbf{C}}$-submodule equivalent to U_2, and thus must contain $(m,0)$-forms.

Any irreducible representation of $\mathfrak{g}_{\mathbf{C}}$ has highest weight $\mu = a_1\mu_1 + \cdots + a_m\mu_m$, where the a_j are nonnegative integers. If the representation appears in the exterior algebra of $V_{\mathbf{C}}$, then all of its weights must be integral linear combinations of the λ_j's. Hence $a_{m-1} \equiv a_m \bmod 2$.

Suppose first that $a_m \geqslant a_{m-1}$. Then we may write

$$\mu = a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_{m-1}(\mu_{m-1} + \mu_m) + b_m(2\mu_m),$$

with $b_m = a_m - a_{m-1}$. Let

$$a := a_1 + 2a_2 + \cdots + (m-1)a_{m-1} + mb_m.$$

The above considerations show that $\wedge^a V_{\mathbf{C}}^b$ contains an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule with highest weight μ which contains $(a,0)$-forms, provided that $b \geqslant a_1 + a_2 + \cdots + a_{m-1} + b_m$.

Next suppose that $a_m < a_{m-1}$, write

$$\mu = a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_m(\mu_{m-1} + \mu_m) + c_m(2\mu_{m-1}),$$

with $c_m = a_{m-1} - a_m$, and let

$$a := a_1 + 2a_2 + \cdots + (m-2)a_{m-2} + (m-1)a_m + mc_m.$$

Then, for $b \geqslant a_1 + a_2 + \cdots + a_{m-2} + a_m + c_m$, $\wedge^a V_{\mathbf{C}}^b$ contains an irreducible $\mathfrak{g}_{\mathbf{C}}$-submodule U with highest weight μ. The complex conjugate of U has highest weight

$$a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_m(\mu_{m-1} + \mu_m) + c_m(2\mu_m).$$

Hence the smallest real $\mathfrak{g}$-submodule of $\wedge^a V_{\mathbf{C}}^b$ containing U contains $(a,0)$-forms. □

Remark 5.2. Case 1 of the above proof is similar to the proofs of Tankeev [T] and Hazama [Ha] of the general Hodge conjecture for these abelian varieties. Their proofs, in fact, show more – that in this case the abelian variety is dominated by itself. In Cases 2 and 3, however, this is no longer true; for example, a 4-dimensional abelian variety of Weil type [W] is not dominated by itself, though it is dominated by its square.

6. Some other cases

In Section 4.2 we listed the abelian varieties for which the general Hodge conjecture is currently known. Of these, only (a) satisfies the hypotheses of Theorem 5.1; the Hodge group fails to be semisimple in the other three cases. However, the conclusions of Theorem 5.1 hold in cases (b) and (c). In Theorem 6.1 below, we show that any product of elliptic curves is dominated by a power of itself. As for the abelian variety in case (c), Theorem 2 of [T] shows that it is dominated by itself. In case (d), the abelian variety is not dominated by any power of itself. It is, however, dominated by the square of an elliptic curve, together with the set of powers of itself. I would like to thank Chad Schoen for explaining this last point to me.

THEOREM 6.1. *If A is a product of elliptic curves then the group theoretic filtration, the arithmetic filtration, and the rational Hodge filtration are all equal, and the general Hodge conjecture is true. A is dominated by any sufficiently large power of A.*

Proof. The usual Hodge conjecture for products of elliptic curves is well known; indeed, the Hodge ring of such an abelian variety is generated by divisors. This was first proved by Tate (unpublished, see [Gr, p. 302]); detailed proofs may be found in [I] and [Mt2, Section 2]. It follows that the Hodge and Lefschetz groups of A coincide. The multiplicativity of the Lefschetz group and Proposition 4.4.1 now imply that it suffices to prove the theorem for a power of an elliptic curve. If the elliptic curve is not of CM-type, this is a special case of Theorem 5.1.

Let $A = E^m$, where E is an elliptic curve of CM-type. Then the Hodge group G of A is a 1-dimensional torus acting on $H^1(E, \mathbf{C})$ as $\chi + \chi^{-1}$, where χ is a character. The representation space of χ is of Hodge type (1, 0), and the representation space of χ^{-1} is of type (0, 1). Any irreducible representation of $G(\mathbf{C})$ has the form χ^n where n is an integer. If $m > n > 0$, then $H^n(E^m, \mathbf{C})$ contains a 1-dimensional subspace on which G acts as χ^n; this subspace consists of $(n, 0)$-forms. Since G is not split, if V is any representation of G defined over $\mathbf{Q}$ such that $V_{\mathbf{C}}$ contains χ^n then it also contains χ^{-n}. It follows that the unusual Hodge conjecture holds for A. The equality of these filtrations with the arithmetic filtration now follows from Proposition 3.2. □

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