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DISCRETE SUBSETS OF PROXIMITY SPACES

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The distinct Hausdorff compactifications δX of a completely regular (Hausdorff) space X are in one-one correspondence with the admissible proximity relations δ on X, or alternatively, with the admissible totally bounded uniform structures for X. (See [**1**], [**2**].) Thus, δX is the Smirnov compactification of (X, δ). Generalized uniform structures $\mathscr{U}$ for X will be described by means of pseudometrics on X (cf. [**5**], [**7**], [**13**]). Let $\sigma \in \mathscr{U}$, where $\mathscr{U}$ is in the proximity class $\pi(\delta)$ associated with (X, δ). Then a subset S of X is *σ-discrete of gauge* $\epsilon > 0$ if $\sigma(x, y) \geqq \epsilon$, for all $x, y \in S$, where $x \neq y$.

In this paper we show that if (X, δ) contains an infinite σ-discrete subset of positive gauge, then $\operatorname{card}(\delta X - X) \geqq 2^c$, where c is the cardinal of the continuum. Results concerning zero-sets of δX in $\delta X - X$ and the Q-closure of (X, δ) are also obtained.

Let $\upsilon_\delta X$ be the real-completion of (X, δ) (see [**8**], [**11**]). Then if $\operatorname{card}(\delta X - X) < 2^c$, it follows that $\upsilon_\delta X = \delta X$ and $\pi(\delta)$ contains only the unique totally bounded uniform structure compatible with δ. Also, if $\upsilon_\delta X \neq \delta X$, then $\operatorname{card}(\delta X - \upsilon_\delta X) \geqq 2^c$.

In (4) we establish that if X and Y are realcomplete metric proximity spaces, then X and Y are uniformly isomorphic if and only if their respective algebras of bounded uniformly continuous real-valued functions are isomorphic.

2. Realcompletions and the Q-closure. Let $P(X)$ be the collection of real-valued proximity functions defined on (X, δ) and $P^*(X)$ be the algebra of bounded members of $P(X)$. Recall that the realcompletion $\upsilon_\delta X$ of (X, δ) is the set of all points in δX to which every member of $P(X)$ can be extended with real values as a p-function. Denote the Smirnov extension of $f \in P(X)$ to δX by f^δ. Throughout this paper the proximity and uniform structures on the real numbers R will be those associated with the standard metric. Definitions and results concerning round filters may be found in [**13**] and notation and terminology for rings of continuous functions will follow that of [**5**].

A proximity space will be called *p-pseudocompact* if $P(X) = P^*(X)$. The theory of p-systems in $P(X)$ is developed in [**10**]. A realcomplete proximity space is realcompact, but a realcompact space need not be realcomplete for every compatible proximity (cf. [**11**] or Example 2.3).

PROPOSITION 2.1. *A proximity space (X, δ) is compact if and only if (X, δ) is realcomplete and p-pseudocompact.*

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Proof. Necessity is obvious. Conversely, if $\mathscr{F}$ is a maximal round filter in (X, δ), then $P(X) = P^*(X)$ implies that $\mathscr{F}$ is real (see [**8**]). Since (X, δ) is realcomplete, $\mathscr{F}$ is fixed. Thus X is compact and the proof is complete.

D. Harris in [**6**] has defined the Q-closure of (X, δ) to be the set of all points p in δX with the property that whenever $f \in C(\delta X)$ and $f(p) = 0$ there exists $q \in X$ such that $f(q) = 0$. Let $Q_\delta(X)$ denote the Q-closure of (X, δ). By Theorem F of [**6**], $p \in Q_\delta(X)$ if and only if $\mathscr{F}^p$ has the countable intersection property, where $\mathscr{F}^p$ is the unique maximal round filter in (X, δ) which converges to p.

The following theorem shows that if (X, δ) is not Q-closed, then $(X, \mathscr{U})$ cannot be complete for all $\mathscr{U} \in \pi(\delta)$. In particular, (X, δ) cannot be realcomplete.

THEOREM 2.2. *Let $\mathscr{U} \in \pi(\delta)$ and* card$X$ *be non-measurable. If $(X, \mathscr{U})$ is complete, then X is Q-closed.*

The proof of Theorem 2.2 consists of showing that each maximal round filter in (X, δ) with the countable intersection property must be a Cauchy filter in $(X, \mathscr{U})$ and is analogous to the proof that (B) implies (C) of Theorem 4.1 of [**9**].

The converse of Theorem 2.2 is false. If $X = (0, 1)$ with the standard metric proximity, then (X, δ) is Q-closed but $\pi(\delta)$ contains only the metric uniform structure $\mathscr{U}_d$ and $(X, \mathscr{U}_d)$ is not complete.

