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ON THE POSITIVE ROOTS OF AN EQUATION INVOLVING MODIFIED BESSEL FUNCTIONS

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We use the Mittag–Leffler partial fractions expansion of $J_{\nu+1}(x)/J_\nu(x)$ to give simple proofs of some recent results due to S. H. Lehnigk concerning the number of positive roots of the equation $(-Br^2+A+q)I_q(r)+rI_{q+1}(r)=0$, where A is real, $B>0$ and $q>-1$.

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1. Introduction

In a recent article [3], S. H. Lehnigk proved three theorems concerning the positive roots of the equation

$$(-Br^2+A+q)I_q(r)+rI_{q+1}(r)=0 \tag{1}$$

where A is real, $B>0$ and $q>-1$. He was led to this function through a study of the delta function initial condition solution of the generalized Feller equation. The main purpose of this note is to use the Mittag–Leffler partial fractions expansion of $J_{\nu+1}(x)/J_\nu(x)$ to give conceptually simpler proofs of Lehnigk's results and to unify the proofs of his three theorems. With little additional effort, the present method leads to some extensions of the results of [3] and to results on the monotonicity of the roots with respect to some of the parameters.

2. Lehnigk's results

In view of the expansion [4, pp. 497–498]

$$\frac{J_{\nu+1}(z)}{J_\nu(z)}=\sum_{n=1}^{\infty}\frac{2z}{j_{\nu n}^2-z^2}, \tag{2}$$

where $\pm j_{\nu n}$ are the zeros of $z^{-\nu}J_\nu(z)$, and the relation [4, p. 77]

$$I_q(r)=e^{-q\pi i/2}J_q(ir), \qquad r>0,$$

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we have

$$\frac{I_{q+1}(r)}{rI_q(r)}=\sum_{n=1}^{\infty}\frac{2}{j_{qn}^2+r^2}, \qquad r>0.$$

Thus the positive roots of equation (1) are the same as those of the equation

$$\sum_{n=1}^{\infty}\frac{2}{j_{qn}^2+r^2}+\frac{A+q}{r^2}=B. \tag{3}$$

Since $q>-1$, we have j_{qn} real [**4**, p. 483] and so $j_{qn}^2>0$, $n=1,2,\ldots$. We consider the three cases dealt with in [**3**]. In what follows, when we assert that a function is increasing (or decreasing), we exclude the possibility that its derivative may vanish at one or more points.

Case (i). $A+q>0$. In this case the left-hand side of (3) decreases from $+\infty$ to 0 as r increases from 0 to $+\infty$. Hence, for $B>0$, equation (3) has exactly one positive simple root. This is Lehnigk's Theorem 3.1.

Case (ii). $A+q=0$. Now the left-hand side of (3) decreases from $[2(1+q)]^{-1}$ to 0. (We can use, for example, the power series expansion of $I_q(r)$ to get the value at $r=0$.) In this way we get [**3**, Theorem 3.2], which asserts that when $A+q=0$, (1) has one simple positive root if $B<[2(1+q)]^{-1}$ and no positive roots otherwise.

Case (iii). $A+q<0$. The left-hand side, $g(r)$, of (3) is not longer a monotonic function of r. We have

$$-r^3g'(r)=\sum_{n=1}^{\infty}\frac{4}{(j_{qn}^2/r^2+1)^2}+2(A+q), \qquad r>0. \tag{4}$$

(The differentiation is justified since the differentiated series converges uniformly in r in compact subsets of $(0,\infty)$.) Now the right-hand side of (4) increases from the negative value $2(A+q)$ to $+\infty$ as r increases from 0 to $+\infty$. This shows that $g(r)$ increases from $-\infty$, achieves a maximum value $\tilde{B}(>0)$ and then decreases to 0. Thus we have Lehnigk's Theorem 3.3 which asserts the existence of exactly one positive $\tilde{B}$ such that (1) has two positive roots if $B<\tilde{B}$, one positive double root if $B=\tilde{B}$ and no positive root if $B>\tilde{B}$. Incidentally, our method shows also that if $A+q<0$ and $B\leqq 0$, then equation (3) has a single positive root.

3. Extensions and further results

An advantage of the present simple method is that certain extensions are immediate. Thus in Case (i), for example, we can replace the r^2 in the left-hand side of (1) by any function which increases from 0 to ∞ and in Case (ii) we can replace r^2 by any function which does not vanish on $(0,\infty)$.

It is clear that, in Case (i), for fixed A and q, the single positive root decreases from $+\infty$ to 0 as B increases from 0 to $+\infty$. Also, since the left-hand side of (3) is an increasing function of A for each fixed B, q and r, we see that the root increases to $+\infty$ as A increases from $-q$ to $+\infty$.

In Case (ii), $A+q=0$, we are dealing with the root of the equation

$$\sum_{n=1}^{\infty} \frac{2}{j_{qn}^2+r^2}=B. \tag{5}$$

Since the left-hand side is a decreasing function of r, we see that the single root decreases from $+\infty$ to 0 as B increases from 0 to $[2(1+q)]^{-1}$. In this case, it is possible to say something about monotonicity in q also. For each fixed r, the left-hand side of (5) is a decreasing function of q. (This is because each of the j_{qn} is an increasing function of q, $q>-1$ [**4**, p. 508].) Hence we see that, for fixed B, the root decreases to zero as q increases from -1 to $1/(2B)-1$.

With regard to Case (iii), our method shows that if we keep A and q fixed and let B decrease from $\tilde{B}$ to 0, the smaller of the two roots decreases to a positive value and the larger increases to $+\infty$. As we continue to decrease B through values $\leqq 0$, the smaller root decreases to 0 while the larger one has disappeared.

4. Related work

The real roots of (1) are the same as the purely imaginary roots of

$$(Br^2+A)J_q(r)+rJ_q'(r)=0. \tag{6}$$

The existence of such roots and bounds for them in various special cases have been considered in [**1**].

The expansion (2) has been found useful also in [**2**] in studying properties of zeros of the function $(\alpha+\delta z)J_\nu(z)+(\beta+\gamma z)J_\nu'(z)$.

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