Example 2.3. Let X be the unit ball in l_2, the space of square summable real sequences, and let δ be the proximity associated with the standard metric for X. Then, as is the case for any metric proximity, $P(X)$ is just the collection of real-valued uniformly continuous functions. Thus, (X, δ) is p-pseudocompact (cf. problem 15 D of [**5**]). Now X is complete in its metric uniform structure so that (X, δ) is Q-closed.

If β is the proximity associated with the Stone-Čech compactification βX of X, then X is pseudocompact if and only if every maximal round filter in (X, β) has the countable intersection property. The analogous result for p-pseudocompactness does not necessarily hold if $\delta \neq \beta$, however, as is evident from Example 2.3.

3. Cardinals of sets in $\delta X - X$. We say that a pseudometric σ for X is compatible with a proximity δ if $\sigma(A, B) = 0$ whenever $A\delta B$, where A, B are subsets of X. If S is a σ-discrete subset of (X, δ) having positive gauge, then S is C-embedded in X. However, an example of [**4**] (p. 157) shows that we need not have $P(S) = P(X)|S$, where S has the discrete proximity, and that not every continuous pseudometric on S can be extended to a pseudometric on X compatible with δ.

THEOREM 3.1. *If (X, δ) contains an infinite σ-discrete subspace S of positive gauge, where σ is compatible with δ, then* card$(\delta X - X) \geqq 2^c$.

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Proof. Let δ_s be the proximity for S inherited from (X, δ) so that (S, δ_s) is discrete. Since the gauge of S is positive, (S, δ_s) is p-homeomorphic with (N, β), where N denotes the natural numbers. Moreover, $P^*(S) = C^*(S)$.

Since S is C^*-embedded in X, $P^*(S) = P^*(X)|S$ implies that S is C^*-embedded in δX, hence in $\mathrm{Cl}_{\delta X}S$. Thus $\mathrm{Cl}_{\delta X}S = \delta_s S$. Let $\mathscr{F}$ be any free maximal round filter in (S, δ_s). Then $\mathscr{F}$ converges to a point x of $\delta_s S - S$. If 2ϵ is the gauge of S and if $x \in X$, then the ϵ-ball about x determined by σ contains at most one point of S. Thus $\mathscr{F}$ cannot converge to x, therefore the limit points of the free maximal round filters in (S, δ_s) are in $\delta X - X$. Now $\delta_s S - S \subseteq \delta X - X$ and $\mathrm{card}\,\delta_s S = \mathrm{card}\,\beta N = 2^c$ implies that $\mathrm{card}\,\delta X - X \geqq 2^c$.

This completes the proof.

COROLLARY 3.2. *If* (X, δ) *satisfies* $\mathrm{card}(\delta X - X) < 2^c$, *then* (X, δ) *is p-pseudocompact and* $v_\delta X = \delta X$.

Proof. If $P(X) \neq P^*(X)$ choose $f \in P(X)$, where f is unbounded on X. Set $\sigma_f(x, y) = |f(x) - f(y)|$, so that σ_f is a pseudometric for X compatible with δ. Since f is unbounded, X contains an infinite subset S which is σ_f-discrete of gauge 1. Theorem 3.1 now yields a contradiction. Thus (X, δ) is p-pseudocompact and the proof is complete.

For metric proximity spaces, the following result applies.

COROLLARY 3.3. *Let* (X, d) *be a metric space with associated proximity* δ. *Then* $\mathrm{card}(\delta X - X) < 2^c$ *if and only if* d *is totally bounded.*

Proof. Necessity is immediate from Theorem 3.1 and sufficiency follows from the fact that δX is the completion of a totally bounded, therefore separable, metric space.

Example 3.4. The converse of Corollary 3.2 is false since the space (X, δ) of Example 2.3 is p-pseudocompact but is a non-totally bounded metric proximity space so that $\mathrm{card}(\delta X - X) \geqq 2^c$. Since X is separable δX must be a continuous image of βN, the Stone-Čech compactification of the natural numbers (cf. 9.A. [**5**]). Thus $\mathrm{card}(\delta X - X) = \mathrm{card}(\beta X - X) = 2^c$. Yet X is not pseudocompact so that any unbounded member of $C(X)$ is a proximity function with respect to (X, β) but not with respect to (X, δ). Thus $\beta X \neq \delta X$. We further observe that since the proximity class $\pi(\delta)$ contains the metric uniform structure which is not totally bounded, then it follows from results of Reed and Thron (cf. Corollary 2.1.3 of [**12**]) that $\pi(\delta)$ has at least c members.

COROLLARY 3.5. *If* $\mathrm{card}(\delta X - X) < 2^c$, *then the proximity class* $\pi(\delta)$ *contains only the unique totally bounded uniform structure.*

The following example shows that the converse of Corollary 3.5 is false.

Example 3.6. Let $\Lambda = \beta R - (\beta N - N)$ (see [**5**]) and take $\delta = \beta$. If σ is any continuous pseudometric for (Λ, β) which is not totally bounded, then Λ contains an infinite σ-discrete subset S of positive gauge. Since S must be

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C-embedded in Λ and Λ is pseudocompact, no such σ can exist. Thus the proximity class $\pi(\beta)$ contains only $\mathscr{C}^*$, the uniform structure determined by $C^*(\Lambda)$. But $\beta\Lambda = \beta R$ and $\text{card}(\beta\Lambda - \Lambda) = 2^c$.

Unlike zero-sets of $\beta X - X$, zero-sets Z of δX contained in $\delta X - X$ may have card $Z < 2^c$. If $X = (0, 1)$ with the usual metric proximity, $\delta X = [0, 1]$ and $Z = \{0, 1\}$ is a zero set of δX. Also, while the realcompactification υX of X contains no G_δ-points of υX, we note that in this example 0 and 1 are G_δ-points of the realcompletion $\upsilon_\delta X = \delta X$ of (X, δ). Clearly, no zero set of δX contained in $\delta X - X$ can meet $Q_\delta(X)$, however.

THEOREM 3.7. *If Z is a zero-set of δX contained in $\delta X - \upsilon_\delta X$, then* card $Z \geqq 2^c$.

Proof. Let Z be a zero-set of some $f^\delta \in C^*(\delta X)$, where $Z \subseteq \delta X - \upsilon_\delta X$. Let $f = f^\delta|X$. We can assume $f > 0$ on X and set $g = f^{-1}$. Then g is unbounded on X and X contains a copy S of N on which g approaches infinity (see Corollary 1.20 [**5**]). Now g has a continuous extension to a function g_1 on $\delta X - Z$. Thus, for each point x of $\delta X - Z$, the neighborhood $\{y \in \delta X - Z| \, |g_1(x) - g_1(y)| < 1\}$ of x contains only finitely many points of S. Thus, all limit points of S lie in Z.

Let p be a limit point of S. Since $p \in Z$, $p \notin \upsilon_\delta X$ and there exists $h \in P(X)$ such that $h^\delta(p)$ is not real. Thus, h is not bounded on S. It follows that S contains a countably infinite subset T such that T is σ_h-discrete of positive gauge, where σ_h is the pseudometric for X determined by h. Now T is p-homeomorphic with N so that δT is homeomorphic with βN. Since $h \in P(X)$, $\text{Cl}_{\delta X}T = \delta T$ and $\text{Cl}_{\delta X}T - T \subseteq Z$. Thus Z contains a copy of $\beta N - N$ so that card $Z \geqq 2^c$ and the proof is complete.

COROLLARY 3.8. *If p is a G_δ-point of δX, where $p \in \delta X - X$, then $p \in \upsilon_\delta X - Q_\delta X$.*

Thus no realcomplete and non-compact (X, δ) can have a Smirnov compactification which satisfies the first countability axiom.

THEOREM 3.9. *If (X, δ) is not p-pseudocompact, then* $\text{card}(\delta X - \upsilon_\delta X) \geqq 2^c$.

Proof. Let f be an unbounded member of $P(X)$ and let σ_f be the pseudometric for (X, δ) determined by f. Since f is unbounded, (X, δ) contains a countably infinite σ_f-discrete subset S of gauge 1. Now σ_f is compatible with δ so that $\text{Cl}_{\delta X}S = \delta S$. Let $p \in \delta S - S$. The Smirnov extension f^δ of f is real-valued on $\upsilon_\delta X$, hence if $p \in \upsilon_\delta X$ the neighborhood $\{x \in \upsilon_\delta X| \, |f^\delta(x) - f^\delta(p)| < 1/2\}$ contains at most one point of S. Thus $p \notin \upsilon_\delta X$. Since $\text{card}(\delta S - S) \geqq 2^c$ and $\delta S - S \subseteq \delta X - \upsilon_\delta X$, the proof is complete.

Example 2.3 shows that $\text{card}(\delta X - X) \geqq 2^c$ can occur when X is p-pseudocompact so that $\delta X = \upsilon_\delta X$. We recall that the non-real maximal p-systems of $P(X)$ are in one-one correspondence with the points of $\delta X - \upsilon_\delta X$.

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COROLLARY 3.10. *If (X, δ) is not p-pseudocompact, then $P(X)$ contains at least 2^c non-real maximal p-systems.*

Example 3.11. Let δ be the standard metric proximity for R, the real numbers. The Smirnov extension of the identity function on R has no real values at any point of $\delta R - R$, hence (R, δ) is realcomplete. Thus $\text{card}(\delta R - R) = 2^c$ and $P(R)$ contains 2^c non-real maximal p-systems. Let $A = \{n | n \in N\}$ and $B = \{n - 1/n | n \in N\}$. Then $\text{Cl}_{\beta R}A \cap \text{Cl}_{\beta R}B = \emptyset$ but $\text{Cl}_{\delta R}A \cap \text{Cl}_{\delta R}B \neq \emptyset$ so that $\beta R \neq \delta R$.

Take $f \in C^*(X)$ and let $K(f)$ be the collection of all compactifications δX to which f has a continuous, real-valued extension f^δ. Chandler and Geller have shown in [**3**] that δX is a minimal element of $K(f)$ if and only if f^δ is $1 - 1$ on $\delta X - X$. Moreover, the proof of Theorems 1 and 2 of [**3**] apply to any δX, so that if card $\delta X - X$ is countable, there exists $f \in P^*(X)$ for which δX is a minimal element of $K(f)$.

COROLLARY 3.12. *If δX is a minimal element in $K(f)$, for some $f \in C^*(X)$, then (X, δ) is p-pseudocompact and X contains no σ-discrete infinite subset of positive gauge, for all compatible pseudometrics σ on (X, δ).*

The converse of Corollary 3.12 is false. For, if $\Lambda = \beta R - (\beta N - N)$ and $\delta = \beta$, then (Λ, β) is $(\beta -)$ pseudocompact and contains no infinite σ-discrete subset of positive gauge for all continuous σ on (Λ, β). But $\text{card}(\beta\Lambda - \Lambda) = 2^c$ so that no f^β can be $1 - 1$ on $\beta\Lambda - \Lambda$.

4. A characterization of uniformly isomorphic spaces. We observe that metric space (X, d) may be complete relative to the metric uniform structure but the associated metric proximity space may not be realcomplete (cf. Example 2.3). Let $U^*(X, d)$ be the algebra of bounded real-valued uniformly continuous functions on (X, d).

THEOREM 4.1. *Let (X, d) and (Y, d_1) be metric spaces where the associated proximity spaces (X, δ) and (Y, δ_1) are realcomplete. Then (X, d) and (Y, d_1) are uniformly isomorphic if and only if $U^*(X, d)$ and $U^*(Y, d_1)$ are isomorphic.*

Proof. Necessity is immediate. Conversely, if $U^*(X, d)$ and $U^*(Y, d_1)$ are isomorphic, then $P^*(X) = U^*(X, d)$ and $P^*(Y) = U^*(Y, d_1)$ implies that $C^*(\delta X)$ and $C^*(\delta_1 Y)$ are isomorphic. Thus δX and $\delta_1 Y$ are homeomorphic under a mapping t. But t carries G_δ-points of δX onto G_δ-points of $\delta_1 Y$ and and by Corollary 3.8 no point of $\delta X - X$ or $\delta_1 Y - Y$ is a G_δ-point. Moreover, each point X of the metric space (X, d) has a countable base of neighborhoods in δX. Hence each point of X is a G_δ-point of δX and similarly each point of Y is a G_δ-point of $\delta_1 Y$. Thus t carries X onto Y. Moreover, t is a p-homeomorphism of δX onto $\delta_1 Y$ hence the restriction t_1 of t to X is a p-homeomorphism of (X, δ)

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onto (Y, δ_1). Since δ and δ_1 are metric proximities it follows that t_1 is a uniform isomorphism of (X, δ) onto (Y, δ_1).

This completes the proof.

Theorem 4.1 remains true if "metric" is replaced by the condition that X and Y satisfy the first countability axiom and the uniform isomorphism is taken with respect to the unique totally bounded uniform structures in the respective proximity classes of δ and δ_1.

"Realcomplete" cannot be replaced by "realcompact" in Theorem 4.1. Take $X = R$ and $Y = R - \{0\}$ and let d and d_1 be the standard metrics for X and Y, respectively. Then $U^*(X, d)$ is isomorphic to $U^*(Y, d_1)$, but X and Y are not homeomorphic. Evidently, X and Y are realcompact but (Y, δ_1) is not realcomplete.

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